A New Model for Describing Heavy-Light Mesons in The Extended Nonrelativistic Quark Model Under a New Modified Potential Containing Cornell, Gaussian And Inverse Square Terms in The Symmetries Of NCQM

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Abstract:

In present work, the modified Schrödinger equation (MSE) is analytically solved for the Heavy-Light Mesons (HLM) under modified quark-antiquark potential containing modified Cornell, Gaussian, and inverse square terms *MCGISTs*, in the symmetries of 3-dimensional noncommutative real space phase (NC: 3D-RSP), using the generalized Bopp's shift method. The energy a spectrum of HLM has been investigated in the framework of extended nonrelativistic quark model ENRQM. Furthermore, the new energy eigenvalues and the corresponding Hamiltonian operator are calculated in (NC: 3D-RSP) symmetries. The masses of the scalar, vector, pseudoscalar, and pseudovector for (B, B_s, D and D_s) mesons have been calculated in (NC: 3D-RSP), and we have shown that the spin-orbital coupling $\left\langle H_{so-hlm}\right\rangle$ generated automatically. Moreover, using the perturbation approach, we found that the perturbative solutions of discrete spectrum can be expressed by the parabolic cylinder functions function, Gamma function, the discreet atomic quantum numbers (j,l,s,m) of the $Q\overline{Q}$ state and (the spin-independent and spin-dependent) parameters (a,b,c,α), in addition to noncommutativity parameters (Θ and $\overline{\theta}$). As a special case this model has been applied to study the S- and P-wave states of B, B_s, D and D_s mesons in NCQM symmetries The total complete degeneracy of new energy levels of HLM was changed to become equals the new value $3n^2$ instead n^2 in ordinary quantum mechanics. Our obtained results are in good agreement with the already existing literatures in NCQM.

Keywords: Schrödinger equation, Heavy-Light Mesons, the quark-antiquark potential containing Cornell, Gaussian and inverse square terms, Bopp's shift method, noncommutative space phase, and the Weyl Moyal star product.

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Introduction

A few decades ago, the study of mesonic systems has been very interesting both theoretically and experimentally and so to understand the hadrons containing a heavy quark. There are several potential models like the Martin, Cornell, Richardson and logarithmic potentials for quark–antiquark bound states, and in particular, the Cornell potential is believed to be the most realistic phenomenological potential model to understand the mesonic systems [1-5]. The researchers M. Moazami *et al.* were studied the Heavy-Light Mesons (HLM) under the combination of vector and scalar potentials and obtained the mass spectra and energy spectra of HLM (B, B_s, D and D_s mesons) under a new potential containing Cornell, Gaussian and inverse square terms (*CGISTs*) [6]. The main objective is to develop the research article and expanding it to the hug symmetry known by noncommutative quantum mechanics (NCQM) in order to achieve a more accurate physical vision so that this study becomes valid in the field of nanotechnology. Among different types of potential model used in various fields of physics, in general, and in quantum chromodynamics theory in particular, here we are going to introduce a new type of potential for the mesonic system, i.e. the combination of modified both Cornell, Gaussian and inverse square terms potentials (*MCGISTs*), this has the following form:

$$V_{hlm}(r)=\underbrace{\frac{a}{r}+\frac{b}{r^2}+k_0\exp\left(-\alpha^2\frac{r^2}{2}\right)+cr}_{\text{Ordinary-QM}}\rightarrow V_{hlm}(\hat{r})=\underbrace{V_{hlm}(r)+\left\{\frac{a}{2r^3}+\frac{b}{r^4}-\frac{c}{2r}+\frac{k_0\alpha^2}{2}\exp\left(-\alpha^2\frac{r^2}{2}\right)\right\}\vec{\mathbf{L}}\vec{\Theta}}_{\text{NCQM}} \tag{1}$$

The above model is used to study the *S*- and *P-wave* states of B, B_s, D and D_s mesons in NCQM symmetries. Thus, the analysis of the consequences of *MCGISTs* in heavy quarkonia is our main goal in this article. On the other hand, to explore the possibility of creating new applications and more profound interpretations in the sub-atomics and nano scales using new version the modified quark-antiquark potential containing MCGISTs. It is important to mention that, the noncommutativity theory was introduced firstly by W. Heisenberg in 1930 [7] and then by H. Syndre in 1947 [8]. The new structure of NCQM based to new canonical commutations relations in Schrödinger, Heisenberg and Interactions pictures (SP, HP and IP), respectively, as follows (Throughout this paper, the natural units $c=\hbar=1$ will be used) [9-13]:

$$\begin{cases}\left[\hat{x}_\mu\overset{*}{,}\hat{p}_\nu\right]=\left[\hat{x}_\mu(t)\overset{*}{,}\hat{p}_\nu(t)\right]=\left[\hat{x}_{I\mu}(t)\overset{*}{,}\hat{p}_{I\nu}(t)\right]=i\delta_{ij}\\ \left[\hat{x}_\mu\overset{*}{,}\hat{x}_\nu\right]=\left[\hat{x}_\mu(t)\overset{*}{,}\hat{x}_\nu(t)\right]=\left[\hat{x}_{I\mu}(t)\overset{*}{,}\hat{x}_{I\nu}(t)\right]=i\theta_{ij}\\ \left[\hat{p}_\mu\overset{*}{,}\hat{p}_\nu\right]=\left[\hat{p}_\mu(t)\overset{*}{,}\hat{p}_\nu(t)\right]=\left[\hat{p}_{I\mu}(t)\overset{*}{,}\hat{p}_{I\nu}(t)\right]=i\bar{\theta}_{ij}\end{cases} \tag{2}$$

the indices ($\mu,\nu\equiv\overline{1,3}$). This means that the principle of uncertainty for Heisenberg was generalized to include rather than both position and momenta $(\hat{x}_\mu,\hat{p}_\nu)$ only to include also two positions $(\hat{x}_\mu,\hat{x}_\nu)$ and two momenta is $(\hat{p}_\mu,\hat{p}_\nu)$ in the same time. The very small two parameters $\theta^{\mu\nu}$ and $\bar{\theta}^{\mu\nu}$ (compared to the energy) are elements of two antisymmetric real matrixes, parameters of noncommutativity and $(*)$ denote to the Weyl Moyal star product, which is generalized between two arbitrary functions $(f,g)(x,p)\rightarrow(\hat{f},\hat{g})(\hat{x},\hat{p})$ to the new form $\hat{f}(\hat{x},\hat{p})\hat{g}(\hat{x},\hat{p})\equiv(f*g)(x,p)$ in (NC: 3D-RSP) symmetries [13-21]:

$$(fg)(x,p)\rightarrow(f*g)(x,p)=\left(fg-\frac{i}{2}\theta^{\mu\nu}\partial_\mu^x f\,\partial_\nu^x g-\frac{i}{2}\bar{\theta}^{\mu\nu}\partial_\mu^p f\,\partial_\nu^p g\right)(x,p) \tag{3}$$

The second and the third terms in the above equation are present the effects of (space-space) and (phase-phase) noncommutativity properties. However, the new operators $\hat{\xi}_H(t)=\left[\hat{x}_\mu\vee\hat{p}_\nu\right](t)$, $\hat{\xi}_I(t)=\left[\hat{x}_{I\mu}\vee\hat{p}_{I\nu}\right](t)$ in (HP and IP, respectively) are depending to the corresponding operator $\hat{\xi}_S=\left[\hat{x}_\mu\vee\hat{p}_\nu\right]$ in SP from the following projections relations:

$$\begin{cases}\xi_H(t)=\exp(i\hat{H}_{hlm}(t-t_0))\xi_S\exp(-i\hat{H}_{hlm}(t-t_0)\\ \xi_I(t)=\exp(i\hat{H}_{ohlm}(t-t_0))\xi_S\exp(-i\hat{H}_{ohlm}(t-t_0)\end{cases}\Rightarrow\begin{cases}\hat{\xi}_H(t)=\exp(i\hat{H}_{nc-hlm}(t-t_0))*\hat{\xi}_S*\exp(-i\hat{H}_{nc-hlm}(t-t_0)\\ \hat{\xi}_I(t)=\exp(i\hat{H}_{nco-hlm}(t-t_0))*\hat{\xi}_S*\exp(-i\hat{H}_{nco-hlm}(t-t_0)\end{cases} \tag{4}$$

Here $\xi_S = (x_\mu \vee p_\nu)$, $\xi_H(t) = (x_\mu \vee p_\nu)(t)$ and $\xi_I(t) = (x_{I\mu} \vee p_{I\nu})(t)$ are the three representations in QM, while the dynamics of new systems $\frac{\mathrm{d}\xi_H(t)}{\mathrm{dt}}$ are described from the following motion equations in NCQM:

$$\frac{\mathrm{d}\xi_H(t)}{\mathrm{dt}} = [\xi_H(t), \hat{H}_{hlm}] \Rightarrow \frac{\mathrm{d}\hat{\xi}_H(t)}{\mathrm{dt}} = \left[\hat{\xi}_H(t) \overset{*}{,} \hat{H}_{nc-hlm}\right] \equiv \hat{\xi}_H(t) * \hat{H}_{nc-hlm} - \hat{H}_{nc-hlm} * \hat{\xi}_H(t) \tag{5}$$

The operators $\hat{H}_{ohlm}$ and $\hat{H}_{hlm}$ are the unperturbed and global Hamiltonian in QM for *CGISTs* while $\hat{H}_{nco-hlm}$ and $\hat{H}_{nc-hlm}$ the corresponding Hamiltonians for *MCGISTs* in the NCQM. This paper consists of five sections, and the organization scheme is given as follows: In next section, we briefly review the ordinary SE with quark-antiquark potential containing *CGISTs* on based to ref. [6]. The Section 3 is devoted to studying the MSE by applying the generalized Bopp's shift method and obtained the modified quark-antiquark potential containing *MCGISTs* and the modified spin-orbital operator. Then, we applied the standard perturbation theory to find the quantum spectrum of the n^{th} excited state which produced by the effects of modified spin-orbital and modified Zeeman interactions. After that, in the fourth section, a discussion of the main results is presented in addition to determine the new formula of mass spectra of the of HLM (B, B_s, D and D_s mesons) in (NC: 3D-RSP) symmetries. Finally, in the last section, summary and conclusions are presented.

Materials and Methods

Overview of the HLM under quark-antiquark potential containing CGISTs

In this section, we shall review the eigenvalues and eigenfunctions for spherically symmetric for the potential includes Cornell, Gaussian and inverse square terms that are among successful interactions of physics [6]:

$$V(r) = \frac{a}{r} + \frac{b}{r^2} + k_0 \exp\left(-\alpha^2 \frac{r^2}{2}\right) + cr \tag{6}$$

The parameters a, b, α and c are the parameters of the potential. The Hamiltonian operator for the scalar, vector, pseudoscalar, and pseudovector (B, B_s, D and D_s) mesons is [6]:

$$\hat{H}(p_i, x_i) = H_0(r) + H' \tag{7}$$

The parent Hamiltonian operator (the principal part) H_0 and the perturbed potential H' as [6]:

$$H_0 = -\frac{\Delta}{2\mu} + \frac{a}{r} + \frac{b}{r^2} \text{ and } H' = k_0 \exp\left(-\alpha^2 \frac{r^2}{2}\right) + cr \tag{8}$$

The reduced mass of quark and anti-quark system equal $\mu = \frac{m_q \overline{m}_q}{m_q + \overline{m}_q}$. The complex eigenfunctions $\Psi_{nlm}(r, \theta, \varphi) = R_{nl}(r) Y_l^m(\theta, \varphi)$ in 3D space while the radial part $R_{nl}(r)$ satisfies the following differential equation:

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+2\mu\left(E_{nl}-\frac{a}{r}-\frac{b}{r^2}-\frac{l(l+1)}{2\mu r^2}\right)\right]R_{nl}(r)=0 \tag{9}$$

Where l and E_{nl} represent angular momentum and the energy while $-l\le m\le +l$. The reference [6] give the wave function and the energy as follow:

$$\begin{aligned}\Psi_{nlm}(r,\theta,\varphi)&=Nr^{-\frac{1}{2}+\frac{1}{2}d(l,\mu)}\exp(-1/2\delta(n,l)r)_1L_n^{d(l,\mu)}(\delta(n,l)r)Y_l^m(\theta,\varphi)\\ E_{n,l}&=\frac{-2\mu a^2}{(2n+1)^2+(2l+1)^2+8\mu b+(2n+2)\sqrt{(2l+1)^2+8\mu b}}\end{aligned} \tag{10}$$

Where N is the normalization constant, $d(l,\mu)=\sqrt{(2l+1)^2+8\mu b}$, $\delta(n,l)=2\sqrt{-2\mu E_{n,l}}$ while $Y_l^m(\theta,\varphi)$ are the spherical harmonics.

Solution of Modified Schrödinger equation for HLM under Modified quark-antiquark potential containing *MCGISTs*

Review of Generalized Bopp's shift method: Some Basic Considerations

*In this section, we shall give an overview or a brief preliminary for HLM under modified quark-antiquark potential containing modified Cornell, Gaussian, and inverse square terms (*MCGISTs*) in (NC: 3D-RSP) symmetries. To perform this task the physical form of modified Schrödinger equation (MSE), it is necessary to replace ordinary 3-dimensional Hamiltonian operators $\hat{H}(x_\mu,p_\mu)$, ordinary complex wave function $\Psi\left(\vec{r}\right)$ and ordinary energy E_{nl} by new Hamiltonian operators $\hat{H}_{nc-hlm}(\hat{x}_\mu,\hat{p}_\mu)$, new complex wave function $\hat{\Psi}\left(\vec{\hat{r}}\right)$ and new values E_{nc-hlm}, respectively. In addition to replace the ordinary product by the Weyl Moyal star product, which allow us to constructing the MSE in (NC-3D: RSP) symmetries as [22-25]:*

$$\hat{H}_{hlm}(x_\mu,p_\mu)\Psi\left(\vec{r}\right)=E_{nl}\Psi\left(\vec{r}\right)\Rightarrow\hat{H}(\hat{x}_\mu,\hat{p}_\mu)*\Psi\left(\vec{\hat{r}}\right)=E_{nc-hlm}\Psi\left(\vec{\hat{r}}\right) \tag{11}$$

The Bopp's shift method has been successfully applied to relativistic and NRQM problems using modified Dirac equation, modified Klein-Gordon equation and MSE, respectively. This method has produced very promising results for a number of situations having a physical, chemical interest. The method reduces to the Dirac equation, Klein-Gordon, and equation Schrödinger under two-simultaneously translations in space and phase. It based on the following new commutators [26-30]:

$$[\hat{x}_\mu,\hat{x}_\nu]=[\hat{x}_\mu(t),\hat{x}_\nu(t)]=i\theta_{\mu\nu}\quad\text{and}\quad[\hat{p}_\mu,\hat{p}_\nu]=[\hat{p}_\mu(t),\hat{p}_\nu(t)]=i\bar{\theta}_{\mu\nu} \tag{12}$$

The new generalized positions and momentum coordinates $(\hat{x}_\mu,\hat{p}_\nu)$ in (NC: 3D-RSP) are defined in terms of the commutative counterparts (x_μ,p_ν) in ordinary quantum mechanics via, respectively [31-33]:

$$\left(x_{\mu},p_{\nu}\right)\Rightarrow\left(\hat{x}_{\mu},\hat{p}_{\nu}\right)=\left(x_{\mu}-\frac{\theta_{\mu\nu}}{2}p_{\nu},p_{\mu}+\frac{\bar{\theta}_{\mu\nu}}{2}x_{\nu}\right) \tag{13}$$

The above equation allows us to obtain the two operators $\left(\hat{r}^2,\hat{p}^2\right)$ in (NC-3D: RSP) symmetries [32-35]:

$$\left(r^2,p^2\right)\Rightarrow\left(\hat{r}^2,\hat{p}^2\right)=\left(r^2-\vec{\mathbf{L}}\vec{\Theta},\, p^2+\vec{\mathbf{L}}\vec{\bar{\boldsymbol{\theta}}}\right) \tag{14}$$

The two couplings $\mathbf{L}\Theta$ and $\vec{\mathbf{L}}\vec{\bar{\boldsymbol{\theta}}}$ are $\left(L_x\Theta_{12}+L_y\Theta_{23}+L_z\Theta_{13}\right)$ and $\left(L_x\bar{\theta}_{12}+L_y\bar{\theta}_{23}+L_z\bar{\theta}_{13}\right)$, respectively and ($L_x$, L_y and L_z) are the three components of angular momentum operator $\vec{L}$ while $\Theta_{\mu\nu}=\theta_{\mu\nu}/2$. Thus, the reduced Schrödinger equation (without star product) can be written as:

$$\hat{H}\left(\hat{x}_{\mu},\hat{p}_{\mu}\right)*\Psi\left(\vec{\hat{r}}\right)=E_{nc-hlm}\Psi\left(\vec{\hat{r}}\right)\Rightarrow H\left(\hat{x}_{\mu},\hat{p}_{\mu}\right)\psi\left(\vec{r}\right)=E_{nc-hlm}\psi\left(\vec{r}\right) \tag{15}$$

The new operator of Hamiltonian $H_{nc-hlm}\left(\hat{x}_{\mu},\hat{p}_{\nu}\right)$ of HLM can be expressed as:

$$H_{hlm}\left(x_{\mu},x_{\mu}\right)\Rightarrow H_{nc-hlm}\left(\hat{x}_{\mu},\hat{p}_{\mu}\right)\equiv H\left(\hat{x}_{\mu}=x_{\mu}-\frac{\theta_{\mu\nu}}{2}p_{\nu},\hat{p}_{\mu}=p_{\mu}+\frac{\bar{\theta}_{\mu\nu}}{2}x_{\nu}\right)=\frac{\hat{p}^2}{2\mu}+V_{hlm}\left(\hat{r}=\sqrt{\left(x_{\mu}-\frac{\theta_{\mu\nu}}{2}p_{\nu}\right)\left(x_{\mu}-\frac{\theta_{\mu\nu}}{2}p_{\nu}\right)}\right)$$
(16

Where $V_{hlm}\left(\hat{r}\right)$ denote to the modified quark-antiquark potential containing *MCGISTs* in (NC: 3D-RSP) symmetries:

$$V_{hlm}\left(r\right)\Rightarrow V_{hlm}\left(\hat{r}\right)=\frac{a}{\hat{r}}+\frac{b}{\hat{r}^2}+k_0\exp\left(-\alpha^2\frac{\hat{r}^2}{2}\right)+c\hat{r} \tag{17}$$

Again, applying Eq. (14) to obtain the four terms ($\frac{a}{\hat{r}}$, $\frac{b}{\hat{r}^2}$, $k_0\exp\left(-\alpha^2\frac{\hat{r}^2}{2}\right)$ and $c\hat{r}$), which will be used to determine the modified quark-antiquark potential containing *MCGISTs* $V_{hlm}\left(\hat{r}\right)$, gives the following results as:

$$\begin{aligned}
\frac{a}{r}&\rightarrow\frac{a}{\hat{r}}=\frac{a}{r}+\frac{a}{2r^3}\vec{\mathbf{L}}\vec{\Theta}+O\left(\Theta^2\right)\\
\frac{b}{r^2}&\rightarrow\frac{b}{\hat{r}^2}=\frac{b}{r^2}+\frac{b}{r^4}\vec{\mathbf{L}}\vec{\Theta}+O\left(\Theta^2\right)\\
cr&\rightarrow c\hat{r}=cr-\frac{c}{2r}\vec{\mathbf{L}}\vec{\Theta}+O\left(\Theta^2\right)
\end{aligned} \tag{18.1}$$

and

$$k_0 \exp\left(-\alpha^2 \frac{r^2}{2}\right) \to k_0 \exp\left(-\alpha^2 \frac{\hat{r}^2}{2}\right)$$
$$= k_0 \exp\left(-\alpha^2 \frac{r^2}{2}\right)\left(1+\frac{\alpha^2}{2}\vec{\mathbf{L}}\vec{\Theta}\right)+O\left(\Theta^2\right) \tag{18.2}$$

Substituting, Eq. (18.1) and (18.2) into Eq. (17), gives the modified quark-antiquark potential containing MCGISTs in (NC-3D: RSP) symmetries as follows:

$$V_{hlm}(r) \to V_{hlm}(\hat{r}) = V_{hlm}(r) + \left\{\frac{a}{2r^3}+\frac{b}{r^4}-\frac{c}{2r}+\frac{k_0\alpha^2}{2}\exp\left(-\alpha^2\frac{r^2}{2}\right)\right\}\vec{\mathbf{L}}\ \vec{\Theta} \tag{19}$$

By making the substitution above equation into Eq. (16), we find the global our working new modified Hamiltonian operator $H_{\text{nc-}hlm}(\hat{r})$ satisfies the equation in (NC: 3D-RSP) symmetries:

$$H_{hlm}(x_\mu, p_\nu) \Rightarrow H_{nc-hlm}\ (\hat{r}) = H_{hlm}(x_\mu, p_\nu) + H_{\text{per-}hlm}(r) \tag{20}$$

the operator $H_{hlm}(x_\mu, p_\nu)$ is just the ordinary Hamiltonian operator for quark-antiquark potential containing CGISTs in commutative quantum mechanics:

$$H_{hlm}(x_\mu, p_\mu) = \frac{p^2}{2\mu}+\frac{a}{r}+\frac{b}{r^2}+k_0 \exp\left(-\alpha^2\frac{r^2}{2}\right)+cr \tag{21}$$

while the rest part $H_{\text{per-}hlm}(r)$ is proportional with two infinitesimals parameters (Θ and $\bar{\theta}$):

$$H_{\text{per-}hlm}(r) = \left(\frac{a}{2r^3}+\frac{b}{r^4}-\frac{c}{2r}+\frac{k_0\alpha^2}{2}\exp\left(-\alpha^2\frac{r^2}{2}\right)\right)\vec{\mathbf{L}}\vec{\Theta}+\frac{\vec{\mathbf{L}}\vec{\bar{\theta}}}{2\mu} \tag{22}$$

Thus, we can consider $H_{\text{per-}hlm}(r)$ as a perturbation terms compared with the parent Hamiltonian operator $H_{hlm}(x_\mu, p_\mu)$ in (NC: 3D-RSP) symmetries.

The exact Modified Spin-Orbital Spectrum for HLM under Modified quark-antiquark potential containing *MCGISTs* in Global (NC: 3D- RSP) Symmetries:

In this subsection, we will apply the same strategy, which we have seen exclusively in some of our published scientific works [32-35]. Under such particular choice, one can easily reproduce both couplings ($\vec{\mathbf{L}}\vec{\Theta}$ and $\vec{\mathbf{L}}\vec{\bar{\theta}}$) to the new physical forms ($\gamma\Theta\vec{L}\vec{S}$ and $\gamma\bar{\theta}\vec{L}\vec{S}$), respectively. Thus, the new forms of $H_{\text{so-}hlm}(r,\Theta,\bar{\theta})$ for Heavy-Light Mesons under modified quark-antiquark potential containing *MCGISTs* as follows:

$$H_{\text{per-}hlm}(r) \to H_{\text{so-}hlm}(r,\Theta,\bar{\theta}) \equiv \gamma\left\{\left(\frac{a}{2r^3}+\frac{b}{r^4}-\frac{c}{2r}+\frac{k_0\alpha^2}{2}\exp\left(-\alpha^2\frac{r^2}{2}\right)\right)+\frac{\bar{\theta}}{2\mu}\right\}\vec{L}\vec{S} \tag{23}$$

Here $\Theta=\sqrt{\Theta_{12}{}^{2}+\Theta_{23}{}^{2}+\Theta_{13}{}^{2}}$, $\bar{\theta}=\sqrt{\bar{\theta}_{12}{}^{2}+\bar{\theta}_{23}{}^{2}+\bar{\theta}_{13}{}^{2}}$ and $\gamma\approx\frac{1}{137}$ is a new constant, which play the role of fine structure constant in the electromagnetic interaction or quantum electrodynamics theory QED, we have chosen the two vectors $\vec{\Theta}$ and $\vec{\vec{\theta}}$ parallel to the spin $\vec{S}$ of Heavy-Light Mesons. Furthermore, the above perturbative terms $H_{\text{per-}h\text{lm}}(r)$ can be rewritten to the following new form:

$$H_{so-hlm}(r,\Theta,\bar{\theta})=\frac{\gamma}{2}\left\{\left(\frac{a}{2r^{3}}+\frac{b}{r^{4}}-\frac{c}{2r}+\frac{k_0\alpha^{2}}{2}\exp\left(-\alpha^{2}\frac{r^{2}}{2}\right)\right)\Theta+\frac{\bar{\theta}}{2\mu}\right\}\left(\vec{J}^{2}-\vec{L}^{2}-\vec{S}^{2}\right) \quad (24)$$

Where $\vec{J}$ and $\vec{S}$ are defined the operators of the total angular momentum and spin of Heavy-Light Mesons such as B, B_s, D and D_s mesons. This operator traduces the coupling between spin $\vec{S}$ and orbital momentum $\vec{L}\vec{S}$. The set ($H_{so-hlm}(r,\Theta,\bar{\theta})$, J^2, L^2, S^2 and J_z) forms a complete of conserved physics quantities and for $\overleftrightarrow{S}=\vec{1}$, the eigenvalues of the spin-orbital coupling operator are $k(l)\equiv\frac{1}{2}\{j(j+1)-l(l+1)-2\}$ corresponding $j=l+1$ (spin great), $j=l$ (spin middle) and $j=l-1$ (spin little), respectively, then, one can form a diagonal (3×3) matrix for modified quark-antiquark potential containing MCGISTs in (NC: 3D-RSP) symmetries, with diagonal element $(H_{so-hlm})_{11}$, $(H_{so-hlm})_{22}$ and $(H_{so-hlm})_{33}$ are given by:

$$\begin{aligned}
(H_{so-hlm})_{11}&=\gamma k_1(l)\left(\left(\frac{a}{2r^{3}}+\frac{b}{r^{4}}-\frac{c}{2r}+\frac{k_0\alpha^{2}}{2}\exp\left(-\alpha^{2}\frac{r^{2}}{2}\right)\right)\Theta+\frac{\bar{\theta}}{2\mu}\right)\text{if } j=l+1\\
(H_{so-hlm})_{22}&=\gamma k_2(l)\left(\left(\frac{a}{2r^{3}}+\frac{b}{r^{4}}-\frac{c}{2r}+\frac{k_0\alpha^{2}}{2}\exp\left(-\alpha^{2}\frac{r^{2}}{2}\right)\right)\Theta+\frac{\bar{\theta}}{2\mu}\right)\text{if } j=l\\
(H_{so-hlm})_{33}&=\gamma k_3(l)\left(\left(\frac{a}{2r^{3}}+\frac{b}{r^{4}}-\frac{c}{2r}+\frac{k_0\alpha^{2}}{2}\exp\left(-\alpha^{2}\frac{r^{2}}{2}\right)\right)\Theta+\frac{\bar{\theta}}{2\mu}\right)\text{if } j=l-1
\end{aligned} \quad (25)$$

Here $(k_1(l),k_2(l),k_3(l))\equiv\frac{1}{2}(l,-2,-2l-2)$. The non-null diagonal elements $(H_{so-hlm})_{11}$, $(H_{so-hlm})_{22}$ and $(H_{so-hlm})_{33}$ of modified Hamiltonian operator $H_{nc-hlm}(\hat{r})$ will change the energy values E_{nl} by creating three new values:

$$\begin{cases}
E_{\text{g-}hlm}=\langle\Psi(r,\theta,\varphi)|(H_{so-hlm})_{11}|\Psi(r,\theta,\varphi)\rangle\\
E_{\text{m-hlm}}=\langle\Psi(r,\theta,\varphi)|(H_{so-hlm})_{22}|\Psi(r,\theta,\varphi)\rangle\\
E_{\text{l-}hlm}=\langle\Psi(r,\theta,\varphi)|(H_{so-hlm})_{33}|\Psi(r,\theta,\varphi)\rangle
\end{cases} \quad (26)$$

We will see them in detail in the next subsection. After profound calculation, one can show that the new radial function $R_{nl}(r)$ satisfying the following differential equation for Heavy-Light Mesons under modified quark-antiquark potential containing *MCGISTs*:

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}+2\mu\left(E_{nc-hlm}-\frac{a}{r}-\frac{b}{r^2}-k_0\exp\left(-\alpha^2\frac{r^2}{2}\right)-cr-\left(\frac{a}{2r^3}+\frac{b}{r^4}-\frac{c}{2r}+\frac{k_0\alpha^2}{2}\exp\left(-\alpha^2\frac{r^2}{2}\right)\right)\vec{\mathbf{L}}\vec{\Theta}-\frac{\vec{\mathbf{L}}\vec{\overline{\theta}}}{2\mu}\right)\right]R_{nl}(r)=0$$
(27)

Through our observation of the expression of $H_{\text{per-}hlm}(r)$, which appear in the equation (23), we see it as proportionate to two infinitesimals parameters (Θ and $\overline{\theta}$), thus, in what follows, we proceed to solve the modified radial part of the MSE that is, equation (27) by applying standard perturbation theory to find acceptable solutions at first order of two parameters Θ and $\overline{\theta}$. The proposed solutions for MSE under modified quark-antiquark potential containing *MCGISTs* includes energy corrections, which produced automatically from two principal physical phoneme's, the first one is the effect of modified spin-orbital interaction and the second is the modified Zeeman effect while the stark effect can be appear in the linear part of modified quark-antiquark potential containing *MCGISTs*.

The exact Modified Spin-Orbital Spectrum for HLM under modified quark-antiquark model containing *MCGISTs* in (NC: 3D- RSP) Symmetries:

The purpose here is to give a complete prescription for determine the energy level of n^{th} the excited state, for HLM such as scalar, vector, pseudoscalar, and pseudovector for (B, B_s, D and D_s) mesons under modified quark-antiquark potential containing MCGISTs. We first find the corrections $E_{g-hlm}(k_1(l),a,b,c,k_0,\alpha,n,l)$, $E_{m-hlm}(k_2(l),a,b,c,k_0,\alpha,n,l)$ and $E_{l-hlm}(k_3(l),a,b,c,k_0,\alpha,n,l)$ which are generated with the non-null diagonal elements $(H_{so-hlm})_{11}$, $(H_{so-hlm})_{22}$ and $(H_{so-hlm})_{33}$ corresponding $j=l+1$ (spin great), $j=l$ (spin middle) and $j=l-1$ (spin little), respectively, at first order of two parameters (Θ and $\overline{\theta}$). Moreover, by applying the perturbative quantum chromodynamics (PQCD) quark model, we obtained the following results:

$$E_{g-hlm}=\gamma N^2 k_1(l)\int_0^{+\infty} r^{1+d(l,\mu)}\exp(-\delta(n,l)r)\left[L_n^{d(l,\mu)}(\delta(n,l)r)\right]^2\left(\left(\frac{a}{2r^3}+\frac{b}{r^4}-\frac{c}{2r}+\frac{k_0\alpha^2}{2}\exp\left(-\alpha^2\frac{r^2}{2}\right)\right)\Theta+\frac{\overline{\theta}}{2\mu}\right)dr$$
$$E_{m-hlm}=\gamma N^2 k_2(l)\int_0^{+\infty} r^{1+d(l,\mu)}\exp(-\delta(n,l)r)\left[L_n^{d(l,\mu)}(\delta(n,l)r)\right]^2\left(\left(\frac{a}{2r^3}+\frac{b}{r^4}-\frac{c}{2r}+\frac{k_0\alpha^2}{2}\exp\left(-\alpha^2\frac{r^2}{2}\right)\right)\Theta+\frac{\overline{\theta}}{2\mu}\right)dr$$
$$E_{l-hlm}=\gamma N^2 k_3(l)\int_0^{+\infty} r^{1+d(l,\mu)}\exp(-\delta(n,l)r)\left[L_n^{d(l,\mu)}(\delta(n,l)r)\right]^2\left(\left(\frac{a}{2r^3}+\frac{b}{r^4}-\frac{c}{2r}+\frac{k_0\alpha^2}{2}\exp\left(-\alpha^2\frac{r^2}{2}\right)\right)\Theta+\frac{\overline{\theta}}{2\mu}\right)dr$$
(28)

We have used the orthogonality property of the spherical harmonics $\int Y_l^m(\theta,\varphi)Y_{l'}^{m'}(\theta,\varphi)\sin(\theta)d\theta d\varphi=\delta_{ll'}\delta_{mm'}$ Now, we can rewrite the above three equations to the simplified new form:

$$E_{g-hlm}(k_1(l),a,b,c,k_0,\alpha,n,l)=\gamma N^2 k_1(l)\left\{\Theta[T_1(n,l,a)+T_2(n,l,b)+T_3(n,l,c)+T_4(n,l,\alpha,k_0)]+\frac{\bar{\theta}}{2\mu}T_5(n,l)\right\}$$
$$E_{m-hlm}(k_2(l),a,b,c,k_0,\alpha,n,l)=\gamma N^2 k_2(l)\left\{\Theta[T_1(n,l,a)+T_2(n,l,b)+T_3(n,l,c)+T_4(n,l,\alpha,k_0)]+\frac{\bar{\theta}}{2\mu}T_5(n,l)\right\}$$
$$E_{l-hlm}(k_3(l),a,b,c,k_0,\alpha,n,l)=\gamma N^2 k_3(l)\left\{\Theta[T_1(n,l,a)+T_2(n,l,b)+T_3(n,l,c)+T_4(n,l,\alpha,k_0)]+\frac{\bar{\theta}}{2\mu}T_5(n,l)\right\} \tag{29}$$

Moreover, the expressions of the 5-factors $T_i\left(i=\overline{1,5}\right)$ are given by:

$$T_1(n,l,a)=\frac{a}{2}\int_0^{+\infty} r^{-2+d(l,\mu)}\exp(-\delta(n,l)r)\left[L_n^{d(l,\mu)}(\delta(n,l)r)\right]^2 dr$$
$$T_2(n,l,b)=b\int_0^{+\infty} r^{-3+d(l,\mu)}\exp(-\delta(n,l)r)\left[L_n^{d(l,\mu)}(\delta(n,l)r)\right]^2 dr$$
$$T_3(n,l,c)=-\frac{c}{2}\int_0^{+\infty} r^{d(l,\mu)}\exp(-\delta(n,l)r)\left[L_n^{d(l,\mu)}(\delta(n,l)r)\right]^2 dr \tag{30}$$
$$T_4(n,l,\alpha,k_0)=\frac{k_0\alpha^2}{2}\int_0^{+\infty} r^{1+d(l,\mu)}\exp(-\delta(n,l)r)\exp\left(-\alpha^2\frac{r^2}{2}\right)\left[L_n^{d(l,\mu)}(\delta(n,l)r)\right]^2 dr$$
$$T_5(n,l)=\int_0^{+\infty} r^{1+d(l,\mu)}\exp(-\delta(n,l)r)\left[L_n^{d(l,\mu)}(\delta(n,l)r)\right]^2 dr$$

S-wave states, for the ground state ($n=0, l=0$), we have $L_0^{d(0,\mu)}(\delta(0,0)r)=1$, a direct simplification to eq. (30), we obtain:

$$T_1(n=0,l=0)=\frac{a}{2}\int_0^{+\infty} r^{d(0,\mu)-1-1}\exp(-\delta(0,0)r)\,dr$$
$$T_2(n=0,l=0)=b\int_0^{+\infty} r^{d(0,\mu)-2-1}\exp(-\delta(0,0)r)\,dr$$
$$T_3(n=0,l=0,c)=-\frac{c}{2}\int_0^{+\infty} r^{d(0,\mu)+1-1}\exp(-\delta(0,0)r)dr \tag{31}$$
$$T_4(n=0,l=0,\alpha,k_0)=\frac{k_0\alpha^2}{2}\int_0^{+\infty} r^{d(0,\mu)+2-1}\exp(-\alpha^2\frac{r^2}{2}-\delta(0,0)r)\,dr$$
$$T_5(n=0,l=0)=\int_0^{+\infty} r^{d(0,\mu)+2-1}\exp(-\delta(0,0)r)\,dr$$

With $d(0,\mu)=\sqrt{8\mu b}$ and $\delta(0,0)=2\sqrt{-2\mu E_{0,0}}$. For the purpose of calculating the 4-factors $T_1(n,a)$, $T_2(n,b)$, $T_3(n,c)$ and $T_5(n,l)$, it is convenient to apply the following special integral [36]:

$$\int_0^{+\infty} x^{\nu-1}\exp(-\lambda x^p)dx=\frac{\lambda^{-\frac{\nu}{p}}}{p}\Gamma\left(\frac{\nu}{p}\right) \tag{32}$$

With conditions ($\mathrm{Re}\lambda\rangle 0$, $\mathrm{Re}\nu\rangle 0$ and $p\rangle 0$) and $\Gamma\left(\frac{\nu}{p}\right)$ the ordinary Gamma function. After straightforward calculations, we can obtain the explicitly results:

$$\begin{aligned}
T_1(n=0,l=0,a)&=\frac{a}{2}\delta(0,0)^{1-d(0,\mu)}\Gamma(d(0,\mu)-1)\\
T_2(n=0,l=0,b)&=b\delta(0,0)^{2-d(0,\mu)}\Gamma(d(0,\mu)-2)\\
T_3(n=0,l=0,c)&=-\frac{c}{2}\delta(0,0)^{-1-d(0,\mu)}\Gamma(d(0,\mu)+1)\\
T_5(n=0,l=0)&=\delta(0,0)^{2-d(0,\mu)}\Gamma(d(0,\mu)+1)
\end{aligned} \tag{33}$$

Now, to obtain $T_4(n,\alpha,k_0)$, we apply the following special integration [36]:

$$\int_0^{+\infty} x^{\nu-1}\exp(-\lambda x^2-\gamma x)dx=(2\lambda)^{-\frac{\nu}{2}}\Gamma(\nu)\exp\left(\frac{\gamma^2}{8\lambda}\right)D_{-\nu}\left(\frac{\gamma}{\sqrt{2\lambda}}\right) \tag{34}$$

Where $D_{-\nu}\left(\frac{\gamma}{\sqrt{2\lambda}}\right)$ and $\Gamma(\nu)$ denote to the Parabolic cylinder functions and Gamma function, respectively. $\mathrm{Rel}(\lambda)\rangle 0$ and $\mathrm{Rel}(\nu\rangle 0)$. After straightforward calculations, we can obtain the explicitly results:

$$T_4(n=0,l=0,\alpha,k_0)=\frac{k_0\alpha^2}{2}\left(\alpha^2\right)^{-\frac{d(0,\mu)+2}{2}}\Gamma(d(0,\mu)+2)\exp\left(\frac{\delta(0,0)^2}{4\alpha^2}\right)D_{-(d(0,\mu)+2)}\left(\frac{\delta(0,0)}{\alpha}\right) \tag{35}$$

Allow us to obtain the exact modifications ($E_{g-hlm}(k_1(0),a,b,c,\alpha,n=0,l=0)$, $E_{m-hlm}(k_2(0),a,b,c,\alpha,n=0,l=0)$ and $E_{l-hlm}(k_3(0),a,b,c,\alpha,n=0,l=0)$) for Heavy-Light Mesons under modified quark-antiquark potential containing MCGISTs, which induced by the effect of modified spin-orbital operator $H_{so-hlm}(r,\Theta,\overline{\theta})$, as:

$$E_{g-hlm}(k_1(0)=0,a,b,c,\alpha,n=0)=0$$
$$E_{m-hlm}(k_2(0)=-1,a,b,c,\alpha,n)=-\gamma N^2\left\{\Theta T(a,b,c,\alpha,n=0,l=0)+\frac{\bar{\theta}}{2\mu}T_5(n=0,l=0)\right\} \quad (36)$$
$$E_{l-hlm}(k_3(0)=-1,a,b,c,\alpha,n)=-\gamma N^2\left\{\Theta T(a,b,c,\alpha,n=0,l=0)+\frac{\bar{\theta}}{2\mu}T_5(n=0,l=0)\right\}$$
with $T(a,b,c,\alpha,n=0,l=0)=T_1(n=0,l=0,a)+T_2(n=0,l=0,b)+T_3(n=0,l=0,c)+T_4(n=0,l=0,\alpha,k_0)$

P-wave states, for this case ($n=0$, $l=1$), we have $L_0^{d(0,\mu)}(\delta(0,0)r)=1$, a direct simplification to eq. (30), we obtain:

$$T_1(n=0,l=1,a)=\frac{a}{2}\int_0^{+\infty}r^{d(1,\mu)-1-1}\exp(-\delta(0,1)r)\,dr$$
$$T_2(n=0,l=1,b)=b\int_0^{+\infty}r^{d(1,\mu)-2-1}\exp(-\delta(0,1)r)\,dr$$
$$T_3(n=0,l=1,c)=-\frac{c}{2}\int_0^{+\infty}r^{d(1,\mu)+1-1}\exp(-\delta(0,1)r)dr \quad (37)$$
$$T_4(n=0,l=1,\alpha,k_0)=\frac{k_0\alpha^2}{2}\int_0^{+\infty}r^{d(1,\mu)+2-1}\exp(-\alpha^2\frac{r^2}{2}-\delta(0,1)r)\,dr$$
$$T_5(n=0,l=1)=\int_0^{+\infty}r^{d(1,\mu)+2-1}\exp(-\delta(0,1)r)\,dr$$

With $d(1,\mu)=\sqrt{9+8\mu b}$ and $\delta(0,1)=2\sqrt{-2\mu E_{0,1}}$. For the purpose of calculating the 4-factors $T_1(n,a)$, $T_2(n,b)$, $T_3(n,c)$ and $T_5(n,l)$, it is convenient to apply the special integral, that is in eq. (32):

$$T_1(n=0,l=1,a)=\frac{a}{2}\delta(0,1)^{1-d(1,\mu)}\Gamma(d(1,\mu)-1),T_2(n=0,l=1,b)=b\delta(0,1)^{2-d(1,\mu)}\Gamma(d(1,\mu)-2)$$
$$T_3(n=0,l=1,c)=-\frac{c}{2}\delta(0,1)^{-1-d(1,\mu)}\Gamma(d(1,\mu)+1) \text{ and } T_5(n=0,l=1,)=\delta(0,1)^{2-d(1,\mu)}\Gamma(d(1,\mu)+1) \quad (38)$$

Now, to obtain $T_4(n,\alpha,k_0)$, we apply the special integration that is in eq. (34):

$$T_4(n=0,l=1,\alpha,k_0)=\frac{k_0\alpha^2}{2}(\alpha^2)^{-\frac{d(1,\mu)+2}{2}}\Gamma(d(1,\mu)+2)\exp\left(\frac{\delta(0,1)^2}{4\alpha^2}\right)D_{-(d(1,\mu)+2)}\left(\frac{\delta(0,1)}{\alpha}\right) \quad (39)$$

Allow us to obtain the exact modifications $E_{g-hlm}(k_1(1),a,b,c,\alpha,n=0,l=1)$, $E_{m-hlm}(k_2(1),a,b,c,\alpha,n=0,l=1)$ and $E_{l-hlm}(k_3(1),a,b,c,\alpha,n=0,l=1)$ for HLM under modified quark-antiquark potential containing MCGISTs, which induced by the effect of modified spin-orbital operator $H_{so-hlm}(r,\Theta,\bar{\theta})$, as:

$$
\begin{aligned}
&E_{g-hlm}(k_1(1),a,b,c,\alpha,n=0,l=1)=\frac{\gamma N^2}{2}\left\{\Theta T(a,b,c,\alpha,n=0,l=1)+\frac{\bar{\theta}}{2\mu}T_5(n=0,l=1)\right\}\\
&E_{m-hlm}(k_2(1),a,b,c,\alpha,n=0,l=1)=-1\gamma N^2\left\{\Theta T(a,b,c,\alpha,n=0,l=1)+\frac{\bar{\theta}}{2\mu}T_5(n=0,l=1)\right\}\\
&E_{l-hlm}(k_3(1),a,b,c,\alpha,n=0,l=1)=-2\gamma N^2\left\{\Theta T(a,b,c,\alpha,n=0,l=1)+\frac{\bar{\theta}}{2\mu}T_5(n=0,l=1)\right\}
\end{aligned}
\tag{40}
$$

with $T(a,b,c,\alpha,n=0,l=1)=T_1(n=0,l=1,a)+T_2(n=0,l=1,b)+T_3(n=0,l=1,c)+T_4(n=0,l=1,\alpha,k_0)$

For, any other $n^{th}(n\neq 0,l)$ excited state, the exact modifications ($E_{g-hlm}(k_1(l),a,b,c,\alpha,n,l)$, $E_{m-hlm}(k_2(l),a,b,c,\alpha,n,l)$ and $E_{l-hlm}(k_3(1),a,b,c,\alpha,n,l)$) for Heavy-Light Mesons under modified quark-antiquark potential containing *MCGISTs* which induced by the effect of modified spin-orbital operator $H_{so-hlm}(r,\Theta,\bar{\theta})$:

$$
\begin{aligned}
&E_{g-hlm}(k_1(l),a,b,c,k_0,\alpha,n,l)=\gamma N^2 k_1(l)\left\{\Theta T(a,b,c,\alpha,n,l)+\frac{\bar{\theta}}{2\mu}T_5(n,l)\right\}\\
&E_{m-hlm}(k_2(l),a,b,c,k_0,\alpha,n,l)=\gamma N^2 k_2(l)\left\{\Theta T(a,b,c,\alpha,n,l)+\frac{\bar{\theta}}{2\mu}T_5(n,l)\right\}\\
&E_{l-hlm}(k_3(l),a,b,c,k_0,\alpha,n,l)=\gamma N^2 k_3(l)\left\{\Theta T(a,b,c,\alpha,n,l)+\frac{\bar{\theta}}{2\mu}T_5(n,l)\right\}
\end{aligned}
\tag{41}
$$

With $T(a,b,c,\alpha,n,l)=T_1(n,l,a)+T_2(n,l,b)+T_3(n,l,c)+T_4(n,l,\alpha,k_0)$ while the perturbed energy $E_{nl}^{(p)}$ corresponded the perturbed potential H' obtained after applying the standard perturbation theory:

$$
E_{nl}^{(p)}=N^2\int_0^{+\infty} r^{1+d(l,\mu)}\exp(-\delta(n,l)r)\left[L_n^{d(l,\mu)}(\delta(n,l)r)\right]^2\left(k_0\exp\left(-\alpha^2\frac{r^2}{2}\right)+cr\right)dr
\tag{42}
$$

For S-wave states, the ground state ($n=0,l=0$), the corresponding energy is $E_{0,0}^{(p)}$, a direct simplification to eq. (42) gives:

$$
E_{nl}^{(p)}=N^2\int_0^{+\infty}\left(k_0 r^{2+d(0,\mu)-1}\exp\left(-\delta(0,0)r-\alpha^2\frac{r^2}{2}\right)+cr^{3+d(0,\mu)-1}\exp(-\delta(0,0)r)\right)dr
\tag{43}
$$

We apply similtaniously, the two integrals (32) and (34), we obtain:

$$
E_{0,0}^{(p)}=N^2\left\{\left(\alpha^2\right)^{-\frac{d(0,\mu)+2}{2}}\Gamma(d(0,\mu)+2)\exp\left(\frac{\delta(0,0)^2}{4\alpha^2}\right)D_{-(d(1,\mu)+2)}\left(\frac{\delta(0,0)}{\alpha}\right)+\delta(0,0)^{3-d(0,\mu)}\Gamma(d(0,\mu)+3)\right\}
\tag{44}
$$

P-wave states, for this case ($n=0,l=1$), the corresponding energy is $E_{01}^{(p)}$ which produced by perturbed potential H', by the same méthode, we obtain:

$$E_{0,1}^{(p)} = N^2\left\{\left(\alpha^2\right)^{-\frac{d(1,\mu)+2}{2}}\Gamma(d(1,\mu)+2)\exp\left(\frac{\delta(0,1)^2}{4\alpha^2}\right)D_{-(d(1,\mu)+2)}\left(\frac{\delta(0,1)}{\alpha}\right)+\delta(0,1)^{3-d(1,\mu)}\Gamma(d(1,\mu)+3)\right\} \tag{45}$$

The exact Modified Magnetic Spectrum for HLM under Modified quark-antiquark potential containing *MCGISTs* in (NC: 3D- RSP) Symmetries:

Further to the important previously obtained results, now, we consider another important physically meaningful phenomena produced by the effect of induced self-uniform magnetic field $\vec{B}$. This field is self-generated from the properties of (space-space) and (phase-phase) noncommutativity influenced on HLM such as scalar, vector, pseudoscalar, and pseudovector for (B, B_s, D and D_s) mesons, under modified quark-antiquark potential containing MCGISTs related to the influence of an external. To avoid the repetition in the theoretical calculations, it is sufficient to apply the two following similtaniously replacements:

$$\begin{cases}\vec{\Theta}\rightarrow\chi\vec{B}\\ \vec{\bar{\theta}}\rightarrow\bar{\sigma}\vec{B}\end{cases}\Rightarrow\left(\left(\frac{a}{2r^3}+\frac{b}{r^4}-\frac{c}{2r}+\frac{k_0\alpha^2}{2}\exp\left(-\alpha^2\frac{r^2}{2}\right)\right)\vec{\Theta}+\frac{\vec{\bar{\theta}}}{2\mu}\right)\vec{L}\text{ will - be - replace - by}:\left(\left(\frac{a}{2r^3}+\frac{b}{r^4}-\frac{c}{2r}+\frac{k_0\alpha^2}{2}\exp\left(-\alpha^2\frac{r^2}{2}\right)\right)\chi+\frac{\bar{\sigma}}{2\mu}\right)\vec{B}\vec{L} \tag{46}$$

Here χ and $\bar{\sigma}$ are two infinitesimal real proportional constants, and we choose the arbitrary uniform external magnetic field $\vec{B}$ parallel to the (Oz) axis, which allow us to introduce the new modified magnetic Hamiltonian $H_{m-hlm}(r,\chi,\bar{\sigma})$ in (NC: 3D-RSP) symmetries as:

$$H_{\text{so-hlm}}(r,\Theta,\bar{\theta})\rightarrow H_{m-hlm}(r,\chi,\bar{\sigma})=\left(\left(\frac{a}{2r^3}+\frac{b}{r^4}-\frac{c}{2r}+\frac{k_0\alpha^2}{2}\exp\left(-\alpha^2\frac{r^2}{2}\right)\right)\chi+\frac{\bar{\sigma}}{2\mu}\right)\left\{\vec{B}\vec{J}-\aleph_z\right\} \tag{47}$$

Here $\aleph_z\equiv-\vec{S}\vec{B}$ denote to Zeeman effect in commutative quantum mechanics, while $\aleph_{\text{mod}-z}\equiv\vec{B}\vec{J}-\aleph_z$ is the new Zeeman effect. To obtain the exact NC magnetic modifications of energy for n^{th} excited states of Heavy-Light Mesons, $E_{mag-hlm}(m,n,l,a,b,c,\alpha,k_0)$ we just replace both two parameters $k_1(l)$ and Θ in the Eq. (41) by the corresponding quantum parameters m and χ, respectively:

$$E_{mag-hlm}\left(m=\overline{-l,+l},n,l,a,b,c,\alpha,k_0\right)=\gamma N^2\left\{\chi T(a,b,c,\alpha,n,l)+\frac{\bar{\sigma}}{2\mu}T_5(n,l)\right\}Bm \tag{48}$$

S-wave states ($n=0,l=0$) and ***P-wave states*** ($n=0,l=1$), the NC magnetic modifications of energy excited of Heavy-Light Mesons $E_{mag-hlm}(m=0,n=0,l=0,a,b,c,\alpha,k_0)$ and $E_{mag-hlm}(m=0,\pm1,n=0,l=1,a,b,c,\alpha,k_0)$, respectively:

$$\begin{aligned}&E_{mag-hlm}\left(m=\overline{-l,+l},n=0,l=0,a,b,c,\alpha,k_0\right)=0\\&E_{mag-hlm}((m=0,\pm1),n,l=1,a,b,c,\alpha,k_0)=\gamma N^2\left\{\chi T(a,b,c,\alpha,n=0,l=1)+\frac{\bar{\sigma}}{2\mu}T_5(n=0,l=1)\right\}Bm\end{aligned} \tag{49}$$

We have $-l \leq m \leq +l$, which allow us to fixing ($2l+1$) values for discreet number m. It should be noted that the results obtained in Eq. (49) could find it by direct calculation $E_{mag-hlm} = \langle \Psi(r,\theta,\varphi) | H_{m-hlm}(r,\chi,\overline{\sigma}) | \Psi(r,\theta,\varphi) \rangle$ that takes the following explicit relation:

$$E_{mag-hlm} = \gamma N^2 \int_0^{+\infty} r^{1+d(l,\mu)} \exp(-\delta(n,l)r)\left[L_n^{d(l,\mu)}(\delta(n,l)r)\right]^2 \left(\left(\frac{a}{2r^3} + \frac{b}{r^4} - \frac{c}{2r} + \frac{k_0\alpha^2}{2}\exp\left(-\alpha^2\frac{r^2}{2}\right)\right)\chi + \frac{\overline{\sigma}}{2\mu}\right) dr$$

(50)

Then we find the corrections produced by the operator $H_{m-hlm}(r,\chi,\overline{\sigma})$ for n^{th} excited states repeating the same calculations in the previous subsection.

Results and Discussion

In the previous sections, we obtained the solution of the modified Schrödinger equation for HLM such as scalar, vector, pseudoscalar, and pseudovector for (B, B_s, D and D_s) mesons under modified quark-antiquark potential containing MCGISTs, which is given in Eq. (22) by using the generalized Bopp's shift method and standard perturbation theory. The energy eigenvalue is calculated in the 3D space-phase. The modified eigenenergies $(E_{\text{nc-ghlm}}, E_{\text{nc-mhlm}}, E_{\text{nc-lhlm}})(m,n,l,a,b,c,\alpha,k_0)$ with spin $\vec{S} = \vec{1}$ for MSE for Heavy-Light Mesons are obtained in this paper on based to our original results presented on the Eqs. (41) and (48), in addition to the ordinary energy E_{nl} for quark-antiquark potential containing CGISTs which presented in the Eq. (10):

$$\begin{aligned}
E_{\text{nc-ghlm}}(m,n,l,a,b,c,\alpha,k_0) &= E_{nl} + E_{nl}^{(p)} + \gamma N^2\left\{(k_1(l)\Theta + \chi Bm)T(a,b,c,\alpha,n,l) - \left(\frac{\overline{\theta}}{2\mu}k_1(l) + \frac{\overline{\sigma}}{2\mu}Bm\right)T_5(n,l)\right\} \\
E_{\text{nc-mhlm}}(m,n,l,a,b,c,\alpha,k_0) &= E_{nl} + E_{nl}^{(p)} + \gamma N^2\left\{(k_2(l)\Theta + \chi Bm)T(a,b,c,\alpha,n,l) - \left(\frac{\overline{\theta}}{2\mu}k_2(l) + \frac{\overline{\sigma}}{2\mu}Bm\right)T_5(n,l)\right\} \\
E_{\text{nc-lchlm}}(m,n,l,a,b,c,\alpha,k_0) &= E_{nl} + E_{nl}^{(p)} + \gamma N^2\left\{(k_3(l)\Theta + \chi Bm)T(a,b,c,\alpha,n,l) - \left(\frac{\overline{\theta}}{2\mu}k_3(l) + \frac{\overline{\sigma}}{2\mu}Bm\right)T_5(n,l)\right\}
\end{aligned} \quad (51.1)$$

Where $E_{nl}^{(p)}$ is the perturbed energy which produced by perturbed potential H'. For ***S-wave states*** ($n=0, l=0$) and ***P-wave states*** ($n=0, l=1$):

$$\begin{aligned}
E_{\text{nc-ghlm}}(m=0,n=0,l=0,a,b,c,\alpha,k_0) &= E_{0,0} + E_{0,0}^{(p)} \\
E_{\text{nc-mhlm}}(m=0,n=0,l=0,a,b,c,\alpha,k_0) &= E_{0,0} + E_{0,0}^{(p)} + \gamma N^2\left\{-2\Theta T(a,b,c,\alpha,n=0,l=0) + \frac{\overline{\theta}}{2\mu}T_5(n,l)\right\} \\
E_{\text{nc-lchlm}}(m=0,n=0,l=0,a,b,c,\alpha,k_0) &= E_{0,0} + E_{0,0}^{(p)} + \gamma N^2\left\{-2\Theta T(a,b,c,\alpha,n=0,l=0) + \frac{\overline{\theta}}{2\mu}T_5(n,l)\right\}
\end{aligned}$$

(51.2)

$$E_{\text{nc-ghlm}}((m=0,\pm1),n=0,l=1,a,b,c,\alpha,k_0)=E_{0,1}+E_{0,1}^{(p)}+\gamma N^2\left\{(1/2\Theta+\chi Bm)T(a,b,c,\alpha,n,l)-\left(\frac{\bar{\theta}}{4\mu}+\frac{\bar{\sigma}}{2\mu}Bm\right)T_5(n,l)\right\}$$

$$E_{\text{nc-mhlm}}((m=0,\pm1),n=0,l=1,a,b,c,\alpha,k_0)=E_{0,1}+E_{0,1}^{(p)}+\gamma N^2\left\{(-\Theta+\chi Bm)T(a,b,c,\alpha,n,l)+\left(\frac{\bar{\theta}}{2\mu}-\frac{\bar{\sigma}}{2\mu}Bm\right)T_5(n,l)\right\}$$

$$E_{\text{nc-lchlm}}((m=0,\pm1),n=0,l=1,a,b,c,\alpha,k_0)=E_{0,1}+E_{0,1}^{(p)}+\gamma N^2\left\{(-2\Theta+\chi Bm)T(a,b,c,\alpha,n,l)+\left(\frac{\bar{\theta}}{\mu}-\frac{\bar{\sigma}}{2\mu}Bm\right)T_5(n,l)\right\}$$

(51.3)

This is one of the main objectives of our research and by noting that, the obtained eigenvalues of energies are real's and then the NC diagonal Hamiltonian $H_{nc-hlm}(x_\mu,p_\mu)$ is Hermitian, furthermore it's possible to writing the three elements $(H_{nc-hlm})_{11}$, $(H_{nc-hlm})_{22}$ and $(H_{nc-hlm})_{33}$ as follows:

$$H_{hlm}(x_\mu,p_\mu)\rightarrow H_{nc-hlm}(x_\mu,p_\mu)\equiv\begin{pmatrix}(H_{nc-hlm})_{11} & 0 & 0\\ 0 & (H_{nc-hlm})_{22} & 0\\ 0 & 0 & (H_{nc-hlm})_{33}\end{pmatrix} \quad (52)$$

Where $(H_{nc-hlm})_{11}=-\frac{\Delta_{nc}}{2\mu}+H_{\text{int}-ghlm}$, $(H_{nc-hlm})_{22}=-\frac{\Delta_{nc}}{2\mu}+H_{\text{int}-mhlm}$ and $(H_{nc-cp})_{33}=-\frac{\Delta_{nc}}{2\mu}+H_{\text{int}-lhlm}$ with $\frac{\Delta_{nc}}{2\mu}=\frac{\Delta-\vec{\bar{\theta}}\vec{L}-\vec{\bar{\sigma}}\vec{L}}{2\mu}$ and the three modified interactions elements ($H_{\text{int}-ghlm}, H_{\text{int}-mhlm}, H_{\text{int}-lhlm}$) are given by:

$$V_{hlm}(r)\rightarrow\begin{cases}H_{\text{int}-ghlm}=\frac{a}{r}+\frac{b}{r^2}+k_0\exp\left(-\alpha^2\frac{r^2}{2}\right)+cr+\gamma(k_1(l)\Theta+\chi\aleph_{\text{mod}-z})\left(\frac{a}{2r^3}+\frac{b}{r^4}-\frac{c}{2r}+\frac{k_0\alpha^2}{2}\exp\left(-\alpha^2\frac{r^2}{2}\right)\right)\\ H_{\text{int}-mhlm}=\frac{a}{r}+\frac{b}{r^2}+k_0\exp\left(-\alpha^2\frac{r^2}{2}\right)+cr+\gamma(k_2(l)\Theta+\chi\aleph_{\text{mod}-z})\left(\frac{a}{2r^3}+\frac{b}{r^4}-\frac{c}{2r}+\frac{k_0\alpha^2}{2}\exp\left(-\alpha^2\frac{r^2}{2}\right)\right)\\ H_{\text{int}-lhlm}=\frac{a}{r}+\frac{b}{r^2}+k_0\exp\left(-\alpha^2\frac{r^2}{2}\right)+cr+\gamma(k_3(l)\Theta+\chi\aleph_{\text{mod}-z})\left(\frac{a}{2r^3}+\frac{b}{r^4}-\frac{c}{2r}+\frac{k_0\alpha^2}{2}\exp\left(-\alpha^2\frac{r^2}{2}\right)\right)\end{cases}$$

(53)

Thus, the ordinary kinetic term for quark-antiquark potential containing CGISTs ($-\frac{\Delta}{2\mu}$) and ordinary interaction $\frac{a}{r}+\frac{b}{r^2}+k_0\exp\left(-\alpha^2\frac{r^2}{2}\right)+cr$ are replaced by new modified form of kinetic term $\frac{\Delta_{nc}}{2\mu}$ and new modified interactions modified to the new form $(H_{\text{int}-ghlm}, H_{\text{int}-mhlm}, H_{\text{int}-lhlm})$ in (NC-3D: RSP) symmetries. On the other hand, it is evident to consider the quantum number m takes ($2l+1$) values and we have also two values for $(j=l\pm1,l)$, thus every state in usually three-dimensional space of energy for heavy quarkonium system and hydrogenic atoms under modified quark-antiquark potential containing *MCGISTs* will be $(3(2l+1))$sub-states. To obtain the total complete degeneracy of energy level of the modified quark-antiquark potential containing *MCGISTs* in (NC-3D: RSP) symmetries, we need to sum for all allowed values of l. Total degeneracy is thus,

$$\sum_{i=0}^{n-1}(2l+1)=n^2 \rightarrow 3\left(\sum_{i=0}^{n-1}(2l+1)\right)\equiv 3n^2 \tag{54}$$

Note that the obtained new energy eigenvalues $(E_{\text{nc-g}h\text{lm}}, E_{\text{nc-m}h\text{lm}}, E_{\text{nc-l}h\text{lm}})(m,n,l,a,b,c,\alpha)$ now depend to new discrete atomic quantum numbers (n,j,l,s) and m in addition to the parameters (a,b,c,α,k_0) of the quark-antiquark potential. It is pertinent to note that when the atoms have $\vec{S}=\vec{0}$, the total operator can be obtains from the interval $|l-s|\leq j\leq |l+s|$, which allow us to obtaining the eigenvalues of the operator $\left(\overleftrightarrow{J}^2-\overleftrightarrow{L}^2-\overleftrightarrow{S}^2\right)$ as $k(j,l,s)\equiv 0$ and then the energy spectrum $(E_{\text{nc-g}h\text{lm}}, E_{\text{nc-m}h\text{lm}}, E_{\text{nc-l}h\text{lm}})(m,n,l,a,b,c,\alpha)$ reads:

$$(E_{\text{nc-g}h\text{lm}}, E_{\text{nc-m}h\text{lm}}, E_{\text{nc-l}h\text{lm}})(m,n,l,a,b,c,\alpha)=E_{nl}+E_{nl}^{(p)}+\gamma N^2\left\{\chi T(a,b,c,\alpha,n,l)-\frac{\bar{\sigma}}{2}T_5(n,l)\right\}Bm \tag{55}$$

Our last application is to calculate the modified mass spectra of the Heavy-Light Mesons such as scalar, vector, pseudoscalar, and pseudovector for (B, B_s, D and D_s) mesons under modified quark-antiquark potential containing *MCGISTs*. In order to achieve this goal, we generalize the traditional formula $M=m_q+m_{\bar{q}}+E_{nl}+E_{nl}^{(p)}$ to the new form:

$$M=m_q+m_{\bar{q}}+E_{nl}+E_{nl}^{(p)}\rightarrow M_{nc-hlm}=m_q+m_{\bar{q}}+\frac{1}{3}(E_{\text{nc-g}h\text{lm}}+E_{\text{nc-m}h\text{lm}}+E_{\text{nc-l}h\text{lm}})(m,n,l,a,b,c,\alpha) \tag{56}$$

Here $\frac{1}{3}(E_{\text{nc-g}h\text{lm}}+E_{\text{nc-m}h\text{lm}}+E_{\text{nc-l}h\text{lm}})(m,n,l,a,b,c,\alpha)$ is the non-polarized energy value. Thus, the modified mass of Heavy-Light Mesons M_{nc-hlm} such as (B, B_s, D and D_s) mesons:

$$M_{nc-hlm}=M+\gamma N^2\begin{cases}\left\{\left(\chi Bm-\frac{l+4}{6}\Theta+\right)T(a,b,c,\alpha,n,l)-\left(\frac{\bar{\sigma}}{2\mu}Bm-\frac{\bar{\theta}(l+4)}{12\mu}\right)T_5(n,l)\right\} & \text{for } \vec{S}=\vec{1}\\ \left\{\chi T(a,b,c,\alpha,n,l)-\frac{\bar{\sigma}}{2}T_4(n,l)\right\}Bm & \text{for } \vec{S}=\vec{0}\end{cases} \tag{57}$$

Thus, the spin-orbital coupling $\langle H_{so-hlm}\rangle$ introduced automatically in the masses of (B, B_s, D and D_s) mesons, we did not consider it an external terms. Here M is the Heavy-Light Mesons under quark-antiquark potential containing *CGISTs* in commutative quantum mechanics, which defined in [6]. If we consider $(\Theta,\chi)\rightarrow(0,0)$, we recover the results of commutative space of ref. [6], which means that our calculations are correct.

Concluding Remarks

In the present work, the 3DMSE is analytically solved using the generalized Bopp's shift method and standard perturbation theory. The quark-antiquark potential is extended to include effect of noncommutativity space phase based on ref. [6]; we resume the main obtained results:

- Ordinary quark-antiquark potential ($\frac{a}{r}+\frac{b}{r^2}+k_0\exp\left(-\alpha^2\frac{r^2}{2}\right)+cr$) were replaced by new modified interactions $\left(H_{\text{int}-ghlm}, H_{\text{int}-mhlm}, H_{\text{int}-lhlm}\right)$ for Heavy-Light Mesons,
- The ordinary kinetic term $-\frac{\Delta}{2\mu}$ modified to the new form $\frac{\Delta_{nc}}{2\mu}=\frac{\Delta-\vec{\overline{\theta}}\vec{L}-\vec{\overline{\sigma}}\vec{L}}{2\mu}$ for Heavy-Light Mesons under influence of modified quark-antiquark potential,
- We obtained the perturbative corrections $\left(E_{\text{nc-ghlm}}, E_{\text{nc-mhlm}}, E_{\text{nc-lhlm}}\right)(m,n,l,a,b,g,h)$ for n^{th} excited state with (spin $\vec{S}=\vec{1}$ and $\vec{S}=\vec{0}$) for MSE for Heavy-Light Mesons under influence modified quark-antiquark potential containing *MCGISTs* are obtained.
- We have shown that the spin-orbital coupling $\langle H_{so-hlm}\rangle$ were introduced automatically in the masses of (B, B_s, D and D_s) mesons

The mass spectra of heavy-light mesons (B, B_s, D and D_s mesons) were calculated in the extended quark model containing *MCGISTs*, the new values M_{nc-hlm} equal the sum of corresponding value M in CQM and two perturbative terms proportional with two parameters (Θ and $\overline{\theta}$). Through the of high-value results, which we have achieved in present work, we hope to extend our recently work physics for further investigations of particles physics and other characteristics of quarkonium.

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