A New Look at a Nonrelativistic Shell Model: Study of the Mirror Nuclei ^{17}O and ^{17}F in the Symmetries of NCQM

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Abstract

In present work, the 3-dimensional modified Schrödinger equation is analytically solved for nuclei ^{17}O and ^{17}F, which can be modeled as a doubly magic isotope $^{17}O = n + (N = Z = 8)$ and$^{17}F = p + (N = Z = 8)$, with one additional nucleon (valence) in the $1d_{5/2}$ level under modified quadratic Hellmann potential in the symmetries of noncommutative quantum mechanics (NCQM), using the generalized Bopp's shift method. The new energy eigenvalues and the corresponding modified Hamiltonian operator are calculated in the 3-dimensional noncommutative real space phase (NC: 3D-RSP) symmetries. It is found that the perturbative solutions of the discrete spectrum can be expressed by the Gamma function, the discreet subatomic quantum numbers (j, l, s, m) and the potential parameters (α, a, b), in addition to the noncommutativity parameters ($\Theta$and$\overline{\theta}$). The total complete degeneracy of new energy levels of the modified quadratic Hellmann potential changed to become equals to the value$2n^2$ instead of the initial values n^2 in ordinary QM. Our results are in good agreement with the already existing literature in NCQM.

Keywords: Schrödinger equation, Mirror Nuclei ^{17}O and^{17}F, the quadratic Hellmann potential, noncommutative space phase and Bopp's shift method.

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1. Introduction

It is well known that the isotopes are atoms of the same element of protons and electrons that have different masses due to differing numbers of neutrons in the nucleus, it have many vital applications in theoretical and practical research. The two nuclei ^{17}O (three known stable isotopes of oxygen) and ^{17}F (Fluorine $_9F$ has 17 known isotopes) are good examples of isotopes. It has received much attention due to its rich experimental results on binding energy, single particle energy, etc. It is useful to calculate these quantities in order to test the microscopic theory by future experiments, as another example of isotopes, the Calcium and Scandium isotopes $^{41}Ca = n + (N = Z = 20)and^{41}Sc = p + (N = Z = 20)$ [1–3]. M. Mousavi *et al.* studied the Mirror Nuclei of ^{17}O and ^{17}Fin relativistic and nonrelativistic shell model, those, isotopes can be modeled as a doubly magic isotope $^{17}O = n + (N = Z = 8)$and$^{17}F = p + (N = Z = 8)$ with one additional nucleon (valence) in the $1d_{5/2}$ level. The ground-state spin and parity of (^{17}O and ^{17}F) are$j^{\Pi} = 5/2^{+}$, which corresponds to the spin and parity of the level where the valence nucleon resides [3-4]. Moreover, they obtained the energy spectrum in relativistic and nonrelativistic shell models [3]. The main objective is to develop the study of ref. [3] and expanding it into a large symmetry known by noncommutative quantum mechanics (NCQM) in order to achieve a more accurate physical vision. On the other hand, to explore the possibility of creating new applications and more profound interpretations in the sub-atomics using new version the modified quadratic Hellmann potential, because these potentials are important nuclear potentials for a description of the interaction between single nucleon in the $1d_{5/2}$ level and whole nuclei, which has the following form:

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$$V_{qh}(r) = \underbrace{-\frac{a}{r} + \frac{b}{r^2} e^{-\alpha r}}_{\text{Ordinary-QM}} \rightarrow V_{qh}(\hat{r}) = \underbrace{V_{qh}(r) + \left(\alpha b \frac{exp(-\alpha r)}{2r^3} + b \frac{exp(-\alpha r)}{r^4} - \frac{a}{2r^3}\right) \vec{L}\vec{\Theta}}_{\text{NCQM}} \quad (1)$$

We refer to this term $\vec{L}\vec{\Theta}$ in the materials and methods section. The new structure of NCQM based to new NC canonical commutations relations in three representations of Schrödinger, Heisenberg and Interactions pictures (SP, HP, and IP), respectively, as follows (Throughout this paper, the natural units $c = \hbar = 1$ will be used) [5-12]:

$$\begin{cases} [\hat{x}_\mu \overset{*}{,} \hat{p}_\nu] = [\hat{x}_\mu(t) \overset{*}{,} \hat{p}_\nu(t)] = [\hat{x}_{I\mu}(t) \overset{*}{,} \hat{p}_{I\nu}(t)] = i\hbar^{eff}\delta_{\mu\nu} \Rightarrow |\Delta\hat{x}_\mu \Delta\hat{p}_\nu| \geq \frac{\hbar^{eff}\delta_{\mu\nu}}{2} \\ [\hat{x}_\mu \overset{*}{,} \hat{x}_\nu] = [\hat{x}_\mu(t) \overset{*}{,} \hat{x}_\nu(t)] = [\hat{x}_{I\mu}(t) \overset{*}{,} \hat{x}_{I\nu}(t)] = i\theta_{\mu\nu} \Rightarrow |\Delta\hat{x}_\mu \Delta\hat{x}_\nu| \geq \left|\frac{\theta_{\mu\nu}}{2}\right| \\ [\hat{p}_\mu \overset{*}{,} \hat{p}_\nu] = [\hat{p}_\mu(t) \overset{*}{,} \hat{p}_\nu(t)] = [\hat{p}_{I\mu}(t) \overset{*}{,} \hat{p}_{I\nu}(t)] = i\overline{\theta}_{\mu\nu} \Rightarrow |\Delta\hat{p}_\mu \Delta\hat{p}_\nu| \geq \left|\frac{\overline{\theta}_{\mu\nu}}{2}\right| \end{cases} \quad (2)$$

where the indices$\mu, \nu \equiv \overline{1,3}$ and $\hbar^{eff} = \hbar\left(1 + \frac{\theta\overline{\theta}}{4\hbar^2}\right)$ is the effective Planck constant. This means that the principle of uncertainty for Heisenberg generalized to include another two new uncertainties related to the positions $(\hat{x}_\mu, \hat{x}_\nu)$ and the momenta's$(\hat{p}_\mu, \hat{p}_\nu)$, in addition to the ordinary uncertainty of$(\hat{x}_\mu, \hat{p}_\nu)$. The very small two parameters $\left(\theta^{\mu\nu}, \overline{\theta}^{\mu\nu}\right) = \varepsilon^{\mu\nu}(\theta, \overline{\theta})$ (compared to the energy) are elements of two antisymmetric real matrixes, parameters of noncommutativity and $(*)$ denote to the Weyl Moyal star product, which is generalized between two arbitrary functions (fg) (x, p) to the new form $(\hat{f}\hat{g})(\hat{x}, \hat{p}) \equiv (f * g)(x, p)$ in (NC: 3D-RSP) symmetries [10-17]:

$$(fg)\,(x, p) \rightarrow (f * g)(x, p) = \left(fg - \frac{i}{2}\theta^{\mu\nu}\,\partial_\mu^x f\,\partial_\nu^x g - \frac{i}{2}\overline{\theta}^{\mu\nu}\,\partial_\mu^p f\,\partial_\nu^p g\right)(x, p) \quad (3)$$

The second and the third terms in Eq. (3) are present the effects of (space-space) and (phase-phase) noncommutativity properties. However, the new operators:$\hat{\xi}_H(t) = (\hat{x}_i \vee \hat{p}_i)(t)$ and$\hat{\xi}_I(t) = (\hat{x}_{Ii} \vee \hat{p}_{Ii})(t)$ in (HP and IP, respectively) are depending to the corresponding new operator $\hat{\xi}_S = \hat{x}_i \vee \hat{p}_i$ in SP from the following projections relations:

$$\begin{cases} \xi_{\mu H}(t) = exp(i\hat{H}_{qh}(t - t_0))\xi_{\mu S}\,exp(-i\hat{H}_{qh}(t - t_0)) \\ \xi_{\mu I}(t) = exp(i\hat{H}_{oqh}(t - t_0))\xi_{\mu S}\,exp(-i\hat{H}_{oqh}(t - t_0)) \end{cases} \Rightarrow \begin{cases} \hat{\xi}_{\mu H}(t) = exp(i\hat{H}_{nc}^{qh}(t - t_0)) * \hat{\xi}_{\mu S} * exp(-i\hat{H}_{nc}^{qh}(t - t_0)) \\ \hat{\xi}_{\mu I}(t) = exp(i\hat{H}_{nco}^{qh}(t - t_0)) * \hat{\xi}_{\mu S} * exp(-i\hat{H}_{nco}^{qh}(t - t_0)) \end{cases} \quad (4)$$

Here$\xi_{\mu S} = x_\mu \vee p_\mu$, $\xi_{\mu H}(t) = (x_\mu \vee p_\mu)(t)$ and $\xi_{\mu I}(t) = (x_{I\mu} \vee p_{I\mu})(t)$ are the three representations in QM, while the dynamics of new systems $\frac{d\xi_H(t)}{\mathrm{dt}}$ are describe from the following motion equations in NCQM:

$$\frac{d\xi_H(t)}{\mathrm{dt}} = [\xi_H(t), \hat{H}_{qh}] + \frac{\partial\xi_H(t)}{\partial t} \Rightarrow \frac{d\hat{\xi}_H(t)}{\mathrm{dt}} = [\hat{\xi}_H(t) \overset{*}{,} \hat{H}_{nc}^{qh}] + \frac{\partial\hat{\xi}_H(t)}{\partial t} \quad (5)$$

the operators $\hat{H}_{oqh}$ and $\hat{H}_{qh}$ are the unperturbed and global Hamiltonian in NRQM for quadratic Hellmann potential while $\hat{H}_{nco}^{qh}$ and $\hat{H}_{nc}^{qh}$ the corresponding Hamiltonians for modified quadratic Hellmann potential in the NCQM. It should be noted that the noncommutativity was introduced firstly by W. Heisenberg in 1930 [18] and then by H. Syndre in 1947 [19]. This paper consists of five sections and the organization scheme is given as follows: In the next section, the theory part, we briefly review the SE with the quadratic Hellmann potential based. Section 3 is devoted to studying the MSE by applying the generalized Bopp's shift method to obtain the modified quadratic Hellmann potential and the modified spin-orbit operator. Then, we apply the standard perturbation theory to find the quantum spectrum of the ground state, (the first, the second and then^{th}) excited states which are produced by the effects of modified spin-orbit and modified Zeeman interactions. After that, in the fourth section, we present the main results in (NC: 3D-RSP) symmetries. Finally, in the last section, a summary and conclusions are presented.

2. Theory

2.1 Overview of the eigenfunctions and the energy eigenvalues for Mirror Nuclei ^{17}O and ^{17}F under the quadratic Hellmann potential in ordinary QM

We shall recall briefly in this section, the nonrelativistic Schrödinger equation of a single nucleon with mass Mmoving in the quadratic Hellmann potential, which describes the interaction between single nucleon in the $1d_{5/2}$ level and whole nuclei. The nuclei$^{17}O$and^{17}Fcan be modeled as a doubly magic isotope $^{17}O=n+(N=Z=8)$and$^{17}F=p+(N=Z=8)$, with one additional nucleon (valence) in the $1d_{5/2}$ level [3]:

$$V_{qh}(r)=-\frac{a}{r}+\frac{b}{r^2}e^{-\alpha r} \tag{6}$$

where the strength parameters a and b are real while αit is related to the range of the potential. If we insert this potential into the Schrödinger equation$\Psi(r,\theta,\phi)=R(r)Y_l^m(\theta,\phi)$, the radial part function $R(r)=\frac{U(r)}{r}$ is given as [3]:

$$\frac{d^2R(r)}{dr^2}+\frac{2}{r}\frac{dR(r)}{dr}+2\mu\left[E+\frac{a}{r}-\frac{b}{r^2}e^{-\alpha r}-\frac{l(l+1)}{2\mu r^2}\right]R(r)=0\rightarrow$$
$$\frac{d^2U(r)}{dr^2}+2\mu\left[E+\frac{a}{r}-\frac{b}{r^2}e^{-\alpha r}-\frac{l(l+1)}{2\mu r^2}\right]U(r)=0 \tag{7}$$

here μ denote to the reduced mass for the single nucleon and whole nuclei. [3] Give the complete wave function:

$$\Psi(r,\theta,\phi)=Nr^{(\sqrt{\chi_{0l}+1/4}-1/2)}\,exp\left(-\sqrt{\chi_{2n}}r\right)L_n^{2\sqrt{\chi_{0l}+1/4}}\left(\left(2+2\sqrt{\chi_{2n}}\right)r\right)Y_l^m(\theta,\phi) \tag{8}$$

In addition, the energyE_{nl} of the potential in Eq. (6) is given by [3]:

$$E_{nl}=-2\mu\frac{(a+b\alpha)^2}{\left[2n+1+2\sqrt{2\mu b+l(l+1)+1/4}\right]^2} \tag{9}$$

where$\chi_{0l}=2\mu+l(l+1)$, $\chi_{2n}=-2\mu E_n$ and N is a normalization constant while n is a natural number accounting for the radial excitation while l is a non-negative integer number which represents the orbital angular momentum.

3. Materials and Methods

3.1 Solution of MSE for modified quadratic Hellmann potential

In this sub-section, we shall give an overview or a brief preliminary for the modified quadratic Hellmann potential in (NC: 3D-RSP) symmetries. To perform this task the physical form of MSE, it is necessary to replace ordinary three-dimensional Hamiltonian operators$\hat{H}_{qh}(x_\mu,p_\mu)$, complex wave function $\Psi(\vec{r})$ and energy E_{nl} by new three Hamiltonian operators$\hat{H}_{nc}^{qh}(\hat{x}_\mu,\hat{p}_\mu)$, new complex wave function$\Psi(\vec{\hat{r}})$ and the new valuesE_{nc}^{qh}, respectively. In addition to replacing the ordinary product by the Weyl Moyal star product, which allows us to construct the MSE in (NC-3D: RSP) symmetries as [20-24]:

$$\hat{H}_{qh}(x_\mu,p_\mu)\psi(\vec{r})=E_{nl}\psi(\vec{r})\Rightarrow\hat{H}_{nc}^{qh}(\hat{x}_\mu,\hat{p}_\mu)*\Psi(\vec{\hat{r}})=E_{nc}^{qh}\Psi(\vec{\hat{r}}) \tag{10}$$

The generalized Bopp's shift method has been successfully applied to relativistic and nonrelativistic noncommutative quantum mechanical problems using modified Dirac equation (MDE), modified Klein-Gordon equation (MKGE) and MSE. This method has produced very promising results for a number of situations having physical, chemical interest. The method reduces MDE, MKGE, and MSE to the Dirac equation, Klein-Gordon and

Schrödinger equations, respectively, under two- simultaneously translations in space and phase. It based on the following new commutators [13-16, 22-27]:

$$\left[\hat{x}_\mu, \hat{x}_\nu\right] = \left[\hat{x}_\mu(t), \hat{x}_\nu(t)\right] = i\theta_{\mu\nu} \text{ and } \left[\hat{p}_\mu, \hat{p}_\nu\right] = \left[\hat{p}_\mu(t), \hat{p}_\nu(t)\right] = i\overline{\theta}_{\mu\nu} \quad (11)$$

The new generalized positions and momentum coordinates $(\hat{x}_\mu, \hat{p}_\nu)$ in (NC: 3D-RSP) are defined in terms of the commutative counterparts (x_μ, p_ν) in ordinary quantum mechanics via, respectively [5-10]:

$$(x_\mu, p_\nu) \Rightarrow (\hat{x}_\mu, \hat{p}_\nu) = \left(x_\mu - \frac{\theta_{\mu\nu}}{2}p_\nu, p_\mu + \frac{\overline{\theta}_{\mu\nu}}{2}x_\nu\right) \quad (12)$$

The above equation allows us to obtain the two operators $(\hat{r}^2, \hat{p}^2)$ in (NC-3D: RSP) symmetries [20-25]:

$$(r^2, p^2) \Rightarrow (\hat{r}^2, \hat{p}^2) = \left(r^2 - \vec{\mathrm{L}}\vec{\Theta}, p^2 + \vec{\mathrm{L}}\vec{\overline{\theta}}\right) \quad (13)$$

The two couplings $\vec{\mathrm{L}}\vec{\Theta}$ and $\vec{\mathrm{L}}\vec{\overline{\theta}}$ are $(L_x\Theta_{12} + L_y\Theta_{23} + L_z\Theta_{13})$ and$(L_x\overline{\theta}_{12} + L_y\overline{\theta}_{23} + L_z\overline{\theta}_{13})$, respectively and ($L_x$, L_y andL_z) are the three components of angular momentum operator$\vec{L}$ while$\Theta_{\mu\nu} = \theta_{\mu\nu}/2$. Thus, the reduced Schrödinger equation (without star product) can be written as:

$$\widehat{H}(\hat{x}_\mu, \hat{p}_\mu) * \Psi\left(\vec{\hat{r}}\right) = E_{nc-qh}\Psi\left(\vec{\hat{r}}\right) \Rightarrow H(\hat{x}_\mu, \hat{p}_\mu)\psi\left(\vec{r}\right) = E_{nc-qh}\psi\left(\vec{r}\right) \quad (14)$$

The new operator of Hamiltonian $H_{nc}^{qh}(\hat{x}_\mu, \hat{p}_\nu)$ can be expressed as:

$$H_{nc}^{qh}(\hat{x}_\mu, \hat{p}_\mu) \equiv H\left(\hat{x}_\mu = x_\mu - \frac{\theta_{\mu\nu}}{2}p_\nu, \hat{p}_\mu = p_\mu + \frac{\bar{\theta}_{\mu\nu}}{2}x_\nu\right) = \frac{\hat{p}^2}{2\mu} + V_{qh}\left(\hat{r} = \sqrt{\left(x_\mu - \frac{\theta_{\mu\nu}}{2}p_\nu\right)\left(x_\mu - \frac{\theta_{\mu\beta}}{2}p_\beta\right)}\right) \quad (15)$$

Where $V_{qh}(\hat{r})$ denote to the modified quadratic Hellmann potential in (NC: 3D-RSP) symmetries:

$$V_{qh}(r) \Rightarrow V_{qh}(\hat{r}) = -\frac{a}{\hat{r}} + \frac{b}{\hat{r}^2}e^{-\alpha\hat{r}} \quad (16)$$

Again, applying Eq. (13) to find the three terms ($C\hat{r}$, $\left(-\frac{b}{\hat{r}}\right)$ and$(-D\hat{r}^2)$), which will be used to determine the modified quadratic Hellmann potential$V_{qh}(\hat{r})$, as follows:

$$\begin{cases} -\frac{a}{r} \to -\frac{a}{\hat{r}} = -\frac{a}{r} - \frac{a}{2r^3}\vec{\mathrm{L}}\vec{\Theta} + O(\Theta^2) \\ \frac{b}{r^2} \to \frac{b}{\hat{r}^2} = \frac{b}{r^2} + \frac{b}{r^4}\vec{\mathrm{L}}\vec{\Theta} + O(\Theta^2) \\ e^{-\alpha r} \to -e^{-\alpha\hat{r}} = e^{-\alpha r} + \alpha\frac{e^{-\alpha r}}{2r}\vec{\mathrm{L}}\vec{\Theta} + O(\Theta^2) \end{cases} \quad (17)$$

A simple calculation gives:

$$b\frac{e^{-\alpha\hat{r}}}{\hat{r}^2} = b\frac{e^{-\alpha r}}{r^2} + \left(\alpha b\frac{e^{-\alpha r}}{2r^3} + b\frac{e^{-\alpha r}}{r^4}\right)\vec{\mathrm{L}}\vec{\Theta} + O(\Theta^2) \quad (18)$$

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Substituting, Eqs. (17) and (18) into Eq. (16), gives the modified quadratic Hellmann potential in (NC-3D: RSP) symmetries as follows:

$$V_{qh}(\hat{r}) = V_{qh}(r) + \left(\alpha b \frac{exp(-\alpha r)}{2r^3} + b\frac{exp(-\alpha r)}{r^4} - \frac{a}{2r^3}\right)\vec{L}\vec{\Theta} \tag{19}$$

By making the substitution above equation into Eq. (15), we find the global our working modified Hamiltonian operator $H_{\mathrm{nc}}^{qh}(\hat{r})$ satisfies the equation in (NC: 3D-RSP) symmetries:

$$H_{qh}(x_\mu, p_\nu) \Rightarrow H_{nc}^{qh}(\hat{r}) = H_{qh}(x_\mu, p_\nu) + H_{\mathrm{per}}^{qh}(r, \Theta, \bar{\theta}) \tag{20}$$

where the operator $H_{qh}(x^\mu, p^\nu)$ is just the ordinary Hamiltonian operator with modified quadratic Hellmann potential in commutative quantum mechanics:

$$H_{qh}(x^\mu, p^\nu) = \frac{p^2}{2\mu} - \frac{a}{r} + \frac{b}{r^2}e^{-\alpha r} \tag{21}$$

while the rest part $H_{\mathrm{per}}^{qh}(r, \Theta, \bar{\theta})$ (the perturbative Hamiltonian operator) is proportional with two infinitesimals parameters (Θ and$\bar{\theta}$):

$$H_{\mathrm{pert}}^{qh}(r, \Theta, \bar{\theta}) = \left(\alpha b \frac{exp(-\alpha r)}{2r^3} + b\frac{exp(-\alpha r)}{r^4} - \frac{a}{2r^3}\right)\vec{L}\vec{\Theta} + \vec{L}\vec{\bar{\theta}} \tag{22}$$

Thus, we can consider $H_{\mathrm{pert}}^{qh}(r, \Theta, \bar{\theta})$ as a perturbation term compared with the principal Hamiltonian operator$H_{qh}(x^\mu, p^\nu)$ in (NC: 3D-RSP) symmetries.

3.2 The exact modified spin-orbit operator for the mirror nuclei ^{17}O and^{17}Funder modified quadratic Hellmann potential:

In this subsection, we will apply the same strategy, which we have seen exclusively in some of our published scientific works [22-30]. Under a particular choice, one can easily reproduce both couplings ($\vec{L}\vec{\Theta}$and$\vec{L}\vec{\bar{\theta}}$) to the new physical forms ($\gamma\Theta\vec{L}\vec{S}and\gamma\bar{\theta}\vec{L}\vec{S}$), respectively. Thus, the perturbative Hamiltonian operator $H_{\mathrm{pert}}^{qh}(r, \Theta, \bar{\theta})$ for mirror nuclei ^{17}O and^{17}F, will be transformed into a modified spin-orbit operator $H_{\text{so-}qh}(r, \Theta, \bar{\theta})$ under modified quadratic Hellmann potential as follows:

$$H_{\text{per-}qh}(r, \Theta, \bar{\theta}) \to H_{\text{so-}qh}(r, \Theta, \bar{\theta}) \equiv \gamma\left\{\left(\alpha b \frac{exp(-\alpha r)}{2r^3} + b\frac{exp(-\alpha r)}{r^4} - \frac{a}{2r^3}\right) + \frac{\bar{\theta}}{2\mu}\right\}\vec{L}\vec{S} \tag{23}$$

Here $\Theta = \sqrt{\Theta_{12}^2 + \Theta_{23}^2 + \Theta_{13}^2}$, $\bar{\theta} = \sqrt{\bar{\theta}_{12}^2 + \bar{\theta}_{23}^2 + \bar{\theta}_{13}^2}and\gamma$ is a new constant, which plays the role of strong coupling constant in the strong interaction or in the quantum chromodynamics theory , we have chosen the two vectors $\vec{\Theta}$ and $\vec{\bar{\theta}}$ parallel to the spin-S of the mirror nuclei ^{17}O and ^{17}F. Furthermore, the above perturbative term $H_{\text{per-}qh}(r)$ can be rewritten to the following new form:

$$H_{so-qh}(r, \Theta, \bar{\theta}) = \frac{\gamma}{2}\left\{\left(\alpha b \frac{exp(-\alpha r)}{2r^3} + b\frac{exp(-\alpha r)}{r^4} - \frac{a}{2r^3}\right)\Theta + \frac{\bar{\theta}}{2\mu}\right\}\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right) \tag{24}$$

where $\vec{J}$ is the total angular momentum of mirror nuclei ^{17}O and^{17}F. This operator, $H_{\text{so-}qh}(r, \Theta, \bar{\theta})$, traduces the coupling between spin-S and orbital momentum$\vec{L}$. The set $(H_{so}^{qh}(r, \Theta, \bar{\theta})$,$J^2$, L^2, $S^2$and $J_z)$ forms a complete of conserved physics quantities. For the single nucleon (neutron or proton) in the mirror nuclei^{17}O and^{17}F, the spin-

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1/2, the eigenvalues of the spin-orbit coupling operator $k(l) \equiv \frac{1}{2}\{j(j+1) - l(l+1) - 3/4\}$corresponding $j = l + 1/2$ (spin up) and $j = l - 1/2$ (spin down), respectively. Then, one can form a diagonal (3×3)matrix for modified Quadratic Hellmann potential in (NC: 3D-RSP) symmetries, with element$(H_{so}^{qh})_{33} = 0$ and the other non-null elements $(H_{so}^{qh})_{11}$ and $(H_{so}^{qh})_{22}$ are given by:

$$(H_{so}^{qh})_{11} = \gamma k_+(l)\left\{\left(\alpha b\frac{exp(-\alpha r)}{2r^3} + b\frac{exp(-\alpha r)}{r^4} - \frac{a}{2r^3}\right)\Theta + \frac{\overline{\theta}}{2\mu}\right\} \text{if } j = l + 1/2$$
$$(H_{so}^{qh})_{22} = \gamma k_-(l)\left\{\left(\alpha b\frac{exp(-\alpha r)}{2r^3} + b\frac{exp(-\alpha r)}{r^4} - \frac{a}{2r^3}\right)\Theta + \frac{\overline{\theta}}{2\mu}\right\} \text{if } j = l - 1/2$$

(25)

Here

$(k_+(l), k_-(l)) \equiv \frac{1}{2}\big((l+1/2)(l+3/2) - l(l+1) - 3/4, (l-1/2)(l+1/21) - l(l+1) - 3/4\big) = \frac{1}{2}(2l, -l-1)$ and j is the total quantum number of single nucleon. The non-null diagonal elements$(H_{so-cp})_{11}$ and $(H_{so-cp})_{22}$ of the modified Hamiltonian operator $H_{nc-qh}(\hat{r})$ will change the energy values $E_{nl}$by creating two new values:

$$\begin{cases} E_{\text{so-uqh}} = \langle \Psi(r,\theta,\phi) \left| (H_{so-qh})_{11} \right| \Psi(r,\theta,\phi) \rangle \\ E_{\text{so-dqh}} = \langle \Psi(r,\theta,\phi) \left| (H_{so-qh})_{22} \right| \Psi(r,\theta,\phi) \rangle \end{cases} \quad (26)$$

We will see them in detail in the next subsection. After profound calculation, one can show that the new radial function $U_{nl}(r)$ satisfying the following differential equation for modified Quadratic Hellmann potential:

$$\frac{d^2U_{nl}(r)}{dr^2} + 2\mu\left[E_{nc-qh} + \frac{a}{r} - \frac{b}{r^2}e^{-\alpha r} - \left(\alpha b\frac{exp(-\alpha r)}{2r^3} + b\frac{exp(-\alpha r)}{r^4} - \frac{a}{2r^3}\right)\vec{\mathrm{L}}\vec{\Theta} - \frac{\vec{\mathrm{L}}\vec{\theta}}{2\mu} - \frac{l(l+1)}{2\mu r^2}\right]U_{nl}(r) = 0$$

(27)

Through our observation of the expression of $H_{\text{per-}qh}(r)$, which appear in the equation (23), we see it as proportionate to two infinitesimals parameters (Θ and$\bar{\theta}$), thus, in what follows, we proceed to solve the modified radial part of the MSE that is in equation (27) by applying standard perturbation theory to find acceptable solutions at first order of two parameters Θ and$\bar{\theta}$. The proposed solutions for MSE under the modified quadratic Hellmann potential includes energy corrections, which produced automatically from two principal physical phoneme's, the first one is the effect of modified spin-orbit interaction and the second is the modified Zeeman effect.

3.3 The exact modified spin-orbit spectrum for mirror *nuclei ^{17}O and ^{17}F under the modified quadratic Hellmann potential*

The purpose here is to give a complete prescription for determine the energy level of the ground state, (the first, the second and then^{th} excited states of mirror nuclei ^{17}O and ^{17}F with one additional nucleon (valence) in the $1d_{5/2}$ level. We first find the corrections ($E_{so-uqh}(n,l,j)$ and$E_{so-dqh}(n,l,j)$) for mirror nuclei ^{17}O and ^{17}F under modified Quadratic Hellmann potential, which have two polarities (up and down) $j = l + 1/2$ and $j = l - 1/2$ (spin down) , respectively, of the single nucleon (neutron or proton) at first order of two parameters (Θ and $\bar{\theta}$). Moreover, by applying the perturbative theory, we obtained the following results:

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$$E_{so-uqh} = \gamma N^2 k_+(l) \int_0^{+\infty} r^{(2\sqrt{\chi_{0l}+1/4}+1)} exp\left(-2\sqrt{\chi_{2n}}r\right)\left[L_n^{2\sqrt{\chi_{0l}+1/4}}\left(\left(2 + 2\sqrt{\chi_{2n}}\right)r\right)\right]^2 \left\{\left(\alpha b \frac{exp(-\alpha r)}{2r^3} + b\frac{exp(-\alpha r)}{r^4} - \frac{a}{2r^3}\right)\Theta + \frac{\bar{\theta}}{2\mu}\right\} dr$$
$$E_{so-dqh} = \gamma N^2 k_-(l) \int_0^{+\infty} r^{(2\sqrt{\chi_{0l}+1/4}+1)} exp\left(-2\sqrt{\chi_{2n}}r\right)\left[L_n^{2\sqrt{\chi_{0l}+1/4}}\left(\left(2 + 2\sqrt{\chi_{2n}}\right)r\right)\right]^2 \left\{\left(\alpha b \frac{exp(-\alpha r)}{2r^3} + b\frac{exp(-\alpha r)}{r^4} - \frac{a}{2r^3}\right)\Theta + \frac{\bar{\theta}}{2\mu}\right\} dr$$

(28)

We have used the orthogonality property of the spherical harmonics $\int Y_l^m(\theta,\phi)Y_{l'}^{m'}(\theta,\phi)\, sin(\theta)\, d\theta d\phi = \delta_{ll'}\delta_{mm'}$. Now, we can rewrite the above equations as a sum of four contributions involving the parameters (Θ and $\bar{\theta}$):

$$E_{so-uqh}(n,l,j) = \gamma N^2 k_+(l)\left\{\Theta[T_1(n,l) + T_2(n,l) + T_3(n,l)] + \frac{\bar{\theta}}{2\mu}T_4(n,l)\right\}$$
$$E_{so-dqh}(n,l,j) = \gamma N^2 k_+(l)\left\{\Theta[T_1(n,l) + T_2(n,l) + T_3(n,l)] + \frac{\bar{\theta}}{2\mu}T_4(n,l)\right\}$$

(29)

the expressions of the 4-factors $T_i\left(i = \overline{1,4}\right)$ are given by:

$$T_1(n,l) = \frac{\alpha b}{2}\int_0^{+\infty} r^{(2\sqrt{\chi_{0l}+1/4}-2)} exp\left(-2\sqrt{\chi_{2n}}r\right) exp(-\alpha r)\left[L_n^{2\sqrt{\chi_{0l}+1/4}}\left(\left(2 + 2\sqrt{\chi_{2n}}\right)r\right)\right]^2 dr$$
$$T_2(n,l) = b\int_0^{+\infty} r^{(2\sqrt{\chi_{0l}'+1/4}-3)} exp\left(-2\sqrt{\chi_{2n}}r\right) exp(-\alpha r)\left[L_n^{2\sqrt{\chi_{0l}'+1/4}}\left(\left(2 + 2\sqrt{\chi_{2n}}\right)r\right)\right]^2 dr$$
$$T_3(n,l) = -\frac{a}{2}\int_0^{+\infty} r^{(2\sqrt{\chi_{0l}+1/4}-2)} exp\left(-2\sqrt{\chi_{2n}}r\right)\left[L_n^{2\sqrt{\chi_{0l}+1/4}}\left(\left(2 + 2\sqrt{\chi_{2n}}\right)r\right)\right]^2 dr$$
$$T_4(n,l) = \int_0^{+\infty} r^{(2\sqrt{\chi_{0l}+1/4}+1)} exp\left(-2\sqrt{\chi_{2n}}r\right)\left[L_n^{2\sqrt{\chi_{0l}+1/4}}\left(\left(2 + 2\sqrt{\chi_{2n}}\right)r\right)\right]^2 dr$$

(30)

The two nucleus ^{17}O and ^{17}F can be modeled as a doubly magic isotope $^{17}O = n + (N = Z = 8)$ and $^{17}F = p + (N = Z = 8)$ with one additional nucleon (valence) in the $1d_{5/2}$ level. If identified with the typical state nX_j [31], we conclude that the additional nucleon or the single nucleon (neutron or proton) correspond to the subatomic quantum numbers $(n = 1, X \equiv d,\ j = 5/2$ and $l = 2)$. We have $L_{n=1}^{2\sqrt{\chi_{02}+1/4}}\left(\left(2 + 2\sqrt{\chi_{21}(l = 2)}\right)r\right) = -\Omega r + \Lambda$, with $\chi_{21}(l = 2) = \frac{4\mu^2(a+b\alpha)^2}{\left[3+2\sqrt{2\mu b+6+1/4}\right]^2}$, $\chi_{02}(l = 2) = 2\mu + 6$, $\Omega = 2\left(2 + 2\sqrt{\chi_{21}(l = 2)}\right)$ and $\Lambda = 2\sqrt{\chi_{02}(l = 2) + 1/4} + 1$. Thus, the expressions of the 4-factors $T_i\left(i = \overline{1,4}\right)$ are given by:

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$$T_1(n=1,l=2)=\frac{\alpha b}{2}\int_0^{+\infty} r^{(\delta_{02}-2)}\, exp\left(-\left(2\sqrt{\chi_{21}}+\alpha\right)r\right)[-\Omega r+\Lambda]^2 dr$$
$$T_2(n=1,l=2)=b\int_0^{+\infty} r^{(\delta_{02}-3)}\, exp\left(-\left(2\sqrt{\chi_{21}}+\alpha\right)r\right)[-\Omega r+\Lambda]^2 dr$$
$$T_3(n=1,l=2)=-\frac{a}{2}\int_0^{+\infty} r^{(\delta_{02}-2)}\, exp\left(-2\sqrt{\chi_{21}}r\right)[-\Omega r+\Lambda]^2 dr$$
$$T_4(n=1,l=2)=\int_0^{+\infty} r^{(\delta_{02}+1)}\, exp(-2\sqrt{\chi_{21}}r)[-\Omega r+\Lambda]^2 dr$$

(31)

Here $\delta_{02}=2\sqrt{\chi_{02}(l=2)+1/4}$. A direct simplification gives:

$$T_1(n=1,l=2)=\frac{\alpha b}{2}\int_0^{+\infty}\left[\Omega^2 r^{\delta_{02}+1-1}+2\Omega\Lambda r^{\delta_{02}-1}+\Lambda^2 r^{\delta_{02}-1-1}\right] exp\left(-\left(2\sqrt{\chi_{21}}+\alpha\right)r\right) dr$$
$$T_2(n=1,l=2)=b\int_0^{+\infty}\left[\Omega^2 r^{\delta_{02}-1}+2\Omega\Lambda r^{\delta_{02}-1-1}+\Lambda^2 r^{\delta_{02}-2-1}\right] exp\left(-\left(2\sqrt{\chi_{21}}+\alpha\right)r\right) dr$$
$$T_3(n=1,l=2)=-\frac{a}{2}\int_0^{+\infty}\left[\Omega^2 r^{\delta_{02}+1-1}+2\Omega\Lambda r^{\delta_{02}-1}+\Lambda^2 r^{\delta_{02}-1-1}\right] exp\left(-2\sqrt{\chi_{21}}r\right) dr$$
$$T_4(n=1,l=2)=\int_0^{+\infty}\left[\Omega^2 r^{\delta_{02}+4-1}+2\Omega\Lambda r^{\delta_{02}+3-1}+\Lambda^2 r^{\delta_{02}+2-1}\right] exp(-2\sqrt{\chi_{21}}r)\, dr$$

(32)

It is convenient to use the following special integral [32]:

$$\int_0^{+\infty} x^{\nu-1}\, exp(-\lambda x^p)\, dx=\frac{\lambda^{-\frac{\nu}{p}}}{p}\Gamma\left(\frac{\nu}{p}\right) \quad (33)$$

With conditions ($Re\,\lambda\rangle 0$, $Re\,\nu\rangle 0$ and$p\rangle 0$) and $\Gamma\left(\frac{\nu}{p}\right)$ the ordinary Gamma function. After straightforward calculations, we can obtain explicit results:

$$T_1(n=1,l=2)=\frac{\alpha b}{2}\left\{\Omega^2\beta_1^{-(\delta_{02}+1)}\Gamma(\delta_{02}+1)+2\Omega\Lambda\beta_1^{-\delta_{02}}\Gamma(\delta_{02})+\Lambda^2\beta_1^{-(\delta_{02}-1)}\Gamma(\delta_{02}-1)\right\}$$
$$T_2(n=1,l=2)=b\left\{\Omega^2\beta_1^{-\delta_{02}}\Gamma(\delta_{02})+2\Omega\Lambda\beta_1^{-(\delta_{02}-1)}\Gamma(\delta_{02}-1)+\Lambda^2\beta_1^{-(\delta_{02}-2)}\Gamma(\delta_{02}-2)\right\}$$
$$T_3(n=1,l=2)=-\frac{a}{2}\left\{\Omega^2\beta_2^{-(\delta_{02}+1)}\Gamma(\delta_{02}+1)+2\Omega\Lambda\beta_2^{-\delta_{02}}\Gamma(\delta_{02})+\Lambda^2\beta_2^{-(\delta_{02}-1)}\Gamma(\delta_{02}-1)\right\}$$
$$T_4(n=1,l=2)=\left\{\Omega^2\beta_2^{-(\delta_{02}+4)}\Gamma(\delta_{02}+4)+2\Omega\Lambda\beta_2^{-(\delta_{02}+3)}\Gamma(\delta_{02}+3)+\Lambda^2\beta_2^{-(\delta_{02}+2)}\Gamma(\delta_{02}+2)\right\}$$

(34)

with $\beta_1=2\sqrt{\chi_{21}}+\alpha$ and $\beta_2=2\sqrt{\chi_{21}}$. Allows us to obtain the exact modifications$E_{\text{so-uqh}}(n=1,l=2,j=5/2)$ and $E_{\text{so-dqh}}(n=1,l=2,j=5/2)$ of the ground state for the nuclei [17]O and [17]F with one additional nucleon (valence) in the $1d_{5/2}$ level:

$$E_{\text{so-uqh}}(n=1,l=2,j=5/2)=2\gamma N^2\left\{\Theta T_{11}(n=1,l=2)+\frac{\bar{\theta}}{2\mu}T_4(n=1,l=2)\right\}$$

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$$E_{\text{so-d}qh}(n=1,l=2,j=5/2) = -\frac{3}{2}N^2\left\{\Theta T_{11}(n=1,l=2)+\frac{\bar{\theta}}{2\mu}T_4(n=1,l=2)\right\} \quad (35)$$

with $T_{11}(n=1,l=2)=T_1(n=1,l=2)+T_2(n=1,l=2)+T_3(n=1,l=2)$. The first excited state corresponds to the one additional nucleon (valence) in the $2S_{1/2}$ level. Thus, the additional nucleon or the single nucleon (neutron or proton) correspond the subatomic quantum numbers ($n=2, X\equiv S$, $j=1/2$ and $l=0$), then, to obtain, the exact modifications $E_{\text{so-u}qh}(n=2,l=0,j=1/2)$ and $E_{\text{so-d}qh}(n=2,l=0,j=1/2)$ for the first excited state are, we replace $L_n^{2\sqrt{\chi_{0l}+1/4}}\left((2+2\sqrt{\chi_{2n}})r\right)$ in eq. (30) by $L_{n=2}^{2\sqrt{\chi_{00}+1/4}}((2+2\sqrt{\chi_{22}})r)=f_1r^2+f_2r+f_3$ with $f_1=\frac{1}{2}(2+2\sqrt{\chi_{22}})^2$, $f_2=-(2\sqrt{\chi_{00}+1/4}+2)(2+2\sqrt{\chi_{22}})$, $f_3=\frac{1}{2}(2\sqrt{\chi_{00}+1/4}+1)(2\sqrt{\chi_{00}+1/4}+2)$, $\chi_{00}(l=0)=2\mu$ and $\chi_{22}(l=0)=\frac{4\mu^2(a+b\alpha)^2}{[5+2\sqrt{2\mu b+1/4}]^2}$, allowed us to obtain $E_{\text{so-u}qh}(n=2,l=0,j=1/2)$ and $E_{\text{so-d}qh}(n=2,l=0,j=1/2)$ as:

$$E_{\text{so-u}qh}(n=2,l=0,j=1/2)=0$$
$$E_{\text{so-d}qh}(n=2,l=0,j=1/2)=-1/2\gamma N^2\left\{\Theta T_{12}(n=2,l=0)+\frac{\bar{\theta}}{2\mu}T_4(n=2,l=0)\right\} \quad (36)$$

with $T_{12}(n=2,l=0)=T_1(n=2,l=0)+T_2(n=2,l=0)+T_3(n=2,l=0)$ while the 4-foctors $T_1(n=2,l=0)$, $T_2(n=2,l=0)$, $T_3(n=2,l=0)$ and $T_4(n=2,l=0)$are given by :

$$T_1(n=2,l=0)=\frac{\alpha b}{2}\int_0^{+\infty} r^{(\delta_{00}-2)}\,exp\left(-\left(2\sqrt{\chi_{22}}+\alpha\right)r\right)[f_1r^2+f_2r+f_3]^2dr$$
$$T_2(n=2,l=0)=b\int_0^{+\infty} r^{(\delta_{00}-3)}\,exp\left(-\left(2\sqrt{\chi_{22}}+\alpha\right)r\right)[f_1r^2+f_2r+f_3]^2dr$$
$$T_3(n=2,l=0)=-\frac{a}{2}\int_0^{+\infty} r^{(\delta_{00}-2)}\,exp\left(-2\sqrt{\chi_{22}}r\right)[f_1r^2+f_2r+f_3]^2dr$$
$$T_4(n=2,l=0)=\int_0^{+\infty} r^{(\delta_{00}+1)}\,exp\left(-2\sqrt{\chi_{22}}r\right)[f_1r^2+f_2r+f_3]^2dr \quad (37)$$

Here $\delta_{00}=2\sqrt{\chi_{00}+1/4}$. A simple calculation gives:

$$T_1(n=2,l=0)=\frac{\alpha b}{2}\int_0^{+\infty}\left[g_1r^{(\delta_{00}+3-1)}+g_2r^{(\delta_{00}+2-1)}+g_3r^{(\delta_{00}+1-1)}+g_4r^{(\delta_{00}-1)}+g_5r^{(\delta_{00}-1-1)}\right]exp\left(-\left(2\sqrt{\chi_{22}}+\alpha\right)r\right)dr$$
$$T_2(n=2,l=0)=b\int_0^{+\infty}\left[g_1r^{(\delta_{00}+2-1)}+g_2r^{(\delta_{00}+1-1)}+g_3r^{(\delta_{00}-1)}+g_4r^{(\delta_{00}-1-1)}+g_5r^{(\delta_{00}-2-1)}\right]exp\left(-\left(2\sqrt{\chi_{22}}+\alpha\right)r\right)dr$$
$$T_3(n=2,l=0)=-\frac{a}{2}\int_0^{+\infty}\left[g_1r^{(\delta_{00}+3-1)}+g_2r^{(\delta_{00}+2-1)}+g_3r^{(\delta_{00}+1-1)}+g_4r^{(\delta_{00}-1)}+g_5r^{(\delta_{00}-2)}\right]exp\left(-2\sqrt{\chi_{22}}r\right)dr$$
$$T_4(n=2,l=0)=\int_0^{+\infty}\left[g_1r^{(\delta_{00}+6-1)}+g_2r^{(\delta_{00}+5-1)}+g_3r^{(\delta_{00}+4-1)}+g_4r^{(\delta_{00}+3-1)}+g_5r^{(\delta_{00}+2-1)}\right]exp\left(-2\sqrt{\chi_{22}}r\right)dr \quad (38)$$

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with$(g_1, g_2, g_3, g_4, g_5) = \left(f_1{}^2, 2f_1f_2, 2f_1f_3 + f_2{}^2, 2f_3f_2, f_3{}^2\right)$. It is convenient to apply the special integral that we have seen in eq. (33), to obtain the 4-factors$T_1(n = 2, l = 0)$, $T_2(n = 2, l = 0)$, $T_3(n = 2, l = 0)$and$T_4(n = 2, l = 0)$ as:

$$\begin{aligned}
T_1(n=2,l=0) &= \frac{\alpha b}{2}\begin{Bmatrix} g_1\lambda_1{}^{-\delta_{00}-3}\Gamma(\delta_{00}+3) + g_2\lambda_1{}^{-\delta_{00}-2}\Gamma(\delta_{00}+2) + g_3\lambda_1{}^{-\delta_{00}-1}\Gamma(\delta_{00}+1) + g_4\lambda_1{}^{-\delta_{00}}\Gamma(\delta_{00}) \\ +g_5\lambda_1{}^{-\delta_{00}+3}\Gamma(\delta_{00}-3) \end{Bmatrix}\\
T_2(n=2,l=0) &= b\begin{Bmatrix} g_1\lambda_1{}^{-\delta_{00}-2}\Gamma(\delta_{00}+2) + g_2\lambda_1{}^{-\delta_{00}-1}\Gamma(\delta_{00}+1) + g_3\lambda_1{}^{-\delta_{00}}\Gamma(\delta_{00}) + g_4\lambda_1{}^{-\delta_{00}+1}\Gamma(\delta_{00}-1) \\ +g_5\lambda_1{}^{-\delta_{00}+2}\Gamma(\delta_{00}-2) \end{Bmatrix}\\
T_3(n=2,l=0) &= -\frac{a}{2}\begin{Bmatrix} g_1\lambda_2{}^{-\delta_{00}-3}\Gamma(\delta_{00}+3) + g_2\lambda_2{}^{-\delta_{00}-2}\Gamma(\delta_{00}+2) + g_3\lambda_2{}^{-\delta_{00}-1}\Gamma(\delta_{00}+1) + g_4\lambda_2{}^{-\delta_{00}}\Gamma(\delta_{00}) \\ +g_5\lambda_2{}^{-\delta_{00}+1}\Gamma(\delta_{00}-1) \end{Bmatrix}\\
T_4(n=2,l=0) &= g_1\lambda_2{}^{-\delta_{00}-6}\Gamma(\delta_{00}+6) + g_2\lambda_2{}^{-\delta_{00}-5}\Gamma(\delta_{00}+5) + g_3\lambda_2{}^{-\delta_{00}-4}\Gamma(\delta_{00}+4) + g_4\lambda_2{}^{-\delta_{00}-3}\Gamma(\delta_{00}+3)\\
&\quad + g_5\lambda_2{}^{-\delta_{00}-2}\Gamma(\delta_{00}+2)
\end{aligned} \tag{39}$$

with $\lambda_1 = 2\sqrt{\chi_{22}} + \alpha$and $\lambda_2 = 2\sqrt{\chi_{22}}$. The second excited state corresponds to the nuclei ^{17}O and $^{17}F$with additional nucleon (valence) in the $1d_{3/2}$ level, thus, the additional nucleon or the single nucleon (neutron or proton) correspond the subatomic quantum numbers $(n = 1, X \equiv d,\ j = 3/2$ and $l = 1)$, we replace $L_n^{2\sqrt{\chi_{0l}+1/4}}\left(\left(2 + 2\sqrt{\chi_{2n}(l)}\right)r\right)$ in eq. (30) by $L_{n=1}^{2\sqrt{\chi_{01}(l=1)+1/4}}\left(\left(2 + 2\sqrt{\chi_{21}(l=1)}\right)r\right) = \Omega_2 r + \Lambda_2$, with $\Omega_2 = \left(2 + 2\sqrt{\chi_{21}(l=1)}\right)$, $\Lambda_2 = 2\sqrt{\chi_{01}(l=1) + 1/4} + 1$, $\chi_{01}(l=1) = 2\mu + 2$ and $\chi_{21}(l=1) = \frac{4\mu^2(a+b\alpha)^2}{\left[5+2\sqrt{2\mu b+2+1/4}\right]^2}$, then, the exact modifications are $E_{\text{so-uqh}}(n = 1, l = 1, j = 3/2)$ and $E_{\text{so-dqh}}(n = 1, l = 1, j = 3/2)$:

$$\begin{aligned}
E_{\text{so-uqh}}(n=1,l=1,j=3/2) &= \gamma N^2\left(\Theta T_{11}(n=1,l=1) + \frac{\bar{\theta}}{2\mu}T_4(n=1,l=1)\right)\\
E_{\text{so-dqh}}(n=1,l=1,j=3/2) &= -\gamma N^2\left(\Theta T_{11}(n=1,l=1) + \frac{\bar{\theta}}{2\mu}T_4(n=1,l=1)\right)
\end{aligned} \tag{40}$$

with$T_{11}(n = 1, l = 1) = T_{11}(n = 1, l = 2)(\Omega \to \Omega_2$ and $\Lambda \to \Lambda_2)$ and $T_4(n = 1, l = 1) = T_4(n = 1, l = 2)(\Omega \to \Omega_2$ and $\Lambda \to \Lambda_2)$. Now, the n^{th} excited states of the nuclei ^{17}O and ^{17}F, with one additional nucleon (valence) in the nX_j level, under modified Quadratic Hellmann potential in global quantum group symmetry (NC: 3D-RSP):

$$\begin{aligned}
E_{so-uqh}(n,l,j) &= \gamma N^2 k_+(l)\left(\Theta T_{1n}(n,l) + \frac{\bar{\theta}}{2\mu}T_4(n,l)\right)\\
E_{so-dqh}(n,l,j) &= \gamma N^2 k_-(l)\left(\Theta T_{1n}(n,l) + \frac{\bar{\theta}}{2\mu}T_4(n,l)\right)
\end{aligned} \tag{41}$$

with$T_{1n}(n,l) = T_1(n,l) + T_2(n,l) + T_3(n,l)$.

3.4 The exact modified magnetic spectrum for mirror nuclei ^{17}O and ^{17}F under the modified quadratic Hellmann potential

Further to the important previously obtained results. Now, we consider another important physically meaningful phenomena produced by the effect of modified quadratic Hellmann potential related to the influence of an external uniform magnetic field $\vec{B}$ for the nuclei ^{17}O and$^{17}F$with one additional nucleon (valence) in the $1d_{5/2}$level, the first excited state$2S_{1/2}$, the second excited state $1d_{3/2}$and the generalized n^{th} excited statesnX_j. To avoid the repetition in the theoretical calculations, it is sufficient to apply the following replacements:

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$$\begin{cases} \vec{\Theta} \rightarrow \chi\vec{B} \\ \vec{\bar{\theta}} \rightarrow \bar{\sigma}\vec{B} \end{cases} \Rightarrow \qquad \left(\alpha b \frac{exp(-\alpha r)}{2r^3} + b\frac{exp(-\alpha r)}{r^4} - \frac{a}{2r^3}\right)\vec{L}\vec{\Theta} + \frac{\vec{L}\vec{\bar{\theta}}}{2\mu}$$

will -be- replace- by:

$$\left\{\alpha b \frac{exp(-\alpha r)}{2r^3} + b\frac{exp(-\alpha r)}{r^4} - \frac{a}{2r^3}\right\}\vec{B}\vec{L} \tag{42}$$

Hereχ and $\bar{\sigma}$ are two infinitesimal constants, and we choose the induced magnetic field $\vec{B}$ parallel to the (Oz) axis, which allow us to introduce the modified magnetic Hamiltonian $H_m^{qh}(r,\chi,\bar{\sigma})$ in (NC: 3D-RSP) symmetries as:

$$H_{so}^{qh}(r,\Theta,\bar{\theta}) \rightarrow H_m^{qh}(r,\chi,\bar{\sigma}) = \left\{\left(\alpha b \frac{exp(-\alpha r)}{2r^3} + b\frac{exp(-\alpha r)}{r^4} - \frac{a}{2r^3}\right)\chi + \frac{\bar{\sigma}}{2\mu}\right\}\left(\vec{B}\vec{J} - \aleph_z\right) \tag{43}$$

Here $\aleph_z \equiv -\vec{S}\vec{B}$ denote to Zeeman effect in commutative quantum mechanics, while $\aleph_{mod-z} \equiv \vec{B}\vec{J} - \aleph_z$is the new Zeeman effect. To obtain the exact NC magnetic modifications of energy for the ground state, the first excited state, the second excites state and n^{th} excited states of the nuclei ^{17}O and ^{17}F ($E_{mag}^{qh}((m=-2,-1{,}0,+1,+2),n=1,j=5/2,l=2)$, $E_{mag}^{qh}(m=0,n=2,j=1/2,l=0)$, $E_{mag}^{qh}((m=-1{,}0,+1),n=1,j=1/2,l=1)$ and $E_{mag}^{qh}(m=\overline{-l,+l},n,j,l)$) and we just replace$k_+(l)$ and Θ in the Eqs. (35) and (36) by the following parameters m andχ, respectively:

$$\begin{aligned}
&1d_{5/2} \rightarrow E_{mag}^{qh}((m=-2,-1{,}0,+1,+2),n=1,j=5/2,l=2) = \gamma N^2\left(T_{11}(n=1,l=2)\chi + \frac{\bar{\sigma}}{2\mu}T_4(n=1,l=2)\right)Bm \\
&2S_{1/2} \rightarrow E_{mag}^{qh}(m=0,n=2,j=1/2,l=0) = 0 \\
&1d_{3/2} \rightarrow E_{mag}^{qh}((m=-1{,}0,+1),n=1,j=1/2,l=1) = \gamma N^2\left(T_{11}(n=1,l=1)\chi + \frac{\bar{\sigma}}{2\mu}T_4(n=1,l=1)\right)Bm \\
&nX_j \rightarrow E_{mag}^{qh}(m=\overline{-l,+l},n,j,l) = \gamma N^2\left(T_{1n}(n,l)\chi + \frac{\bar{\sigma}}{2\mu}T_4(n,l)\right)Bm
\end{aligned} \tag{44}$$

We have$-l \le m \le +l$, which allows us to fix $(2l+1)$ values for discreet numbers$m$. It should be noted that the results obtained in Eq. (44) could find it by direct calculation $E_{mag}^{qh}(m,n,j,l) = \langle\Psi(r,\theta,\phi)|H_{m-qh}(r,\chi,\bar{\sigma})|\Psi(r,\theta,\phi)\rangle$that takes the following explicit relation:

$$E_{mag}^{qh}(m,n,j,l) = \gamma N^2 mB\int_0^{+\infty} r^{(2\sqrt{\chi_0+1/4}+1)}\, exp(-2\sqrt{\chi_2}r)\left[L_n^{\sqrt{\chi_0+1/4}}((2+2\sqrt{\chi_2})r)\right]^2\left\{\begin{pmatrix}\alpha b\frac{exp(-\alpha r)}{2r^3} + \\ +b\frac{exp(-\alpha r)}{r^4} - \frac{a}{2r^3}\end{pmatrix}\chi + \frac{\bar{\sigma}}{2\mu}\right\}dr \tag{45}$$

Then we find the corrections produced by the operator $H_{m-qh}(r,\chi,\bar{\sigma})$ for the ground state and other excited states repeating the same calculations in the previous subsection.

4. Main Theoretical results

In the previous sub-sections, we obtained the solution of the MSE for the nuclei ^{17}O and$^{17}F$with one additional nucleon (valence) in the ground state $1d_{5/2}$, the first excited state $2S_{1/2}$, the second excited state $1d_{3/2}$and the generalized n^{th} excited states nX_j under modified quadratic Hellmann potential, which given in Eq. (19), by using the generalized Bopp's shift method and standard perturbation theory. The energy eigenvalue is calculated in the 3D space-phase. The modified eigenenergies$(E_{\text{nc-uqh}}, E_{\text{nc-dqh}})(n=1,(m=0,\pm1,\pm2), j=5/2, l=2)$, $(E_{\text{nc-uqh}}, E_{\text{nc-dqh}})(n=2,(m=0), j=1/2, l=0)$, $(E_{\text{nc-uqh}}, E_{\text{nc-dqh}})(n=1,(m=0,\pm1), j=3/2, l=1)$ and $(E_{\text{nc-uqh}}, E_{\text{nc-dqh}})(n,(m=\overline{-l,+l}), j, l)$ with spin-1/2 for single nucleon are obtained in this paper on based to our original results presented on the Eqs. (35), (36), (40), (41) and (44), in addition to the ordinary energy E_{nl} for Quadratic Hellmann potential which presented in the Eq. (9):

$$\begin{aligned}
&E_{\text{nc-uqh}}\left(n=1,\left(m=\overline{-2,+2}\right), j=5/2, l=2\right)\\
&\quad = E_{12}+2\gamma N^2\left\{\Theta T_{11}(n=1,l=2)+\frac{\bar{\theta}}{2\mu}T_4(n=1,l=2)\right\}\\
&\quad +\gamma N^2\left\{T_{11}(n=1,l=2)\chi+\frac{\bar{\sigma}}{2\mu}T_4(n=1,l=2)\right\}Bm\\
&E_{\text{nc-dqh}}\left(n=1,\left(m=\overline{-2,+2}\right), j=5/2, l=2\right)=E_{12}-3/2\gamma N^2\left\{\Theta T_{11}(n=1,l=2)+\frac{\bar{\theta}}{2\mu}T_4(n=1,l=2)\right\}+\\
&\quad \gamma N^2\left\{T_{11}(n=1,l=2)\chi+\frac{\bar{\sigma}}{2\mu}T_4(n=1,l=2)\right\}Bm
\end{aligned} \tag{46}$$

$$\begin{aligned}
&E_{\text{nc-uqh}}(n=2,(m=0), j=1/2, l=0)=E_{20}\\
&E_{\text{nc-dqh}}(n=2,(m=0), j=1/2, l=0)=E_{20}-1/2\gamma N^2\left\{\Theta T_{12}(n=1,l=0)+\frac{\bar{\theta}}{2\mu}T_4(n=1,l=0)\right\}
\end{aligned} \tag{47}$$

$$\begin{aligned}
&E_{\text{nc-uqh}}(n=1,(m=0,\pm1), j=3/2, l=1)\\
&\quad = E_{11}+\gamma N^2\left\{\Theta T_{11}(n=1,l=1)+\frac{\bar{\theta}}{2\mu}T_4(n=1,l=1)\right\}\\
&\quad +\gamma N^2\left\{T_{11}(n=1,l=1)\chi+\frac{\bar{\sigma}}{2\mu}T_4(n=1,l=1)\right\}Bm\\
&E_{\text{nc-dqh}}(n=1,(m=0,\pm1), j=3/2, l=1)=E_{11}-\gamma N^2\left\{\Theta T_{11}(n=1,l=1)+\frac{\bar{\theta}}{2\mu}T_4(n=1,l=1)\right\}+\\
&\quad \gamma N^2\left\{T_{11}(n=1,l=1)\chi+\frac{\bar{\sigma}}{2\mu}T_4(n=1,l=1)\right\}
\end{aligned} \tag{48}$$

and

$$\begin{aligned}
&E_{\text{nc-uqh}}\left(n,\left(m=\overline{-l,+l}\right), j, l\right)\\
&\quad = E_{nl}+l\gamma N^2\left\{\Theta T_{1n}(n,l)+\frac{\bar{\theta}}{2\mu}T_4(n,l)\right\}+\gamma N^2\left\{T_{1n}(n,l)\chi+\frac{\bar{\sigma}}{2\mu}T_4(n,l)\right\}Bm\\
&E_{\text{nc-dqh}}\left(n,\left(m=\overline{-l,+l}\right), j, l\right)=E_{nl}-\frac{\frac{l+1}{2}}{2\gamma N^2\left\{\Theta T_{1n}(n,l)+\frac{\bar{\theta}}{2\mu}T_4(n,l)\right\}}+\gamma N^2\left\{T_{1n}(n,l)\chi+\frac{\bar{\sigma}}{2\mu}T_4(n,l)\right\}Bm
\end{aligned} \tag{49}$$

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Where E_{12}, E_{20} and E_{11} are the energy of the ground state $1d_{5/2}$, the first excited state $2S_{1/2}$, the second excited state $1d_{3/2}$ for mirror nuclei ^{17}O and ^{17}Fin the symmetries of quantum mechanics under quadratic Hellmann potential:

$$E_{12} = -2\mu \frac{(a+b\alpha)^2}{\left[3+2\sqrt{2\mu b+25/4}\right]^2}$$
$$E_{20} = -2\mu \frac{(a+b\alpha)^2}{\left[5+2\sqrt{2\mu b+1/4}\right]^2}$$
$$E_{11} = -2\mu \frac{(a+b\alpha)^2}{\left[3+2\sqrt{2\mu b+9/4}\right]^2} \tag{50}$$

This is one of the main objectives of our research and by noting that, the obtained eigenvalues of energies are real's and then the NC diagonal Hamiltonian $H_{nc}^{qh}(x_\mu, p_\mu) \equiv diag\left(\left(H_{nc}^{qh}\right)_{11}, \left(H_{nc}^{qh}\right)_{22}, \left(H_{nc}^{qh}\right)_{33} \right)$ is Hermitian, furthermore it's possible to writ the three elements $\left(H_{nc}^{qh}\right)_{11}$, $\left(H_{nc}^{qh}\right)_{22}$ and $\left(H_{nc}^{qh}\right)_{33}$ as follows:

$$\left(H_{nc}^{qh}\right)_{11} = -\frac{\Delta_{nc}}{2\mu} + H_{int-uqh}, \left(H_{nc-dqh}\right)_{22} = -\frac{\Delta_{nc}}{2\mu} + H_{int-dqh} \text{ and } \left(H_{nc}^{qh}\right)_{33} = -\frac{\Delta}{2\mu} + H_{qh} \tag{51}$$

Where $\frac{\Delta_{nc}}{2\mu} = \frac{\Delta - \vec{L}\vec{\theta} - \vec{L}\vec{\sigma}}{2\mu}$ and the three modified interactions elements $\left(H_{int-uqh}, H_{int-dqh}, H_{qh}\right)$ are given by:

$$V_{qh}(r) \to \begin{cases} H_{int-uqh} = -\frac{a}{r} + \frac{b}{r^2} e^{-\alpha r} + \gamma(k_+(l)\Theta + \chi\aleph_{mod-z}) \left(\alpha b \frac{exp(-\alpha r)}{2r^3} + b\frac{exp(-\alpha r)}{r^4} - \frac{a}{2r^3}\right) \\ \qquad \text{if additional nucleon polarised up} \\ H_{int-dqh} = -\frac{a}{r} + \frac{b}{r^2} e^{-\alpha r} + \gamma(k_-(l)\Theta + \chi\aleph_{mod-z}) \left(\alpha b \frac{exp(-\alpha r)}{2r^3} + b\frac{exp(-\alpha r)}{r^4} - \frac{a}{2r^3}\right) \\ \qquad \text{if additional nucleon polarised down} \\ H_{qh} = -\frac{a}{r} + \frac{b}{r^2} e^{-\alpha r} \qquad \text{for non polarised additional nucle} \end{cases} \tag{52}$$

Thus, the ordinary kinetic term for quadratic Hellmann potential $\left(-\frac{\Delta}{2\mu}\right)$ and ordinary interaction $(-\frac{a}{r} + \frac{b}{r^2} e^{-\alpha r})$ are replaced by modified form of kinetic term $\left(-\frac{\Delta_{nc}}{2\mu}\right)$ and modified interactions $\left(H_{int-uqh}, H_{int-dqh}, H_{qh}\right)$ in (NC-3D: RSP) symmetries. On the other hand, it is evident to consider the quantum number m takes $(2l+1)$ values and we have also two values for$(j = l \pm 1/2)$, thus every state in usually three-dimensional space of energy for mirror nuclei ^{17}O and^{17}Funder modified quadratic Hellmann potential will be $\left(2(2l+1)\right)$sub-states. To obtain the total complete degeneracy of energy level of the modified quadratic Hellmann potential in (NC-3D: RSP) symmetries, we need to sum for all allowed values of$l$. Total degeneracy is thus,

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$$\underbrace{\sum_{l=0}^{n-1}(2l+1)=n^2}_{\text{NRQM}} \rightarrow \underbrace{2\sum_{l=0}^{n-1}(2l+1)\equiv 2n^2}_{\text{NCNRQM}} \tag{53}$$

If we consider the two simultaneously limits$(\Theta,\overline{\theta}) \rightarrow (0,0)$, we recover the results of the ref. [3] for the quadratic Hellmann potential in ordinary QM, which means that our calculations are correct. The novelty in this work that the generalized Bopp's shift method successfully applies to find the solution of the 3-radial MSE for mirror nuclei ^{17}O and^{17}Fin the subatomic scales, in symmetries of NCQM.

5. Conclusions

In the present work, the 3-dimensional modified Schrodinger equation is analytically solved using the generalized Bopp's shift method and standard perturbation theory. The modified quadratic Hellmann potential is extended to include the effect of the noncommutativity space phase based on ref. [3]; we resume the main obtained results:

- Ordinary modified Quadratic Hellmann potential $(-\frac{a}{r}+\frac{b}{r^2}e^{-\alpha r})$ was replaced by modified interactions $H_{int-ucp}$ and$H_{int-dcp}$ for mirror nuclei ^{17}O and ^{17}F, The ordinary kinetic term $-\frac{\Delta}{2\mu}$ for Quadratic Hellmann potential extended to the new form $\frac{\Delta_{nc}}{2\mu}=\frac{\Delta-\vec{L}\vec{\theta}-\vec{L}\vec{\sigma}}{2\mu}$ for mirror nuclei ^{17}O and^{17}Funder influence of modified Quadratic Hellmann potential,
- We obtained the perturbative corrections $(E_{\text{nc-}uqh}, E_{\text{nc-}dqh})(n=1,(m=0,\pm1,\pm2),j=5/2,l=2)$, $(E_{\text{nc-}uqh}, E_{\text{nc-}dqh})(n=2,(m=0),j=1/2,l=0)$, $(E_{\text{nc-}uqh}, E_{\text{nc-}dqh})(n=1,(m=0,\pm1),j=3/2,l=1)$ and $(E_{\text{nc-}uqh}, E_{\text{nc-}dqh})(n,(m=\overline{-l,+l}),j,l)$ for the nuclei ^{17}O and ^{17}F with one additional nucleon (valence) in the ground state $1d_{5/2}$, the first excited state $2S_{1/2}$, the second excited state $1d_{3/2}$and the generalized n^{th} excited states nX_j under influence of modified Quadratic Hellmann potential.

Through the of high value results, which we have achieved in present work, we hope to extend our recently work physics for further investigations of subatomic scales and other characteristics of mirror nuclei ^{17}O and^{17}F.

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