A Study On Generalized (r, s, t)-Numbers

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Abstract. In this paper, we investigate the generalized (r, s, t) sequence and we deal with, in detail, three special cases which we call them (r, s, t), Lucas (r, s, t) and modified (r, s, t) sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related to these sequences.

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1 Introduction

The sequence of Fibonacci numbers $\{F_n\}$ and the sequence of Lucas numbers $\{L_n\}$ are defined by

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2, \quad F_0 = 0, F_1 = 1,$$

and

$$L_n = L_{n-1} + L_{n-2},\ n \geq 2, \quad L_0 = 2, L_1 = 1$$

respectively. The generalizations of Fibonacci and Lucas sequences lead to several nice and interesting sequences.

The generalized Fibonacci sequence (Horadam sequence or 2-step Fibonacci sequence)

$$\{V_n(V_0, V_1; r, s)\}_{n\geq 0}$$

(or shortly $\{V_n\}_{n\geq 0}$) is defined (by Horadam [7]) as follows:

$$V_n = rV_{n-1} + sV_{n-2}, \qquad V_0 = a, V_1 = b,\ n \geq 2 \tag{1}$$

where V_0, V_1 are arbitrary complex (or real) numbers and r, s are real numbers, see also Horadam [6,8,9].

In this paper, we investigate the third order generalization of Fibonacci numbers. The generalized (r, s, t) sequence (or generalized Tribonacci sequence or generalized 3-step Fibonacci sequence)

$$\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$$

(or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \qquad W_0 = a, W_1 = b, W_2 = c,\ n \geq 3 \tag{2}$$

where W_0, W_1, W_2 are arbitrary complex (or real) numbers and r, s, t are real numbers.

This sequence has been studied by many authors, see for example [1,2,3,4,5,12,13,14,16,18,28,30,31].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (2) holds for all integer n.

As examples, we can define three special cases of $\{W_n\}_{n\geq 0}$ as follows:

$$\begin{array}{rcl} third order Pell - Perrin & : & \{R_n^{(3)}\} = \{W_n(3,0,2;2,1,1)\}, \\ adjusted Padovan & : & \{U_n\} = \{W_n(0,1,0;0,1,1)\}, \\ adjusted Pell - Padovan & : & \{M_n\} = \{W_n(0,1,0;0,2,1)\}. \end{array}$$

In other words, third order Pell-Perrin sequence$\{R_n^{(3)}\}_{n\geq 0}$, adjusted Padovan sequence $\{U_n\}_{n\geq 0}$ and adjusted Pell-Padovan sequence $\{M_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$R_{n+3}^{(3)} = 2R_{n+2}^{(3)} + R_{n+1}^{(3)} + R_n^{(3)}, \quad R_0^{(3)} = 3, R_1^{(3)} = 0, R_2^{(3)} = 2, \tag{3}$$

$$U_{n+3} = U_{n+1} + U_n, \quad U_0 = 0, U_1 = 1, U_2 = 0, \tag{4}$$

$$M_{n+3} = 2M_{n+1} + M_n, \quad M_0 = 0, M_1 = 1, M_2 = 0. \tag{5}$$

The sequences $\{R_n^{(3)}\}_{n\geq 0}$, $\{U_n\}_{n\geq 0}$ and $\{M_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{array}{rcl} R_{-n}^{(3)} & = & -R_{-(n-1)}^{(3)} - 2R_{-(n-2)}^{(3)} + R_{-(n-3)}^{(3)}, \\ U_{-n} & = & -U_{-(n-1)} + U_{-(n-3)}, \\ M_{-n} & = & -2M_{-(n-1)} + M_{-(n-3)}, \end{array}$$

for $n = 1, 2, 3, ...$ respectively.

In literature, for example, the following names and notations (see Table 1) are used for the special case of r, s, t and initial values (we also include newly defined third order Pell-Perrin, adjusted Padovan and adjusted Pell-Padovan sequences in the Table).

Table 1 A few special case of generalized (r,s,t) (generalized Tribonacci) sequence

No	*Sequences(Numbers)*	*Notation*	OEIS [15]	References
1	*Tribonacci*	$\{T_n\} = \{W_n(0,1,1;1,1,1)\}$	A000073, A057597	[19]
2	*Tribonacci − Lucas*	$\{K_n\} = \{W_n(3,1,3;1,1,1)\}$	A001644, A073145	[19]
3	*Tribonacci − Perrin*	$\{M_n\} = \{W_n(3,0,2;1,1,1)\}$		[19]
4	modified Tribonacci	$\{U_n\} = \{W_n(1,1,1;1,1,1)\}$		[19]
5	modified Tribonacci-Lucas	$\{G_n\} = \{W_n(4,4,10;1,1,1)\}$		[19]
6	adjusted Tribonacci-Lucas	$\{H_n\} = \{W_n(4,2,0;1,1,1)\}$		[19]
7	*thirdorder*Pell	$\{P_n^{(3)}\} = \{W_n(0,1,2;2,1,1)\}$	A077939, A077978	[20]
8	*thirdorder*Pell-Lucas	$\{Q_n^{(3)}\} = \{W_n(3,2,6;2,1,1)\}$	A276225, A276228	[20]
9	*thirdordermodified*Pell	$\{E_n^{(3)}\} = \{W_n(0,1,1;2,1,1)\}$	A077997, A078049	[20]
10	*thirdorder*Pell-Perrin	$\{R_n^{(3)}\} = \{W_n(3,0,2;2,1,1)\}$		
11	*Padovan(Cordonnier)*	$\{P_n\} = \{W_n(1,1,1;0,1,1)\}$	A000931	[21]
12	*Perrin(Padovan − Lucas)*	$\{E_n\} = \{W_n(3,0,2;0,1,1)\}$	A001608, A078712	[21]
13	*Padovan − Perrin*	$\{S_n\} = \{W_n(0,0,1;0,1,1)\}$	A000931, A176971	[21]
14	modified Padovan	$\{A_n\} = \{W_n(3,1,3;0,1,1)\}$		[21]
15	adjusted Padovan	$\{U_n\} = \{W_n(0,1,0;0,1,1)\}$		
16	*Pell − Padovan*	$\{R_n\} = \{W_n(1,1,1;0,2,1)\}$	A066983, A128587	[22]
17	*Pell − Perrin*	$\{C_n\} = \{W_n(3,0,2;0,2,1)\}$		[22]
18	*thirdorder*Fibonacci-Pell	$\{G_n\} = \{W_n(1,0,2;0,2,1)\}$		[22]
19	*thirdorder*Lucas-Pell	$\{B_n\} = \{W_n(3,0,4;0,2,1)\}$		[22]
20	adjusted *Pell − Padovan*	$\{M_n\} = \{W_n(0,1,0;0,2,1)\}$		
21	*Jacobsthal − Padovan*	$\{Q_n\} = \{W_n(1,1,1;0,1,2)\}$	A159284	[23]
22	*Jacobsthal − Perrin(−Lucas)*	$\{L_n\} = \{W_n(3,0,2;0,1,2)\}$	A072328	[23]
23	adjusted *Jacobsthal − Padovan*	$\{K_n\} = \{W_n(0,1,0;0,1,2)\}$		[23]
24	modified *Jacobsthal − Padovan*	$\{M_n\} = \{W_n(3,1,3;0,1,2)\}$		[23]
25	*Narayana*	$\{N_n\} = \{W_n(0,1,1;1,0,1)\}$	A078012	[24]
26	Narayana-Lucas	$\{U_n\} = \{W_n(3,1,1;1,0,1)\}$	A001609	[24]
27	Narayana-Perrin	$\{H_n\} = \{W_n(3,0,2;1,0,1)\}$		[24]
28	*thirdorderJacobsthal*	$\{J_n^{(3)}\} = \{W_n(0,1,1;1,1,2)\}$	A077947	[25]
29	*thirdorderJacobsthal − Lucas*	$\{j_n^{(3)}\} = \{W_n(2,1,5;1,1,2)\}$	A226308	[25]
30	modified *thirdorderJacobsthal − Lucas*	$\{K_n^{(3)}\} = \{W_n(3,1,3;1,1,2)\}$		[25]
31	*thirdorderJacobsthal*−Perrin	$\{Q_n^{(3)}\} = \{W_n(3,0,2;1,1,2)\}$		[25]
32	3-primes	$\{G_n\} = \{W_n(0,1,2;2,3,5)\}$		[26]
33	Lucas 3-primes	$\{H_n\} = \{W_n(3,2,10;2,3,5)\}$		[26]
34	modified 3-primes	$\{E_n\} = \{W_n(0,1,1;2,3,5)\}$		[26]
35	reverse 3-primes	$\{N_n\} = \{W_n(0,1,5;5,3,2)\}$		[27]
36	reverse Lucas 3-primes	$\{S_n\} = \{W_n(3,5,31;5,3,2)\}$		[27]
37	reverse modified 3-primes	$\{U_n\} = \{W_n(0,1,4;5,3,2)\}$		[27]

Here, OEIS stands for On-line Encyclopedia of Integer Sequences.

As $\{W_n\}$ is a third-order recurrence sequence (difference equation), it's characteristic equation is

$$x^3 - rx^2 - sx - t = 0 \tag{6}$$

whose roots are

$$\begin{aligned} \alpha &= \alpha(r,s,t) = \frac{r}{3} + A + B, \\ \beta &= \beta(r,s,t) = \frac{r}{3} + \omega A + \omega^2 B, \\ \gamma &= \gamma(r,s,t) = \frac{r}{3} + \omega^2 A + \omega B, \end{aligned}$$

where

$$\begin{aligned} A &= \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta}\right)^{1/3}, B = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta}\right)^{1/3}, \\ \Delta &= \Delta(r,s,t) = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}, \ \omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3). \end{aligned}$$

Note that we have the following identities

$$\begin{aligned} \alpha+\beta+\gamma &= r, \\ \alpha\beta+\alpha\gamma+\beta\gamma &= -s, \\ \alpha\beta\gamma &= t. \end{aligned}$$

If $\Delta(r,s,t)>0$, then the Equ. (6) has one real (α) and two non-real solutions with the latter being conjugate complex. So, in this case, it is well known that the generalized (r,s,t) numbers (the generalized Tribonacci numbers) can be expressed, for all integers n, using Binet's formula

$$W_n = \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{7}$$

where

$$p_1 = W_2-(\beta+\gamma)W_1+\beta\gamma W_0, p_2 = W_2-(\alpha+\gamma)W_1+\alpha\gamma W_0, p_3 = W_2-(\alpha+\beta)W_1+\alpha\beta W_0.$$

(7) can be written in the following form:

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n$$

where

$$A_1 = \frac{W_2-(\beta+\gamma)W_1+\beta\gamma W_0}{(\alpha-\beta)(\alpha-\gamma)}, A_2 = \frac{W_2-(\alpha+\gamma)W_1+\alpha\gamma W_0}{(\beta-\alpha)(\beta-\gamma)}, A_3 = \frac{W_2-(\alpha+\beta)W_1+\alpha\beta W_0}{(\gamma-\alpha)(\gamma-\beta)}.$$

Note that

$A_1A_2A_3 = \frac{1}{4r^3t-r^2s^2+18rst-4s^3+27t^2}(t^2W_0^3+(t+rs)W_1^3+W_2^3+rtW_0^2W_2-2rW_2^2W_1+(r^2-s)W_1^2W_2-sW_2^2W_0+(s^2+rt)$ $W_0W_1^2+2stW_0^2W_1-(3t-rs)\,W_0W_1W_2)$.

Note that the Binet form of a sequence satisfying (6) for non-negative integers is valid for all integers n (see [10], this result of Howard and Saidak [10] is even true in the case of higher-order recurrence relations). We have the following formula: for $n=1,2,3,...$ we have

$$W_{-n} = \frac{1}{t^n}\frac{\beta^n\gamma^n(\beta-\gamma)\,p_1-\alpha^n\gamma^n(\alpha-\gamma)\,p_2+\alpha^n\beta^n(\alpha-\beta)\,p_3}{\alpha^n(\beta-\gamma)\,p_1-\beta^n(\alpha-\gamma)\,p_2+\gamma^n(\alpha-\beta)\,p_3}W_n. \tag{8}$$

(8) can be obtained from the following computations:

$$\begin{aligned} W_{-n} &= \frac{p_1\alpha^{-n}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{p_2\beta^{-n}}{(\beta-\alpha)(\beta-\gamma)} + \frac{p_3\gamma^{-n}}{(\gamma-\alpha)(\gamma-\beta)} = MW_n \\ &\Leftrightarrow \\ M &= \frac{\frac{1}{\alpha^n}\frac{p_1}{(\alpha-\beta)(\alpha-\gamma)} - \frac{1}{\beta^n}\frac{p_2}{(\alpha-\beta)(\beta-\gamma)} + \frac{1}{\gamma^n}\frac{p_3}{(\alpha-\gamma)(\beta-\gamma)}}{\alpha^n\frac{p_1}{(\alpha-\beta)(\alpha-\gamma)} - \beta^n\frac{p_2}{(\alpha-\beta)(\beta-\gamma)} + \gamma^n\frac{p_3}{(\alpha-\gamma)(\beta-\gamma)}} \\ &= \frac{1}{t^n}\frac{\beta^n\gamma^n(\beta-\gamma)\,p_1-\alpha^n\gamma^n(\alpha-\gamma)\,p_2+\alpha^n\beta^n(\alpha-\beta)\,p_3}{\alpha^n(\beta-\gamma)\,p_1-\beta^n(\alpha-\gamma)\,p_2+\gamma^n(\alpha-\beta)\,p_3}. \end{aligned}$$

We can also give Binet's formula of the generalized (r,s,t) numbers (the generalized Tribonacci numbers) for the negative subscripts as follows: for $n=1,2,3,...$ we have

$$W_{-n} = \frac{\alpha^2-r\alpha-s}{t}\frac{\alpha^{1-n}p_1}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^2-r\beta-s}{t}\frac{\beta^{1-n}p_2}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^2-r\gamma-s}{t}\frac{\gamma^{1-n}p_3}{(\gamma-\alpha)(\gamma-\beta)}. \tag{9}$$

(9) can be obtained from the following computations:

$$
\begin{aligned}
W_{-n} &= -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)} \\
&= -\frac{s}{t}\left(\frac{p_1\alpha^{-(n-1)}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{p_2\beta^{-(n-1)}}{(\beta-\alpha)(\beta-\gamma)} + \frac{p_3\gamma^{-(n-1)}}{(\gamma-\alpha)(\gamma-\beta)}\right) \\
&\quad -\frac{r}{t}\left(\frac{p_1\alpha^{-(n-2)}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{p_2\beta^{-(n-2)}}{(\beta-\alpha)(\beta-\gamma)} + \frac{p_3\gamma^{-(n-2)}}{(\gamma-\alpha)(\gamma-\beta)}\right) \\
&\quad +\frac{1}{t}\left(\frac{p_1\alpha^{-(n-3)}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{p_2\beta^{-(n-3)}}{(\beta-\alpha)(\beta-\gamma)} + \frac{p_3\gamma^{-(n-3)}}{(\gamma-\alpha)(\gamma-\beta)}\right) \\
&= \left(\frac{-\frac{s}{t}p_1\alpha^{-(n-1)}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{-\frac{r}{t}p_1\alpha^{-(n-2)}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\frac{1}{t}p_1\alpha^{-(n-3)}}{(\alpha-\beta)(\alpha-\gamma)}\right) \\
&\quad +\left(\frac{-\frac{s}{t}p_2\beta^{-(n-1)}}{(\beta-\alpha)(\beta-\gamma)} + \frac{-\frac{r}{t}p_2\beta^{-(n-2)}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\frac{1}{t}p_2\beta^{-(n-3)}}{(\beta-\alpha)(\beta-\gamma)}\right) \\
&\quad +\left(\frac{-\frac{s}{t}p_3\gamma^{-(n-1)}}{(\gamma-\alpha)(\gamma-\beta)} + \frac{-\frac{r}{t}p_3\gamma^{-(n-2)}}{(\gamma-\alpha)(\gamma-\beta)} + \frac{\frac{1}{t}p_3\gamma^{-(n-3)}}{(\gamma-\alpha)(\gamma-\beta)}\right) \\
&= \left(\frac{-\frac{s}{t}p_1\alpha^{1-n}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{-\frac{r}{t}p_1\alpha\alpha^{1-n}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\frac{1}{t}p_1\alpha^2\alpha^{1-n}}{(\alpha-\beta)(\alpha-\gamma)}\right) \\
&\quad +\left(\frac{-\frac{s}{t}p_2\beta^{1-n}}{(\beta-\alpha)(\beta-\gamma)} + \frac{-\frac{r}{t}p_2\beta\beta^{1-n}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\frac{1}{t}p_2\beta^2\beta^{1-n}}{(\beta-\alpha)(\beta-\gamma)}\right) \\
&\quad +\left(\frac{-\frac{s}{t}p_3\gamma^{1-n}}{(\gamma-\alpha)(\gamma-\beta)} + \frac{-\frac{r}{t}p_3\gamma\gamma^{1-n}}{(\gamma-\alpha)(\gamma-\beta)} + \frac{\frac{1}{t}p_3\gamma^2\gamma^{1-n}}{(\gamma-\alpha)(\gamma-\beta)}\right) \\
&= \frac{\alpha^2-r\alpha-s}{t}\frac{p_1\alpha^{1-n}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^2-r\beta-s}{t}\frac{p_2\beta^{1-n}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^2-r\gamma-s}{t}\frac{p_3\gamma^{1-n}}{(\gamma-\alpha)(\gamma-\beta)}
\end{aligned}
$$

where

$$
\begin{aligned}
-\frac{s}{t} - \frac{r}{t}\alpha + \frac{1}{t}\alpha^2 &= \frac{\alpha^2-r\alpha-s}{t}, \\
-\frac{s}{t} - \frac{r}{t}\beta + \frac{1}{t}\beta^2 &= \frac{\beta^2-r\beta-s}{t} \\
-\frac{s}{t} - \frac{r}{t}\gamma + \frac{1}{t}\gamma^2 &= \frac{\gamma^2-r\gamma-s}{t}.
\end{aligned}
$$

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} W_n x^n$ of the sequence W_n. The following lemma is a special case of a well known formula of generating functions of the generalized m-step Fibonacci numbers which can be found in the literature (see for example [29]). For completeness, we include the proof.

Lemma 1.1 *Suppose that $f_{W_n}(x) = \sum\limits_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized (r,s,t) sequence (the generalized Tribonacci sequence) $\{W_n\}_{n\geq 0}$. Then, $\sum\limits_{n=0}^{\infty} W_n x^n$ is given by*

$$
\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2}{1 - rx - sx^2 - tx^3}. \tag{10}
$$

Proof. Using the definition of generalized Tribonacci numbers, and substracting $rx\sum_{n=0}^{\infty} W_n x^n$, $sx^2\sum_{n=0}^{\infty} W_n x^n$ and $tx^3\sum_{n=0}^{\infty} W_n x^n$ from $\sum_{n=0}^{\infty} W_n x^n$ we obtain

$$
(1 - rx - sx^2 - tx^3)\sum_{n=0}^{\infty} W_n x^n = \sum_{n=0}^{\infty} W_n x^n - rx\sum_{n=0}^{\infty} W_n x^n - sx^2\sum_{n=0}^{\infty} W_n x^n - tx^3\sum_{n=0}^{\infty} W_n x^n
$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} W_n x^n - r\sum_{n=0}^{\infty} W_n x^{n+1} - s\sum_{n=0}^{\infty} W_n x^{n+2} - t\sum_{n=0}^{\infty} W_n x^{n+3} \\
&= \sum_{n=0}^{\infty} W_n x^n - r\sum_{n=1}^{\infty} W_{n-1} x^{n} - s\sum_{n=2}^{\infty} W_{n-2} x^{n} - t\sum_{n=3}^{\infty} W_{n-3} x^{n} \\
&= (W_0 + W_1 x + W_2 x^2) - r(W_0 x + W_1 x^2) - sW_0 x^2 \\
&\quad + \sum_{n=3}^{\infty} (W_n - rW_{n-1} - sW_{n-2} - tW_{n-3}) x^n \\
&= W_0 + W_1 x + W_2 x^2 - rW_0 x - rW_1 x^2 - sW_0 x^2 \\
&= W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2.
\end{aligned}
$$

Rearranging above equation, we obtain

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2}{1 - rx - sx^2 - tx^3}. \quad \square$$

We next find Binet's formula of the generalized (r,s,t) sequence (the generalized Tribonacci sequence) $\{W_n\}$ by the use of generating function for W_n.

Theorem 1.2 *(Binet's formula of the generalized (r,s,t) numbers (the generalized Tribonacci numbers)) For all integers n, we have*

$$W_n = \frac{q_1 \alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{q_2 \beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{q_3 \gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{11}$$

where

$$
\begin{aligned}
q_1 &= W_0 \alpha^2 + (W_1 - rW_0)\alpha + (W_2 - rW_1 - sW_0), \\
q_2 &= W_0 \beta^2 + (W_1 - rW_0)\beta + (W_2 - rW_1 - sW_0), \\
q_3 &= W_0 \gamma^2 + (W_1 - rW_0)\gamma + (W_2 - rW_1 - sW_0).
\end{aligned}
$$

Proof. Let

$$h(x) = 1 - rx - sx^2 - tx^3.$$

Then for some α, β and γ we write

$$h(x) = (1-\alpha x)(1-\beta x)(1-\gamma x)$$

i.e.,

$$1 - rx - sx^2 - tx^3 = (1-\alpha x)(1-\beta x)(1-\gamma x). \tag{12}$$

Hence $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$ are the roots of $h(x)$. This gives α, β, and γ as the roots of

$$h(\frac{1}{x}) = 1 - \frac{r}{x} - \frac{s}{x^2} - \frac{t}{x^3} = 0.$$

This implies $x^3 - rx^2 - sx - t = 0$. Now, by (10) and (12), it follows that

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2}{(1-\alpha x)(1-\beta x)(1-\gamma x)}.$$

Then we write

$$\frac{W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2}{(1-\alpha x)(1-\beta x)(1-\gamma x)} = \frac{B_1}{(1-\alpha x)} + \frac{B_2}{(1-\beta x)} + \frac{B_3}{(1-\gamma x)}. \tag{13}$$

So

$$W_0+(W_1-rW_0)x+(W_2-rW_1-sW_0)x^2=B_1(1-\beta x)(1-\gamma x)+B_2(1-\alpha x)(1-\gamma x)+B_3(1-\alpha x)(1-\beta x).$$

If we consider $x=\frac{1}{\alpha}$, we get $W_0+(W_1-rW_0)\frac{1}{\alpha}+(W_2-rW_1-sW_0)\frac{1}{\alpha^2}=B_1(1-\frac{\beta}{\alpha})(1-\frac{\gamma}{\alpha})$. This gives

$$B_1=\frac{\alpha^2(W_0+(W_1-rW_0)\frac{1}{\alpha}+(W_2-rW_1-sW_0)\frac{1}{\alpha^2})}{(\alpha-\beta)(\alpha-\gamma)}=\frac{W_0\alpha^2+(W_1-rW_0)\alpha+(W_2-rW_1-sW_0)}{(\alpha-\beta)(\alpha-\gamma)}.$$

Similarly, we obtain

$$B_2=\frac{W_0\beta^2+(W_1-rW_0)\beta+(W_2-rW_1-sW_0)}{(\beta-\alpha)(\beta-\gamma)},B_3=\frac{W_0\gamma^2+(W_1-rW_0)\gamma+(W_2-rW_1-sW_0)}{(\gamma-\alpha)(\gamma-\beta)}.$$

Thus (13) can be written as

$$\sum_{n=0}^{\infty}W_nx^n=B_1(1-\alpha x)^{-1}+B_2(1-\beta x)^{-1}+B_3(1-\gamma x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty}W_nx^n=B_1\sum_{n=0}^{\infty}\alpha^nx^n+B_2\sum_{n=0}^{\infty}\beta^nx^n+B_3\sum_{n=0}^{\infty}\gamma^nx^n=\sum_{n=0}^{\infty}(B_1\alpha^n+B_2\beta^n+B_3\gamma^n)x^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$W_n=B_1\alpha^n+B_2\beta^n+B_3\gamma^n$$

where

$$\begin{aligned}B_1&=\frac{W_0\alpha^2+(W_1-rW_0)\alpha+(W_2-rW_1-sW_0)}{(\alpha-\beta)(\alpha-\gamma)},\\B_2&=\frac{W_0\beta^2+(W_1-rW_0)\beta+(W_2-rW_1-sW_0)}{(\beta-\alpha)(\beta-\gamma)},\\B_3&=\frac{W_0\gamma^2+(W_1-rW_0)\gamma+(W_2-rW_1-sW_0)}{(\gamma-\alpha)(\gamma-\beta)},\end{aligned}$$

and then we get (11). □

Note that from (7) and (11) we have

$$\begin{aligned}W_2-(\beta+\gamma)W_1+\beta\gamma W_0&=W_0\alpha^2+(W_1-rW_0)\alpha+(W_2-rW_1-sW_0),\\W_2-(\alpha+\gamma)W_1+\alpha\gamma W_0&=W_0\beta^2+(W_1-rW_0)\beta+(W_2-rW_1-sW_0),\\W_2-(\alpha+\beta)W_1+\alpha\beta W_0&=W_0\gamma^2+(W_1-rW_0)\gamma+(W_2-rW_1-sW_0).\end{aligned}$$

In this paper, we define and investigate, in detail, three special cases of the generalized (r,s,t) sequence $\{W_n\}$ which we call them (r,s,t), Lucas (r,s,t) and modified (r,s,t) sequences. (r,s,t) sequence$\{G_n\}_{n\geq0}$, Lucas (r,s,t) sequence $\{H_n\}_{n\geq0}$ and modified (r,s,t) sequence $\{E_n\}_{n\geq0}$ are defined, respectively, by the third-order recurrence relations

$$\begin{aligned}G_{n+3}&=rG_{n+2}+sG_{n+1}+tG_n,\quad G_0=0,G_1=1,G_2=r,\\H_{n+3}&=rH_{n+2}+sH_{n+1}+tH_n,\quad H_0=3,H_1=r,H_2=2s+r^2,\\E_{n+3}&=rE_{n+2}+sE_{n+1}+tE_n,\quad E_0=1,E_1=r-1,E_2=-r+s+r^2.\end{aligned}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{E_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$G_{-n} = -\frac{s}{t}G_{-(n-1)} - \frac{r}{t}G_{-(n-2)} + \frac{1}{t}G_{-(n-3)}, \tag{14}$$

$$H_{-n} = -\frac{s}{t}H_{-(n-1)} - \frac{r}{t}H_{-(n-2)} + \frac{1}{t}H_{-(n-3)} \tag{15}$$

and

$$E_{-n} = -\frac{s}{t}E_{-(n-1)} - \frac{r}{t}E_{-(n-2)} + \frac{1}{t}E_{-(n-3)} \tag{16}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (14), (15) and (16) hold for all integers n.

Next, we present the first few values of the (r, s, t), Lucas (r, s, t) and modified (r, s, t) numbers with positive and negative subscripts:

Table 1. The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4
G_n	0	1	r	r^2+s	$r^3+2sr+t$
G_{-n}		0	$\frac{1}{t}$	$-\frac{s}{t^2}$	$-\frac{1}{t^4}(rt^2-s^2t)$
H_n	3	r	$2s+r^2$	$r^3+3sr+3t$	$r^4+4r^2s+4tr+2s^2$
H_{-n}		$-\frac{s}{t}$	$\frac{1}{t^2}(s^2-2rt)$	$\frac{1}{t^3}(-s^3+3rst+3t^2)$	$\frac{1}{t^4}(2r^2t^2-4rs^2t+s^4-4st^2)$
E_n	1	$r-1$	$-r+s+r^2$	$r^3-r^2+2sr-s+t$	$r^4-r^3+3r^2s-2rs+2tr+s^2-t$
E_{-n}		0	$-\frac{1}{t}$	$\frac{1}{t^2}(s+t)$	$\frac{1}{t^3}(rt-st-s^2)$

Some special cases of (r, s, t) sequence $\{G_n(0, 1, r; r, s, t)\}_{n\geq 0}$ and Lucas (r, s, t) sequence $\{H_n(3, r, 2s + r^2; r, s, t)\}_{n\geq 0}$ are as follows:

1. $G_n(0, 1, 1; 1, 1, 1) = T_n$, Tribonacci sequence,
2. $H_n(3, 1, 3; 1, 1, 1) = K_n$, Tribonacci-Lucas sequence,
3. $G_n(0, 1, 2; 2, 1, 1) = P_n$, third order Pell sequence,
4. $H_n(3, 2, 6; 2, 1, 1) = Q_n$, third order Pell-Lucas sequence,
5. $G_n(0, 1, 0; 0, 1, 1) = U_n$, adjusted Padovan sequence,
6. $H_n(3, 0, 2; 0, 1, 1) = E_n$, Perrin (Padovan-Lucas) sequence,
7. $G_n(0, 1, 0; 0, 2, 1) = M_n$, adjusted $Pell - Padovansequence$
8. $H_n(3, 0, 4; 0, 2, 1) = B_n$, $thirdorder$Lucas-Pell sequence,
9. $G_n(0, 1, 0; 0, 1, 2) = K_n$, adjusted $Jacobsthal - Padovansequence$,
10. $H_n(3, 0, 2; 0, 1, 2) = L_n$, Jacobsthal-Perrin (-Lucas) sequence,
11. $G_n(0, 1, 1; 1, 0, 1) = N_n$, Narayana sequence,
12. $H_n(3, 1, 1; 1, 0, 1) = U_n$, Narayana-Lucas sequence,
13. $G_n(0, 1, 1; 1, 1, 2) = J_n$, third order Jacobsthal sequence,

14. $H_n(3,1,3;1,1,2) = j_n$, modified third order Jacobsthal-Lucas sequence,
15. $G_n(0,1,2;2,3,5) = G_n$, 3-primes sequence,
16. $H_n(3,2,10;2,3,5) = H_n$, Lucas 3-primes sequence.
17. $G_n(0,1,5;5,3,2) = N_n$, reverse 3-primes sequence,
18. $H_n(3,5,31;5,3,2) = S_n$, reverse Lucas 3-primes sequence.

For all integers n, (r,s,t), Lucas (r,s,t) and modified (r,s,t) numbers (using initial conditions in (7) or (11)) can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \\ H_n &= \alpha^n + \beta^n + \gamma^n, \\ E_n &= \frac{(\alpha-1)\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(\beta-1)\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{(\gamma-1)\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \end{aligned}$$

respectively. Note that for all n we have

$$E_n = G_{n+1} - G_n,$$

and

$$\begin{aligned} G_{-n} &= \frac{(\gamma-\beta)\alpha^{1-n} + (\alpha-\gamma)\beta^{1-n} + (\beta-\alpha)\gamma^{1-n}}{(\gamma-\beta)\alpha^{n+1} + (\alpha-\gamma)\beta^{n+1} + (\beta-\alpha)\gamma^{n+1}} G_n, n \geq 1, \\ E_{-n} &= \frac{\alpha^{-n+1}(\beta-\gamma)(\alpha-1) - \beta^{-n+1}(\alpha-\gamma)(\beta-1) + \gamma^{-n+1}(\alpha-\beta)(\gamma-1)}{\alpha^{n+1}(\beta-\gamma)(\alpha-1) - \beta^{n+1}(\alpha-\gamma)(\beta-1) + \gamma^{n+1}(\alpha-\beta)(\gamma-1)} E_n, n \geq 1. \end{aligned}$$

Lemma 1.1 gives the following results as particular examples (generating functions of (r,s,t), Lucas (r,s,t) and modified (r,s,t) numbers).

Corollary 1.3 *Generating functions of (r,s,t), Lucas (r,s,t) and modified (r,s,t) numbers are*

$$\begin{aligned} \sum_{n=0}^{\infty} G_n x^n &= \frac{x}{1-rx-sx^2-tx^3}, \\ \sum_{n=0}^{\infty} H_n x^n &= \frac{3-2rx-sx^2}{1-rx-sx^2-tx^3}, \\ \sum_{n=0}^{\infty} E_n x^n &= \frac{1-x}{1-rx-sx^2-tx^3}, \end{aligned}$$

respectively.

Proof. In Lemma 1.1, take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = r$, $W_n = H_n$ with $H_0 = 3, H_1 = r, H_2 = 2s + r^2$, and $W_n = E_n$ with $E_0 = 1, E_1 = r-1, E_2 = -r+s+r^2$, respectively. □

2 Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized (r,s,t) sequence $\{W_n\}_{n\geq 0}$.

Theorem 2.1 (Simson Formula of Generalized Grahaml Numbers) *For all integers n, we have*

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = t^n \begin{vmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{vmatrix}. \quad (17)$$

Proof. (17) is given in Soykan [17]. □

The previous theorem gives the following results as particular examples.

Corollary 2.2 *For all integers n, Simson formula of (r,s,t), Lucas (r,s,t) and modified (r,s,t) numbers are given as*

$$\begin{aligned} \begin{vmatrix} G_{n+2} & G_{n+1} & G_n \\ G_{n+1} & G_n & G_{n-1} \\ G_n & G_{n-1} & G_{n-2} \end{vmatrix} &= -t^{n-1}, \\ \begin{vmatrix} H_{n+2} & H_{n+1} & H_n \\ H_{n+1} & H_n & H_{n-1} \\ H_n & H_{n-1} & H_{n-2} \end{vmatrix} &= (-4r^3t + r^2s^2 - 18rst + 4s^3 - 27t^2)t^{n-2}, \\ \begin{vmatrix} E_{n+2} & E_{n+1} & E_n \\ E_{n+1} & E_n & E_{n-1} \\ E_n & E_{n-1} & E_{n-2} \end{vmatrix} &= (1-r-s-t)t^{n-1}, \end{aligned}$$

respectively.

3 Some Identities

In this section, we obtain some identities of (r,s,t), Lucas (r,s,t) and modified (r,s,t) numbers. First, we can give a few basic relations between $\{G_n\}$ and $\{H_n\}$.

Lemma 3.1 *The following equalities are true:*

$$\begin{aligned} t^3H_n &= (-s^3+3t^2+3rst)G_{n+4} + (rs^3-5rt^2+s^2t-3r^2st)G_{n+3} \quad (18) \\ &\quad +(-4st^2+2r^2t^2+s^4-4rs^2t)G_{n+2}, \\ t^2H_n &= (-2rt+s^2)G_{n+3} - (rs^2-2r^2t+st)G_{n+2} + (-s^3+3t^2+3rst)G_{n+1}, \\ tH_n &= -sG_{n+2} + (3t+rs)G_{n+1} + (-2rt+s^2)G_n, \\ H_n &= 3G_{n+1} - 2rG_n - sG_{n-1}, \\ H_n &= rG_n + 2sG_{n-1} + 3tG_{n-2}, \end{aligned}$$

and

$(-4s^3t+4r^3t^2+27t^3+18rst^2-r^2s^2t)G_n=(r^2s-3rt+4s^2)H_{n+4}-(4rs^2+r^3s-r^2t+6st)H_{n+3}+(2r^3t-r^2s^2-4s^3+9t^2+10rst)H_{n+2}$,

$(4r^3t-r^2s^2-4s^3+27t^2+18rst)G_n=-(6s+2r^2)H_{n+3}+(9t+7rs+2r^3)H_{n+2}+(r^2s-3rt+4s^2)H_{n+1}$,

$(4r^3t-r^2s^2-4s^3+27t^2+18rst)G_n=(9t+rs)H_{n+2}-(r^2s+3rt+2s^2)H_{n+1}-(2r^2t+6st)H_n$,

$(4r^3t-r^2s^2-4s^3+27t^2+18rst)G_n=-(-6rt+2s^2)H_{n+1}+(rs^2-2r^2t+3st)H_n+(9t^2+rst)H_{n-1}$,

$(4r^3t-r^2s^2-4s^3+27t^2+18rst)G_n=(-rs^2+4r^2t+3st)H_n+(-2s^3+9t^2+7rst)H_{n-1}-(-6rt^2+2s^2t)H_{n-2}$.

Proof. Note that all the identities hold for all integers n. We prove (18). To show (18), writing

$$H_n=a\times G_{n+4}+b\times G_{n+3}+c\times G_{n+2}$$

and solving the system of equations

$$\begin{aligned}H_0&=a\times G_4+b\times G_3+c\times G_2\\H_1&=a\times G_5+b\times G_4+c\times G_3\\H_2&=a\times G_6+b\times G_5+c\times G_4\end{aligned}$$

we find that $a=\frac{1}{t^3}(-s^3+3t^2+3rst), b=\frac{1}{t^3}(rs^3-5rt^2+s^2t-3r^2st), c=\frac{1}{t^3}(-4st^2+2r^2t^2+s^4-4rs^2t)$. The other equalities can be proved similarly. □

Note that all the identities in the above lemma can be proved by induction as well.

Next, we present a few basic relations between $\{G_n\}$ and $\{E_n\}$.

Lemma 3.2 *The following equalities are true:*

$$\begin{aligned}t^2E_n&=(s+t)G_{n+4}-(t+rs+rt)G_{n+3}-(-rt+st+s^2)G_{n+2},\\tE_n&=-G_{n+3}+rG_{n+2}+(s+t)G_{n+1},\\E_n&=G_{n+1}-G_n,\\E_n&=(r-1)G_n+sG_{n-1}+tG_{n-2},\end{aligned}$$

and

$$\begin{aligned}(r+s+t-1)G_n&=E_{n+2}-(r-1)E_{n+1}-(r+s-1)E_n,\\(r+s+t-1)G_n&=E_{n+1}-(r-1)E_n+tE_{n-1},\\(r+s+t-1)G_n&=E_n+(s+t)E_{n-1}+tE_{n-2}.\end{aligned}$$

Proof. We prove the first identity. The other identities can be proved similarly. Writing

$$E_n=a\times G_{n+4}+b\times G_{n+3}+c\times G_{n+2}$$

and solving the system of equations

$$\begin{aligned}E_0&=a\times G_4+b\times G_3+c\times G_2\\E_1&=a\times G_5+b\times G_4+c\times G_3\\E_2&=a\times G_6+b\times G_5+c\times G_4\end{aligned}$$

we find that $a=\frac{1}{t^2}(s+t), b=-\frac{1}{t^2}(t+rs+rt), c=-\frac{1}{t^2}(-rt+st+s^2)$. □

We give a few basic relations between $\{H_n\}$ and $\{E_n\}$.

Lemma 3.3 *The following equalities are true:*

$t(r+s+t-1)H_n=(-s+3t+rs-2rt+s^2)E_{n+2}-(-3t+rs^2+r^2s-2r^2t-rs+5rt+st)E_{n+1}+(-rs^2+2r^2t-2rt-st+s^2-s^3+3t^2+3rst)E_n,$

$(r+s+t-1)H_n=-(2r+s-3)E_{n+1}+(-2r+2s+3t+rs+2r^2)E_n+(-s+3t+rs-2rt+s^2)E_{n-1},$

$(r+s+t-1)H_n=(r+2s+3t)E_n+(2s+3t-rs-2rt)E_{n-1}-(-3t+2rt+st)E_{n-2},$

and

$t\left(4r^3t-r^2s^2-4s^3+27t^2+18rst\right)E_n=-(r^2s+2r^2t-3rt+6st+4s^2)H_{n+4}+(4rs^2+r^3s-r^2t+2r^3t+6st+9t^2+7rst)H_{n+3}+(-3rt^2-2r^3t+4s^2t+r^2s^2+4s^3-9t^2+r^2st-10rst)H_{n+2},$

$(4r^3t-r^2s^2-4s^3+27t^2+18rst)E_n=((6s+9t+rs+2r^2)H_{n+3}-(9t+r^2s+7rs+3rt+2r^3+2s^2)H_{n+2}-(r^2s+2r^2t-3rt+6st+4s^2)H_{n+1},$

$(4r^3t-r^2s^2-4s^3+27t^2+18rst)E_n=-(9t+rs-6rt+2s^2)H_{n+2}+(rs^2+r^2s-2r^2t+3rt+3st+2s^2)H_{n+1}+(2r^2t+6st+9t^2+rst)H_n,$

$(4r^3t-r^2s^2-4s^3+27t^2+18rst)E_n=(-rs^2+4r^2t-6rt+3st+2s^2)H_{n+1}-(rs^2-2r^2t+3st+2s^3-9t^2-7rst)H_n-(-6rt^2+2s^2t+9t^2+rst)H_{n-1},$

$(4r^3t-r^2s^2-4s^3+27t^2+18rst)E_n=-(-rs^2+4r^2t-4r^3t+r^2s^2+3st+2s^3-9t^2-10rst)H_n+(-rs^3+6rt^2+s^2t+2s^3-9t^2+4r^2st-7rst)H_{n-1}+(-6rt^2+3st^2+2s^2t+4r^2t^2-rs^2t)H_{n-2}.$

Proof. We prove the last identity. The other identities can be proved similarly. Writing

$$E_n=a\times H_n+b\times H_{n-1}+c\times H_{n-2}$$

and solving the system of equations

$$\begin{aligned}E_0&=a\times H_0+b\times H_{-1}+c\times H_{-2}\\E_1&=a\times H_1+b\times H_0+c\times H_{-1}\\E_2&=a\times H_2+b\times H_1+c\times H_0\end{aligned}$$

we find that

$$\begin{aligned}a&=\frac{-(-rs^2+4r^2t-4r^3t+r^2s^2+3st+2s^3-9t^2-10rst)}{4r^3t-r^2s^2-4s^3+27t^2+18rst},\\b&=\frac{(-rs^3+6rt^2+s^2t+2s^3-9t^2+4r^2st-7rst)}{4r^3t-r^2s^2-4s^3+27t^2+18rst},\\c&=\frac{(-6rt^2+3st^2+2s^2t+4r^2t^2-rs^2t)}{4r^3t-r^2s^2-4s^3+27t^2+18rst}.\end{aligned}$$ □

Next, we give a few basic relations between $\{G_n\}$ and $\{W_n\}$.

Lemma 3.4 *The following equalities are true:*

(a) $(W_2^3+(t+rs)W_1^3+t^2W_0^3+(r^2-s)W_1^2W_2-2rW_1W_2^2-sW_0W_2^2+rtW_0^2W_2+(s^2+rt)W_0W_1^2+2stW_0^2W_1+(rs-3t)W_0W_1W_2)G_n=(rW_1^2+tW_0^2-W_1W_2+sW_0W_1)W_{n+2}+(W_2^2-rW_1W_2-sW_0W_2-tW_0W_1)W_{n+1}+(tW_1^2-tW_0W_2)W_n.$

(b) $(W_2^3+(t+rs)W_1^3+t^2W_0^3+(r^2-s)W_1^2W_2-2rW_1W_2^2-sW_0W_2^2+rtW_0^2W_2+(s^2+rt)W_0W_1^2+2stW_0^2W_1+(rs-3t)W_0W_1W_2)G_n=(W_2^2+r^2W_1^2+rtW_0^2-2rW_1W_2-sW_0W_2+(rs-t)W_0W_1)W_{n+1}+((t+rs)W_1^2+stW_0^2-sW_1W_2-tW_0W_2+s^2W_0W_1)W_n+(rtW_1^2+t^2W_0^2-tW_1W_2+stW_0W_1)W_{n-1}.$

(c) $(W_2^3+(t+rs)W_1^3+t^2W_0^3+(r^2-s)W_1^2W_2-2rW_1W_2^2-sW_0W_2^2+rtW_0^2W_2+(s^2+rt)W_0W_1^2+2stW_0^2W_1+(rs-3t)W_0W_1W_2)G_n=(rW_2^2+(r^3+t+rs)W_1^2+t(s+r^2)W_0^2-(s+2r^2)W_1W_2-(t+rs)W_0W_2+(r^2s+s^2-rt)W_0W_1)W_n+(sW_2^2+r(rs+t)W_1^2+t(t+rs)W_0^2-(t+2rs)W_1W_2-s^2W_0W_2+rs^2W_0W_1)W_{n-1}+(tW_2^2+r^2tW_1^2+rt^2W_0^2-2rtW_1W_2-stW_0W_2+t(rs-t)W_0W_1)W_{n-2}.$

(d) $tW_n=(W_2-rW_1-sW_0)G_{n+2}+(-rW_2+r^2W_1+(t+rs)W_0)G_{n+1}+(-sW_2+(t+rs)W_1+(s^2-rt)W_0)G_n.$

(e) $W_n=W_0G_{n+1}+(W_1-rW_0)G_n+(W_2-rW_1-sW_0)G_{n-1}.$

(f) $W_n=W_1G_n+(W_2-rW_1)G_{n-1}+tW_0G_{n-2}.$

Proof. We prove (f). The other identities can be proved similarly. Writing

$$W_n=a\times G_n+b\times G_{n-1}+c\times G_{n-2}$$

and solving the system of equations

$$\begin{aligned}W_0&=a\times G_0+b\times G_{-1}+c\times G_{-2}\\W_1&=a\times G_1+b\times G_0+c\times G_{-1}\\W_2&=a\times G_2+b\times G_1+c\times G_0\end{aligned}$$

we find that $a=W_1,b=W_2-rW_1,c=tW_0$. □

Now, we present a few basic relations between $\{H_n\}$ and $\{W_n\}$.

Lemma 3.5 *The following equalities are true:*

(a) $(W_2^3+(t+rs)W_1^3+t^2W_0^3+(r^2-s)W_1^2W_2-2rW_1W_2^2-sW_0W_2^2+rtW_0^2W_2+(s^2+rt)W_0W_1^2+2stW_0^2W_1+(rs-3t)W_0W_1W_2)H_n=(3W_2^2+(r^2-s)W_1^2+rtW_0^2-4rW_1W_2-2sW_0W_2+(rs-3t)W_0W_1)W_{n+2}+(-2rW_2^2+3tW_1^2-2sW_1W_2-3tW_0W_2+3rsW_1^2+2stW_0^2+2r^2W_1W_2+2s^2W_0W_1+rsW_0W_2+2rtW_0W_1)W_{n+1}+(-sW_2^2+(s^2+rt)W_1^2+3t^2W_0^2+(rs-3t)W_1W_2+2rtW_0W_2+4stW_0W_1)W_n.$

(b) $(W_2^3+(t+rs)W_1^3+t^2W_0^3+(r^2-s)W_1^2W_2-2rW_1W_2^2-sW_0W_2^2+rtW_0^2W_2+(s^2+rt)W_0W_1^2+2stW_0^2W_1+(rs-3t)W_0W_1W_2)H_n=(r^3W_1^2+rW_2^2+3tW_1^2+r^2tW_0^2-2sW_1W_2-3tW_0W_2+2rsW_1^2+2stW_0^2-2r^2W_1W_2+2s^2W_0W_1-rsW_0W_2-rtW_0W_1+r^2sW_0W_1)W_{n+1}+(3t^2W_0^2+2sW_2^2+r^2sW_1^2-3tW_1W_2+rtW_1^2-2s^2W_0W_2-3rsW_1W_2+2rtW_0W_2+stW_0W_1+rstW_0^2+rs^2W_0W_1)W_n+(3tW_2^2+t(r^2-s)W_1^2+rt^2W_0^2-4rtW_1W_2-2stW_0W_2+t(rs-3t)W_0W_1)W_{n-1}.$

(c) $(W_2^3+(t+rs)W_1^3+t^2W_0^3+(r^2-s)W_1^2W_2-2rW_1W_2^2-sW_0W_2^2+rtW_0^2W_2+(s^2+rt)W_0W_1^2+2stW_0^2W_1+(rs-3t)W_0W_1W_2)H_n=((r^2+2s)W_2^2+r(r^3+3rs+4t)W_1^2+t(r^3+3rs+3t)W_0^2-(3t+2r^3+5rs)W_1W_2-(r^2s+rt+2s^2)W_0W_2+(r^3s-r^2t+3rs^2+st)W_0W_1)W_n+((3t+rs)W_2^2+(r^3s+2rs^2+2st+r^2t)W_1^2+t(rt+r^2s+2s^2)W_0^2-2(2rt+s^2+r^2s)W_1W_2-s(rs+5t)W_0W_2+(2s^3+r^2s^2-3t^2)W_0W_1)W_{n-1}+(rtW_2^2+t(r^3+3t+2rs)W_1^2+t^2(2s+r^2)W_0^2-2t(s+r^2)W_1W_2-t(3t+rs)W_0W_2+t(2s^2+r^2s-rt)W_0W_1)W_{n-2}.$

(d) $(4r^3t-r^2s^2-4s^3+27t^2+18rst)W_n=(-2(r^2+3s)W_2+(2r^3+9t+7rs)W_1+(4s^2-3rt+r^2s)W_0)H_{n+2}+((2r^3+9t+7rs)W_2-2(r^4+4r^2s+6tr+s^2)W_1-(4rs^2+6ts-tr^2+r^3s)W_0)H_{n+1}+((-3rt+4s^2+r^2s)W_2-(4rs^2+r^3s-r^2t+6st)W_1+(-r^2s^2+2r^3t+9t^2-4s^3+10rst)W_0)H_n.$

(e) $(4r^3t-r^2s^2-4s^3+27t^2+18rst)W_n=((9t+rs)W_2-(r^2s+3tr+2s^2)W_1-2t(3s+r^2)W_0)H_{n+1}+(-(r^2s+2s^2+3rt)W_2+(r^3s+3rs^2+r^2t+3st)W_1+t(9t+2r^3+7rs)W_0)H_n+(-2t(r^2+3s)W_2+t(2r^3+9t+7rs)W_1+t(4s^2+r^2s-3rt)W_0)H_{n-1}.$

(f) $(4r^3t-r^2s^2-4s^3+27t^2+18rst)W_n=(2(3rt-s^2)W_2+(3st+rs^2-2r^2t)W_1+t(9t+rs)W_0)H_n+((rs^2-2r^2t+3st)W_2+(2r^3t-r^2s^2+4rst-2s^3+9t^2)W_1-t(2s^2+3rt+r^2s)W_0)H_{n-1}+(t(9t+rs)W_2-t(r^2s+3rt+2s^2)W_1-2t^2(3s+r^2)W_0)H_{n-2}.$

Proof. We prove (f). The other identities can be proved similarly. Writing

$$W_n=a\times H_n+b\times H_{n-1}+c\times H_{n-2}$$

and solving the system of equations

$$\begin{aligned}W_0&=a\times H_0+b\times H_{-1}+c\times H_{-2}\\W_1&=a\times H_1+b\times H_0+c\times H_{-1}\\W_2&=a\times H_2+b\times H_1+c\times H_0\end{aligned}$$

we find that

$$\begin{aligned}a&=\frac{2(3rt-s^2)W_2+(3st+rs^2-2r^2t)W_1+t(9t+rs)W_0}{4r^3t-r^2s^2-4s^3+27t^2+18rst},\\b&=\frac{(rs^2-2r^2t+3st)W_2+(2r^3t-r^2s^2+4rst-2s^3+9t^2)W_1-t(2s^2+3rt+r^2s)W_0}{4r^3t-r^2s^2-4s^3+27t^2+18rst},\\c&=\frac{t(9t+rs)W_2-t(r^2s+3rt+2s^2)W_1-2t^2(3s+r^2)W_0}{4r^3t-r^2s^2-4s^3+27t^2+18rst}.\quad\square\end{aligned}$$

Next, we give a few basic relations between $\{E_n\}$ and $\{W_n\}$.

Lemma 3.6 *The following equalities are true:*

(a) $(W_2^3+(t+rs)W_1^3+t^2W_0^3-2rW_1W_2^2+(r^2-s)W_1^2W_2-sW_0W_2^2+rtW_0^2W_2+2stW_0^2W_1+(s^2+rt)W_0W_1^2+(-3t+rs)W_0W_1W_2)E_n=(W_2^2+r(r-1)W_1^2+(rt-t)W_0^2+(1-2r)W_1W_2-sW_0W_2+(rs-s-t)W_0W_1)W_{n+2}+(-W_2^2+(t+rs)W_1^2+stW_0^2+(r-s)W_1W_2+(s-t)W_0W_2+(t+s^2)W_0W_1)W_{n+1}+(t(r-1)W_1^2+t^2W_0^2-tW_1W_2+tW_0W_2+stW_0W_1)W_n.$

(b) $(W_2^3+(t+rs)W_1^3+t^2W_0^3-2rW_1W_2^2+(r^2-s)W_1^2W_2-sW_0W_2^2+rtW_0^2W_2+2stW_0^2W_1+(s^2+rt)W_0W_1^2+(-3t+rs)W_0W_1W_2)E_n=((r-1)W_2^2+(r^3-r^2+rs+t)W_1^2+t(r^2+s-r)W_0^2+(-2r^2+2r-s)W_1W_2+(s-rs-t)W_0W_2+(r^2s-rs-tr+s^2+t)W_0W_1)W_{n+1}+(sW_2^2+(r-1)(t+rs)W_1^2+t(rs+t-s)W_0^2+(s-t-2rs)W_1W_2+(t-s^2)W_0W_2+s^2(r-1)W_0W_1)W_n+t(W_2^2+r(r-1)W_1^2+t(r-1)W_0^2+(1-2r)W_1W_2-sW_0W_2+(rs-s-t)W_0W_1)W_{n-1}.$

(c) $(W_2^3+(t+rs)W_1^3+t^2W_0^3-2rW_1W_2^2+(r^2-s)W_1^2W_2-sW_0W_2^2+rtW_0^2W_2+2stW_0^2W_1+(s^2+rt)W_0W_1^2+(-3t+rs)W_0W_1W_2)E_n=((r^2-r+s)W_2^2+(-t+2r^2s-rs+2rt-r^3+r^4)W_1^2+t(-s+t+2rs-r^2+r^3)W_0^2+(s-t-3rs+2r^2-2r^3)W_1W_2-rs(r-1)W_0W_2+(-s^2+t-rt)W_0W_2+(2rs^2-r^2s+r^3s-r^2t+rt-s^2)W_0W_1)W_n+$

$((rs-s+t)W_2^2+(r^2+s-r)(t+rs)W_1^2+t(r^2s+s^2-rs+rt-t)W_0^2+(-2r^2s-s^2+2rs-2tr+t)W_1W_2+s(s-2t-rs)W_0W_2+(s^3-t^2-rs^2+r^2s^2)W_0W_1)W_{n-1}+(t(r-1)W_2^2+t(r^3-r^2+rs+t)W_1^2+t^2(r^2-r+s)W_0^2+t(2r-2r^2-s)W_1W_2+t(s-t-rs)W_0W_2+t(r^2s+s^2-rs-rt+t)W_0W_1)W_{n-2}$

(d) $t\,(r+s+t-1)\,W_n = \left((1-r-s)W_2+(r^2+rs-r+t)W_1+(s^2+rs-rt-s+t)W_0\right)E_{n+2}+((r^2+rs+t-r)W_2+(r^2-r^3-r^2s-2rt+t)W_1+(r^2t-rs^2-r^2s+rs-2rt-st+t)W_0)E_{n+1}+((s^2-s+t+rs-rt)W_2+(r^2t-rs^2-r^2s+rs-2rt-st+t)W_1+(-s^3+r^2t-rs^2+s^2+t^2+2rst-rt-st)W_0)E_n.$

(e) $(r+s+t-1)W_n=(W_2+(1-r)W_1+(1-r-s)W_0)E_{n+1}+((1-r)W_2+(1-2r+r^2)W_1+(r^2+rs+t-r)W_0)E_n+\left(W_2-(s+r)W_2+(r^2+rs+t-r)W_1+(s^2+rs+t-s-rt)W_0\right)E_{n-1}.$

(f) $(r+s+t-1)W_n=(W_2+(1-r)W_1+tW_0)E_n+((1-r)W_2+(r^2-r+s+t)W_1+t(1-r)W_0)E_{n-1}+(tW_2+t(1-r)W_1+t(1-r-s)W_0)E_{n-2}.$

Proof. We prove (f). The other identities can be proved similarly. Writing

$$W_n=a\times E_n+b\times E_{n-1}+c\times E_{n-2}$$

and solving the system of equations

$$\begin{aligned}W_0&=a\times E_0+b\times E_{-1}+c\times E_{-2}\\W_1&=a\times E_1+b\times E_0+c\times E_{-1}\\W_2&=a\times E_2+b\times E_1+c\times E_0\end{aligned}$$

we find that

$$\begin{aligned}a&=\frac{W_2+(1-r)W_1+tW_0}{r+s+t-1},\\b&=\frac{(1-r)W_2+(r^2-r+s+t)W_1+t(1-r)W_0}{r+s+t-1},\\c&=\frac{tW_2+t(1-r)W_1+t(1-r-s)W_0}{r+s+t-1}.\ \Box\end{aligned}$$

We now present a few special identities for the modified (r,s,t) sequence $\{E_n\}$.

Theorem 3.7 *(Catalan's identity) For all integers n and m, the following identity holds*

$$E_{n+m}E_{n-m}-E_n^2=G_{m+n+1}\left(G_{-m+n+1}-G_{-m+n}\right)+G_{m+n}\left(G_{-m+n}-G_{-m+n+1}\right)-\left(G_{n+1}-G_n\right)^2.$$

Proof. We use the identity

$$E_n=G_{n+1}-G_n.\ \Box$$

Note that for $m=1$ in Catalan's identity, we get the Cassini's identity for the modified (r,s,t) sequnce.

Corollary 3.8 *(Cassini's identity) For all integers numbers n and m, the following identity holds*

$$E_{n+1}E_{n-1}-E_n^2=(G_{n+2}-G_{n+1})(G_n-G_{n-1})-(G_{n+1}-G_n)^2.$$

The d'Ocagne's, Gelin-Cesàro's and Melham's identities can also be obtained by using $E_n=G_{n+1}-G_n$. The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham's identities of modified (r,s,t) sequence $\{E_n\}$.

Theorem 3.9 *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$E_{m+1}E_n - E_mE_{n+1} = G_{m+2}\left(G_{n+1}-G_n\right) + G_{m+1}\left(G_n - G_{n+2}\right) + G_m\left(G_{n+2}-G_{n+1}\right).$$

(b) *(Gelin-Cesàro's identity)*

$$E_{n+2}E_{n+1}E_{n-1}E_{n-2} - E_n^4 = (G_{n+3}-G_{n+2})(G_{n+2}-G_{n+1})(G_n-G_{n-1})(G_{n-1}-G_{n-2}) - (G_{n+1}-G_n)^4.$$

(c) *(Melham's identity)*

$$E_{n+1}E_{n+2}E_{n+6} - E_{n+3}^3 = (G_{n+2}-G_{n+1})(G_{n+3}-G_{n+2})(G_{n+7}-G_{n+6}) - (G_{n+4}-G_{n+3})^3.$$

Proof. Use the identity $E_n = G_{n+1} - G_n$. □

4 Linear Sums

The following theorem presents sum formulas of generalized (r, s, t) numbers.

Theorem 4.1 *For all integers m and j, we have*

$$\sum_{k=0}^{n} W_{mk+j} = \frac{W_{mn+m+j} + W_{mn-m+j}t^m + (1-H_{-m})t^mW_{mn+j} - W_{m+j} - W_{j-m}t^m + (H_m-1)W_j}{H_m + (1-H_{-m})t^m - 1}. \tag{19}$$

Proof. *Note that*

$$\begin{aligned}\sum_{k=0}^{n} W_{mk+j} &= W_{mn+j} + \sum_{k=0}^{n-1} W_{mk+j} = W_{mn+j} + \sum_{k=0}^{n-1}(A_1\alpha^{mk+j} + A_2\beta^{mk+j} + A_3\gamma^{mk+j}) \\ &= W_{mn+j} + A_1\alpha^j\left(\frac{\alpha^{mn}-1}{\alpha^m-1}\right) + A_2\beta^j\left(\frac{\beta^{mn}-1}{\beta^m-1}\right) + A_3\gamma^j\left(\frac{\gamma^{mn}-1}{\gamma^m-1}\right).\end{aligned}$$

Simplifying the last equalities in the last two expression imply (19) as required. □

Note that (19) can be written in the following form:

$$\sum_{k=1}^{n} W_{mk+j} = \frac{W_{mn+m+j} + W_{mn-m+j}t^m + (1-H_{-m})t^mW_{mn+j} - W_{m+j} - W_{j-m}t^m + t^m(H_{-m}-1)W_j}{H_m + (1-H_{-m})t^m - 1}.$$

As special cases of the above theorem, we have the following corollary.

Corollary 4.2 *For all integers m and j, we have*

$$\begin{aligned}\sum_{k=0}^{n} G_{mk+j} &= \frac{G_{mn+m+j} + G_{mn-m+j}t^m + (1-H_{-m})t^mG_{mn+j} - G_{m+j} - G_{j-m}t^m + (H_m-1)G_j}{H_m + (1-H_{-m})t^m - 1}, \\ \sum_{k=0}^{n} H_{mk+j} &= \frac{H_{mn+m+j} + H_{mn-m+j}t^m + (1-H_{-m})t^mH_{mn+j} - H_{m+j} - H_{j-m}t^m + (H_m-1)H_j}{H_m + (1-H_{-m})t^m - 1}, \\ \sum_{k=0}^{n} E_{mk+j} &= \frac{E_{mn+m+j} + E_{mn-m+j}t^m + (1-H_{-m})t^mE_{mn+j} - E_{m+j} - E_{j-m}t^m + (H_m-1)E_j}{H_m + (1-H_{-m})t^m - 1}.\end{aligned}$$

For the sake of page length, we only present sum formulas of the generalized Tribonacci numbers (take $r = 1, s = 1, t = 1$) and the generalized Narayana numbers (take $r = 1, s = 0, t = 1$). Firstly, as special cases of the above theorem, we have the following corollary for the generalized Tribonacci numbers (we consider Tribonacci, Tribonacci-Lucas and Tribonacci-Perrin numbers).

Corollary 4.3 *The following identities hold:*

1. $m = 1$, $j = 0$.
   - **(a)** $\sum_{k=0}^{n} T_k = \frac{1}{2}(T_{n+1} + 2T_n + T_{n-1} - 1)$.
   - **(b)** $\sum_{k=0}^{n} K_k = \frac{1}{2}(K_{n+1} + 2K_n + K_{n-1})$.
   - **(c)** $\sum_{k=0}^{n} M_k = \frac{1}{2}(M_{n+1} + 2M_n + M_{n-1} + 1)$.
2. $m = -1$, $j = 0$.
   - **(a)** $\sum_{k=0}^{n} T_{-k} = \frac{1}{2}(-T_{-n+1} - T_{-n-1} + 1)$.
   - **(b)** $\sum_{k=0}^{n} K_{-k} = \frac{1}{2}(-K_{-n+1} - K_{-n-1} + 6)$.
   - **(c)** $\sum_{k=0}^{n} M_{-k} = \frac{1}{2}(-M_{-n+1} - M_{-n-1} + 5)$.
3. $m = 3$, $j = 2$.
   - **(a)** $\sum_{k=0}^{n} T_{3k+2} = \frac{1}{2}(T_{3n+5} - 4T_{3n+2} + T_{3n-1} - 1)$.
   - **(b)** $\sum_{k=0}^{n} K_{3k+2} = \frac{1}{2}(K_{3n+5} - 4K_{3n+2} + K_{3n-1} - 2)$.
   - **(c)** $\sum_{k=0}^{n} M_{3k+2} = \frac{1}{2}(M_{3n+5} - 4M_{3n+2} + M_{3n-1} - 1)$.
4. $m = -3$, $j = -4$.
   - **(a)** $\sum_{k=0}^{n} T_{-3k-4} = \frac{1}{2}(-T_{-3n-1} + 6T_{-3n-4} - T_{-3n-7} + 1)$.
   - **(b)** $\sum_{k=0}^{n} K_{-3k-4} = \frac{1}{2}(-K_{-3n-1} + 6K_{-3n-4} - K_{-3n-7} + 4)$.
   - **(c)** $\sum_{k=0}^{n} M_{-3k-4} = \frac{1}{2}(-M_{-3n-1} + 6M_{-3n-4} - M_{-3n-7} + 3)$.

Secondly, as special cases of the above theorem, we have the following corollary for the generalized Narayana numbers (we consider Narayana, Narayana-Lucas and Narayana-Perrin numbers).

Corollary 4.4 *The following identities hold:*

1. $m = 1$, $j = 0$.
   - **(a)** $\sum_{k=0}^{n} N_k = N_{n+1} + N_n + N_{n-1} - 1$.
   - **(b)** $\sum_{k=0}^{n} U_k = U_{n+1} + U_n + U_{n-1} - 1$.
   - **(c)** $\sum_{k=0}^{n} H_k = H_{n+1} + H_n + H_{n-1} - 2$.
2. $m = -1$, $j = 0$.
   - **(a)** $\sum_{k=0}^{n} N_{-k} = -N_{-n+1} - N_{-n-1} + 1$.

(b) $\sum_{k=0}^{n} U_{-k} = -U_{-n+1} - U_{-n-1} + 4.$

(c) $\sum_{k=0}^{n} H_{-k} = -H_{-n+1} - H_{-n-1} + 5.$

3. $m = 3,\ j = 2.$

(a) $\sum_{k=0}^{n} N_{3k+2} = N_{3n+5} - 2N_{3n+2} + N_{3n-1}.$

(b) $\sum_{k=0}^{n} U_{3k+2} = U_{3n+5} - 2U_{3n+2} + U_{3n-1} - 3.$

(c) $\sum_{k=0}^{n} H_{3k+2} = H_{3n+5} - 2H_{3n+2} + H_{3n-1} - 3.$

4. $m = -3,\ j = -4.$

(a) $\sum_{k=0}^{n} N_{-3k-4} = -N_{-3n-1} + 3N_{-3n-4} - N_{-3n-7}.$

(b) $\sum_{k=0}^{n} U_{-3k-4} = -U_{-3n-1} + 3U_{-3n-4} - U_{-3n-7} + 3.$

(c) $\sum_{k=0}^{n} H_{-3k-4} = -H_{-3n-1} + 3H_{-3n-4} - H_{-3n-7} + 1.$

5 Matrices related with Generalized (r, s, t) numbers

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{20}$$

For matrix formulation (20), see [11]. In fact, Kalman gave the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ r & s & t \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \end{pmatrix}.$$

We define the square matrix A of order 3 as:

$$A = A_{rst} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = t$. From (2) we have

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} W_{n+1} \\ W_n \\ W_{n-1} \end{pmatrix} \tag{21}$$

and from (20) (or using (21) and induction) we have

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}.$$

If we take $W_n = G_n$ in (21) we have

$$\begin{pmatrix} G_{n+2} \\ G_{n+1} \\ G_n \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} G_{n+1} \\ G_n \\ G_{n-1} \end{pmatrix}. \tag{22}$$

We also define

$$B_n = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} & tG_n \\ G_n & sG_{n-1} + tG_{n-2} & tG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} & tG_{n-2} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} W_{n+1} & sW_n + tW_{n-1} & tW_n \\ W_n & sW_{n-1} + tW_{n-2} & tW_{n-1} \\ W_{n-1} & sW_{n-2} + tW_{n-3} & tW_{n-2} \end{pmatrix}.$$

Theorem 5.1 *For all integer $m, n \geq 0$, we have the following properties:*

(a) $B_n = A^n$.

(b) $C_1 A^n = A^n C_1$.

(c) $C_{n+m} = C_n B_m = B_m C_n$.

Proof.

(a) By expanding the vectors on the both sides of (22) to 3-colums and multiplying the obtained on the right-hand side by A, we get

$$B_n = AB_{n-1}.$$

By induction argument, from the last equation, we obtain

$$B_n = A^{n-1}B_1.$$

But $B_1 = A$. It follows that $B_n = A^n$.

(b) Using (a) and definition of C_1, (b) follows.

(c) We have

$$\begin{aligned} AC_{n-1} &= \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} W_n & sW_{n-1} + tW_{n-2} & tW_{n-1} \\ W_{n-1} & sW_{n-2} + tW_{n-3} & tW_{n-2} \\ W_{n-2} & sW_{n-3} + tW_{n-4} & tW_{n-3} \end{pmatrix} \\ &= \begin{pmatrix} W_{n+1} & sW_n + tW_{n-1} & tW_n \\ W_n & sW_{n-1} + tW_{n-2} & tW_{n-1} \\ W_{n-1} & sW_{n-2} + tW_{n-3} & tW_{n-2} \end{pmatrix} = C_n. \end{aligned}$$

i.e. $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1}C_1$. Now

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^m C_1 = A^{n-1}C_1 A^m = C_n B_m$$

and similarly

$$C_{n+m} = B_m C_n. \square$$

Some properties of matrix A^n can be given as

$$\begin{aligned} A^n &= rA^{n-1} + sA^{n-2} + tA^{n-3}, \\ A^{n+m} &= A^n A^m = A^m A^n, \\ \det(A^n) &= t^n, \end{aligned}$$

for all integers m and n.

Theorem 5.2 *For $m, n \geq 0$, we have*

$$W_{n+m} = W_n G_{m+1} + W_{n-1}(sG_m + tG_{m-1}) + tW_{n-2}G_m \quad (23)$$

$$= W_n G_{m+1} + (sW_{n-1} + tW_{n-2})\, G_m + tW_{n-1}G_{m-1} \quad (24)$$

Proof. From the equation $C_{n+m} = C_n B_m = B_m C_n$ we see that an element of C_{n+m} is the product of row C_n and a column B_m. From the last equation we say that an element of C_{n+m} is the product of a row C_n and column B_m. We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and $C_n B_m$. This completes the proof. □

Remark 5.3 *By induction, it can be proved that for all integers $m, n \leq 0$, (23) holds. So for all integers m, n, (23) is true.*

Corollary 5.4 *For all integers m, n, we have the following properties:*

$$G_{n+m} = G_n G_{m+1} + G_{n-1}(sG_m + tG_{m-1}) + tG_{n-2}G_m, \quad (25)$$

$$H_{n+m} = H_n G_{m+1} + H_{n-1}(sG_m + tG_{m-1}) + tH_{n-2}G_m, \quad (26)$$

$$E_{n+m} = E_n G_{m+1} + E_{n-1}(sG_m + tG_{m-1}) + tE_{n-2}G_m. \quad (27)$$

6 Special Matrix Formulas

In this section, we present some specific matrix relations of third-order numbers (generalized (r, s, t) numbers). Firstly, we present some formulas for the generalized Tribonacci numbers.

Corollary 6.1 *For all integers n, we have the following formulas for the generalized Tribonacci numbers.*

(a) *Tribonacci Numbers.*

$$A_{111}^n = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} T_{n+1} & T_n + T_{n-1} & T_n \\ T_n & T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & T_{n-2} + T_{n-3} & T_{n-2} \end{pmatrix}.$$

(b) *Tribonacci-Lucas Numbers.*

$$A_{111}^n = \frac{1}{22}\begin{pmatrix} 5K_{n+3} - 3K_{n+2} - 4K_{n+1} & K_{n+2} + 6K_{n+1} - 3K_n & 5K_{n+2} - 3K_{n+1} - 4K_n \\ 5K_{n+2} - 3K_{n+1} - 4K_n & K_{n+1} + 6K_n - 3K_{n-1} & 5K_{n+1} - 3K_n - 4K_{n-1} \\ 5K_{n+1} - 3K_n - 4K_{n-1} & K_n + 6K_{n-1} - 3K_{n-2} & 5K_n - 3K_{n-1} - 4K_{n-2} \end{pmatrix}.$$

(c) *Tribonacci-Perrin Numbers.*

$$A_{111}^{n}=\frac{1}{41}\begin{pmatrix} 9M_{n+3}-2M_{n+2}-6M_{n+1} & 3M_{n+2}+13M_{n+1}-2M_{n} & 9M_{n+2}-2M_{n+1}-6M_{n} \\ 9M_{n+2}-2M_{n+1}-6M_{n} & 3M_{n+1}+13M_{n}-2M_{n-1} & 9M_{n+1}-6M_{n-1}-2M_{n} \\ 9M_{n+1}-6M_{n-1}-2M_{n} & 3M_{n}+13M_{n-1}-2M_{n-2} & 9M_{n}-2M_{n-1}-6M_{n-2} \end{pmatrix}.$$

(d) *Modified Tribonacci Numbers.*

$$A_{111}^{n}=\frac{1}{2}\begin{pmatrix} U_{n+3}-U_{n+2} & U_{n+2}-U_{n} & U_{n+2}-U_{n+1} \\ U_{n+2}-U_{n+1} & U_{n+1}-U_{n-1} & U_{n+1}-U_{n} \\ U_{n+1}-U_{n} & U_{n}-U_{n-2} & U_{n}-U_{n-1} \end{pmatrix}.$$

(e) *Modified Tribonacci-Lucas Numbers.*

$$A_{111}^{n}=\frac{1}{22}\begin{pmatrix} 2G_{n+3}+G_{n+2}-6G_{n+1} & -4G_{n+2}+9G_{n+1}+G_{n} & 2G_{n+2}+G_{n+1}-6G_{n} \\ 2G_{n+2}+G_{n+1}-6G_{n} & -4G_{n+1}+9G_{n}+G_{n-1} & 2G_{n+1}+G_{n}-6G_{n-1} \\ 2G_{n+1}+G_{n}-6G_{n-1} & -4G_{n}+9G_{n-1}+G_{n-2} & 2G_{n}+G_{n-1}-6G_{n-2} \end{pmatrix}.$$

(f) *Adjusted Tribonacci-Lucas Numbers.*

$$A_{111}^{n}=\frac{1}{176}\begin{pmatrix} 28H_{n+3}-8H_{n+2}+4H_{n+1} & 32H_{n+2}+16H_{n+1}-8H_{n} & 28H_{n+2}-8H_{n+1}+4H_{n} \\ 28H_{n+2}-8H_{n+1}+4H_{n} & 32H_{n+1}+16H_{n}-8H_{n-1} & 28H_{n+1}-8H_{n}+4H_{n-1} \\ 28H_{n+1}-8H_{n}+4H_{n-1} & 32H_{n}+16H_{n-1}-8H_{n-2} & 28H_{n}-8H_{n-1}+4H_{n-2} \end{pmatrix}.$$

Proof. Take $r=1,s=1,t=1$ in Theorem 5.1 (a). Then in this case, $G_n=T_n$ with $T_0=0,T_1=1,T_2=1$.

(a) In Theorem 5.1 (a), we take $G_n=T_n$ with $T_0=0,T_1=1,T_2=1$.

(b) Take $W_n=K_n$ with $K_0=3,K_1=1,K_2=3$. Note that, from Lemma 3.4 (a), we have

$$22T_n=5K_{n+2}-3K_{n+1}-4K_n.$$

Using the last equation and Theorem 5.1 (a), we get required result.

(c) Take $W_n=M_n$ with $M_0=3,M_1=0,M_2=2$. Note that, from Lemma 3.4 (a), we have

$$41T_n=9M_{n+2}-2M_{n+1}-6M_n.$$

Using the last equation and Theorem 5.1 (a), we get required result.

(d) Take $W_n=U_n$ with $U_0=1,U_1=1,U_2=1$. Note that, from Lemma 3.4 (a), we have

$$2T_n=U_{n+2}-U_{n+1}.$$

Using the last equation and Theorem 5.1 (a), we get required result.

(e) Take $W_n=G_n$ with $G_0=4,G_1=4,G_2=10$. Note that, from Lemma 3.4 (a), we have

$$22T_n=2G_{n+2}+G_{n+1}-6G_n.$$

Using the last equation and Theorem 5.1 (a), we get required result.

(f) Take $W_n=H_n$ with $H_0=4,H_1=2,H_2=0$. Note that, from Lemma 3.4 (a), we have

$$176T_n=28H_{n+2}-8H_{n+1}+4H_n.$$

Using the last equation and Theorem 5.1 (a), we get required result. □

Next, we give some formulas for the generalized third-order Pell numbers.

Corollary 6.2 *For all integers n, we have the following formulas for the generalized third-order Pell numbers.*

(a) *Third-Order Pell Numbers.*

$$A_{211}^n = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} P_{n+1} & P_n + P_{n-1} & P_n \\ P_n & P_{n-1} + P_{n-2} & P_{n-1} \\ P_{n-1} & P_{n-2} + P_{n-3} & P_{n-2} \end{pmatrix}.$$

(b) *Third-Order Pell-Lucas Numbers.*

$$A_{211}^n = \frac{1}{87}\begin{pmatrix} 11Q_{n+3} - 12Q_{n+2} - 14Q_{n+1} & -3Q_{n+2} + 27Q_{n+1} - 12Q_n & 11Q_{n+2} - 12Q_{n+1} - 14Q_n \\ 11Q_{n+2} - 12Q_{n+1} - 14Q_n & -3Q_{n+1} + 27Q_n - 12Q_{n-1} & 11Q_{n+1} - 12Q_n - 14Q_{n-1} \\ 11Q_{n+1} - 12Q_n - 14Q_{n-1} & -3Q_n + 27Q_{n-1} - 12Q_{n-2} & 11Q_n - 12Q_{n-1} - 14Q_{n-2} \end{pmatrix}.$$

(c) *Third-Order Modified Pell Numbers.*

$$A_{211}^n = \frac{1}{3}\begin{pmatrix} E_{n+3} - E_{n+2} + E_{n+1} & 2E_{n+2} - 2E_{n+1} - E_n & E_{n+2} - E_{n+1} + E_n \\ E_{n+2} - E_{n+1} + E_n & 2E_{n+1} - 2E_n - E_{n-1} & E_{n+1} - E_n + E_{n-1} \\ E_{n+1} - E_n + E_{n-1} & 2E_n - 2E_{n-1} - E_{n-2} & E_n - E_{n-1} + E_{n-2} \end{pmatrix}.$$

(d) *Third-Order Pell-Perrin Numbers.*

$$A_{211}^n = \frac{1}{59}\begin{pmatrix} 9R_{n+3} - 2R_{n+2} - 6R_{n+1} & 3R_{n+2} + 19R_{n+1} - 2R_n & 9R_{n+2} - 2R_{n+1} - 6R_n \\ 9R_{n+2} - 2R_{n+1} - 6R_n & 3R_{n+1} + 19R_n - 2R_{n-1} & 9R_{n+1} - 2R_n - 6R_{n-1} \\ 9R_{n+1} - 2R_n - 6R_{n-1} & 3R_n + 19R_{n-1} - 2R_{n-2} & 9R_n - 2R_{n-1} - 6R_{n-2} \end{pmatrix}.$$

Proof. Take $r = 2, s = 1, t = 1$ in Theorem 5.1 (a). Then, in this case, $G_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2$.

(a) In Theorem 5.1 (a), we take $G_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2$.

(b) Take $W_n = Q_n$ with $Q_0 = 3, Q_1 = 2, Q_2 = 6$. Note that, from Lemma 3.4 (a), we have

$$87P_n = 11Q_{n+2} - 12Q_{n+1} - 14Q_n.$$

Using the last equation and Theorem 5.1 (a), we get required result.

(c) Take $W_n = E_n$ with $E_0 = 0, E_1 = 1, E_2 = 1$. Note that, from Lemma 3.4 (a), we have

$$3P_n = E_{n+2} - E_{n+1} + E_n.$$

Using the last equation and Theorem 5.1 (a), we get required result.

(d) Take $W_n = R_n$ with $R_0 = 3, R_1 = 0, R_2 = 2$. Note that, from Lemma 3.4 (a), we have

$$59P_n = 9R_{n+2} - 2R_{n+1} - 6R_n.$$

Using the last equation and Theorem 5.1 (a), we get required result. □

Now, we present some formulas for the generalized Padovan numbers.

Corollary 6.3 *For all integers n, we have the following formulas for the generalized Padovan numbers.*

(a) *Padovan Numbers.*

$$A_{011}^{n}=\begin{pmatrix}0&1&1\\1&0&0\\0&1&0\end{pmatrix}^{n}=\begin{pmatrix}P_{n+3}-P_{n+2}&P_{n+2}-P_{n}&P_{n+2}-P_{n+1}\\P_{n+2}-P_{n+1}&P_{n+1}-P_{n-1}&P_{n+1}-P_{n}\\P_{n+1}-P_{n}&P_{n}-P_{n-2}&P_{n}-P_{n-1}\end{pmatrix}.$$

(b) *Perrin Numbers.*

$$A_{011}^{n}=\frac{1}{23}\begin{pmatrix}9E_{n+3}-2E_{n+2}-6E_{n+1}&3E_{n+2}+7E_{n+1}-2E_{n}&9E_{n+2}-2E_{n+1}-6E_{n}\\9E_{n+2}-2E_{n+1}-6E_{n}&3E_{n+1}+7E_{n}-2E_{n-1}&9E_{n+1}-2E_{n}-6E_{n-1}\\9E_{n+1}-2E_{n}-6E_{n-1}&3E_{n}+7E_{n-1}-2E_{n-2}&9E_{n}-2E_{n-1}-6E_{n-2}\end{pmatrix}.$$

(c) *Padovan-Perrin Numbers.*

$$A_{011}^{n}=\begin{pmatrix}S_{n+2}&S_{n+1}+S_{n}&S_{n+1}\\S_{n+1}&S_{n}+S_{n-1}&S_{n}\\S_{n}&S_{n-1}+S_{n-2}&S_{n-1}\end{pmatrix}.$$

(d) *Modified Padovan Numbers.*

$$A_{011}^{n}=\frac{1}{19}\begin{pmatrix}9A_{n+3}-3A_{n+2}-8A_{n+1}&A_{n+2}+6A_{n+1}-3A_{n}&9A_{n+2}-3A_{n+1}-8A_{n}\\9A_{n+2}-3A_{n+1}-8A_{n}&A_{n+1}+6A_{n}-3A_{n-1}&9A_{n+1}-3A_{n}-8A_{n-1}\\9A_{n+1}-3A_{n}-8A_{n-1}&A_{n}+6A_{n-1}-3A_{n-2}&9A_{n}-3A_{n-1}-8A_{n-2}\end{pmatrix}.$$

(e) *Adjusted Padovan Numbers.*

$$A_{011}^{n}=\begin{pmatrix}U_{n+1}&U_{n}+U_{n-1}&U_{n}\\U_{n}&U_{n-1}+U_{n-2}&U_{n-1}\\U_{n-1}&U_{n-2}+U_{n-3}&U_{n-2}\end{pmatrix}.$$

Proof. Take $r=0,s=1,t=1$ in Theorem 5.1 (a). Then in this case, $G_n=U_n$ with $U_0=0,U_1=1,U_2=0$.

(a) Take $W_n=P_n$ with $P_0=1,P_1=1,P_2=1$. Note that, from Lemma 3.4 (a), we have

$$G_n=P_{n+2}-P_{n+1}.$$

Using the last equation and Theorem 5.1 (a), we get required result.

(b) Take $W_n=E_n$ with $E_0=3,E_1=0,E_2=2$. Note that, from Lemma 3.4 (a), we have

$$23G_n=9E_{n+2}-2E_{n+1}-6E_n.$$

Using the last equation and Theorem 5.1 (a), we get required result.

(c) Take $W_n=S_n$ with $S_0=0,S_1=0,S_2=1$. Note that, from Lemma 3.4 (a), we have

$$G_n=S_{n+1}.$$

Using the last equation and Theorem 5.1 (a), we get required result.

(d) Take $W_n=A_n$ with $A_0=3,A_1=1,A_2=3$. Note that, from Lemma 3.4 (a), we have

$$19G_n=9A_{n+2}-3A_{n+1}-8A_n.$$

Using the last equation and Theorem 5.1 (a), we get required result.

(e) In Theorem 5.1 (a), we take $G_n = U_n$ with $U_0 = 0, U_1 = 1, U_2 = 0$. □

Now, we present some formulas for the generalized Pell-Padovan numbers.

Corollary 6.4 *For all integers n, we have the following formulas for the generalized Pell-Padovan numbers.*

(a) *Pell-Padovan Numbers.*

$$A_{021}^n = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{2}\begin{pmatrix} R_{n+3}-R_{n+2} & 2R_{n+2}-R_{n+1}-R_n & R_{n+2}-R_{n+1} \\ R_{n+2}-R_{n+1} & 2R_{n+1}-R_n-R_{n-1} & R_{n+1}-R_n \\ R_{n+1}-R_n & 2R_n-R_{n-1}-R_{n-2} & R_n-R_{n-1} \end{pmatrix}.$$

(b) *Pell-Perrin Numbers.*

$$A_{021}^n = \frac{1}{11}\begin{pmatrix} 9C_{n+3}-8C_{n+2}-6C_{n+1} & 12C_{n+2}-7C_{n+1}-8C_n & 9C_{n+2}-8C_{n+1}-6C_n \\ 9C_{n+2}-8C_{n+1}-6C_n & 12C_{n+1}-7C_n-8C_{n-1} & 9C_{n+1}-8C_n-6C_{n-1} \\ 9C_{n+1}-8C_n-6C_{n-1} & 12C_n-7C_{n-1}-8C_{n-2} & 9C_n-8C_{n-1}-6C_{n-2} \end{pmatrix}.$$

(c) *Third-Order Fibonacci-Pell Numbers.*

$$A_{021}^n = \begin{pmatrix} G_{n+3}-2G_{n+1} & G_{n+1} & G_{n+2}-2G_n \\ G_{n+2}-2G_n & G_n & G_{n+1}-2G_{n-1} \\ G_{n+1}-2G_{n-1} & G_{n-1} & G_n-2G_{n-2} \end{pmatrix}.$$

(d) *Third-Order Lucas-Pell Numbers.*

$$A_{021}^n = \frac{1}{5}\begin{pmatrix} -9B_{n+3}+8B_{n+2}+12B_{n+1} & -6B_{n+2}+7B_{n+1}+8B_n & -9B_{n+2}+8B_{n+1}+12B_n \\ -9B_{n+2}+8B_{n+1}+12B_n & -6B_{n+1}+7B_n+8B_{n-1} & -9B_{n+1}+8B_n+12B_{n-1} \\ -9B_{n+1}+8B_n+12B_{n-1} & -6B_n+7B_{n-1}+8B_{n-2} & -9B_n+8B_{n-1}+12B_{n-2} \end{pmatrix}.$$

(e) *Adjusted Pell-Padovan Numbers.*

$$A_{021}^n = \begin{pmatrix} M_{n+1} & 2M_n+M_{n-1} & M_n \\ M_n & 2M_{n-1}+M_{n-2} & M_{n-1} \\ M_{n-1} & 2M_{n-2}+M_{n-3} & M_{n-2} \end{pmatrix}.$$

Proof. Take $r = 0, s = 2, t = 1$ in Theorem 5.1 (a). Then in this case, $G_n = M_n$ with $M_0 = 0, M_1 = 1, M_2 = 0$.

(a) Take $W_n = R_n$ with $R_0 = 1, R_1 = 1, R_2 = 1$. Note that, from Lemma 3.4 (a), we have

$$2G_n = (R_{n+2} - R_{n+1}).$$

Using the last equation and Theorem 5.1 (a), we get required result.

(b) Take $W_n = C_n$ with $C_0 = 3, C_1 = 0, C_2 = 2$. Note that, from Lemma 3.4 (a), we have

$$11G_n = (9C_{n+2} - 8C_{n+1} - 6C_n).$$

Using the last equation and Theorem 5.1 (a), we get required result.

(c) Take $W_n = G_n$ with $G_0 = 1, G_1 = 0, G_2 = 2$ (in this case we denote the original (r, s, t) numbers G_n by Y_n to prevent confusion). Note that, from Lemma 3.4 (a), we have

$$Y_n = G_{n+2} - 2G_n.$$

Using the last equation and Theorem 5.1 (a), we get required result.

(d) Take $W_n = B_n$ with $B_0 = 3, B_1 = 0, B_2 = 4$. Note that, from Lemma 3.4 (a), we have

$$5G_n = (-9B_{n+2} + 8B_{n+1} + 12B_n).$$

Using the last equation and Theorem 5.1 (a), we get required result.

(e) In Theorem 5.1 (a), we take $G_n = M_n$ with $M_0 = 0, M_1 = 1, M_2 = 0$. □

Now, we present some formulas for the generalized Jacobsthal-Padovan numbers.

Corollary 6.5 *For all integers n, we have the following formulas for the generalized Jacobsthal-Padovan numbers.*

(a) *Jacobsthal-Padovan Numbers.*

$$A_{012}^n = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{2}\begin{pmatrix} Q_{n+3} - Q_{n+2} & Q_{n+2} + Q_{n+1} - 2Q_n & 2(Q_{n+2} - Q_{n+1}) \\ Q_{n+2} - Q_{n+1} & Q_{n+1} + Q_n - 2Q_{n-1} & 2(Q_{n+1} - Q_n) \\ Q_{n+1} - Q_n & Q_n + Q_{n-1} - 2Q_{n-2} & 2(Q_n - Q_{n-1}) \end{pmatrix}.$$

(b) *Jacobsthal-Perrin Numbers.*

$$A_{012}^n = \frac{1}{52}\begin{pmatrix} 9L_{n+3} - L_{n+2} - 6L_{n+1} & 3L_{n+2} + 17L_{n+1} - 2L_n & 2(9L_{n+2} - L_{n+1} - 6L_n) \\ 9L_{n+2} - L_{n+1} - 6L_n & 3L_{n+1} + 17L_n - 2L_{n-1} & 2(9L_{n+1} - L_n - 6L_{n-1}) \\ 9L_{n+1} - L_n - 6L_{n-1} & 3L_n + 17L_{n-1} - 2L_{n-2} & 2(9L_n - L_{n-1} - 6L_{n-2}) \end{pmatrix}.$$

(c) *Adjusted Jacobsthal-Padovan Numbers.*

$$A_{012}^n = \begin{pmatrix} K_{n+1} & K_n + 2K_{n-1} & 2K_n \\ K_n & K_{n-1} + 2K_{n-2} & 2K_{n-1} \\ K_{n-1} & K_{n-2} + 2K_{n-3} & 2K_{n-2} \end{pmatrix}.$$

(d) *Modified Jacobsthal-Padovan Numbers.*

$$A_{012}^n = \frac{1}{46}\begin{pmatrix} 9M_{n+3} - 3M_{n+2} - 8M_{n+1} & M_{n+2} + 15M_{n+1} - 6M_n & 2(9M_{n+2} - 3M_{n+1} - 8M_n) \\ 9M_{n+2} - 3M_{n+1} - 8M_n & M_{n+1} + 15M_n - 6M_{n-1} & 2(9M_{n+1} - 3M_n - 8M_{n-1}) \\ 9M_{n+1} - 3M_n - 8M_{n-1} & M_n + 15M_{n-1} - 6M_{n-2} & 2(9M_n - 3M_{n-1} - 8M_{n-2}) \end{pmatrix}.$$

Proof. Take $r = 0, s = 1, t = 2$ in Theorem 5.1 (a). Then in this case, $G_n = K_n$ with $K_0 = 0, K_1 = 1, K_2 = 0$.

(a) Take $W_n = Q_n$ with $Q_0 = 1, Q_1 = 1, Q_2 = 1$. Note that, from Lemma 3.4 (a), we have

$$2G_n = Q_{n+2} - Q_{n+1}.$$

Using the last equation and Theorem 5.1 (a), we get required result.

(b) Take $W_n = L_n$ with $L_0 = 3, L_1 = 0, L_2 = 2$. Note that, from Lemma 3.4 (a), we have

$$52G_n = 9L_{n+2} - L_{n+1} - 6L_n.$$

Using the last equation and Theorem 5.1 (a), we get required result.

(c) In Theorem 5.1 (a), we take $G_n = K_n$ with $K_0 = 0, K_1 = 1, K_2 = 0$.

(d) Take $W_n = M_n$ with $M_0 = 3, M_1 = 1, M_2 = 3$. Note that, from Lemma 3.4 (a), we have

$$46G_n = 9M_{n+2} - 3M_{n+1} - 8M_n.$$

Using the last equation and Theorem 5.1 (a), we get required result. □

Next, we give some formulas for the generalized Narayana numbers.

Corollary 6.6 *For all integers n, we have the following formulas for the generalized Narayana numbers.*

(a) *Narayana Numbers.*

$$A_{101}^n = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} N_{n+1} & N_{n-1} & N_n \\ N_n & N_{n-2} & N_{n-1} \\ N_{n-1} & N_{n-3} & N_{n-2} \end{pmatrix}.$$

(b) *Narayana-Lucas Numbers.*

$$A_{101}^n = \frac{1}{31}\begin{pmatrix} 9U_{n+3} - 3U_{n+2} - 2U_{n+1} & 9U_{n+1} - 3U_n - 2U_{n-1} & 9U_{n+2} - 3U_{n+1} - 2U_n \\ 9U_{n+2} - 3U_{n+1} - 2U_n & 9U_n - 3U_{n-1} - 2U_{n-2} & 9U_{n+1} - 3U_n - 2U_{n-1} \\ 9U_{n+1} - 3U_n - 2U_{n-1} & 9U_{n-1} - 3U_{n-2} - 2U_{n-3} & 9U_n - 3U_{n-1} - 2U_{n-2} \end{pmatrix}.$$

(c) *Narayana-Perrin Numbers*

$$A_{101}^n = \frac{1}{53}\begin{pmatrix} 9H_{n+3} + 4H_{n+2} - 6H_{n+1} & 9H_{n+1} + 4H_n - 6H_{n-1} & 9H_{n+2} + 4H_{n+1} - 6H_n \\ 9H_{n+2} + 4H_{n+1} - 6H_n & 9H_n + 4H_{n-1} - 6H_{n-2} & 9H_{n+1} + 4H_n - 6H_{n-1} \\ 9H_{n+1} + 4H_n - 6H_{n-1} & 9H_{n-1} + 4H_{n-2} - 6H_{n-3} & 9H_n + 4H_{n-1} - 6H_{n-2} \end{pmatrix}.$$

Proof. Take $r = 1, s = 0, t = 1$ in Theorem 5.1 (a). Then in this case, $G_n = N_n$ with $N_0 = 0, N_1 = 1, N_2 = 1$.

(a) In Theorem 5.1 (a), we take $G_n = N_n$ with $N_0 = 0, N_1 = 1, N_2 = 1$.

(b) Take $W_n = U_n$ with $U_0 = 3, U_1 = 1, U_2 = 1$. Note that, from Lemma 3.4 (a), we have

$$31N_n = 9U_{n+2} - 3U_{n+1} - 2U_n.$$

Using the last equation and Theorem 5.1 (a), we get required result.

(c) Take $W_n = H_n$ with $H_0 = 3, H_1 = 0, H_2 = 2$. Note that, from Lemma 3.4 (a), we have

$$53N_n = 9H_{n+2} + 4H_{n+1} - 6H_n.$$

Using the last equation and Theorem 5.1 (a), we get required result. □

Next, we give some formulas for the generalized third-order Jacobsthal numbers.

Corollary 6.7 *For all integers n, we have the following formulas for the generalized third-order Jacobsthal numbers.*

(a) *Third-Order Jacobsthal Numbers.*

$$A_{112}^n = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} J_{n+1} & J_n + 2J_{n-1} & 2J_n \\ J_n & J_{n-1} + 2J_{n-2} & 2J_{n-1} \\ J_{n-1} & J_{n-2} + 2J_{n-3} & 2J_{n-2} \end{pmatrix}.$$

(b) *Third-Order Jacobsthal-Lucas Numbers.*

$$A_{112}^{n}=\frac{1}{12}\begin{pmatrix} j_{n+3}+j_{n+2}-3j_{n+1} & -2j_{n+2}+6j_{n+1}+2j_{n} & 2(j_{n+2}+j_{n+1}-3j_{n}) \\ j_{n+2}+j_{n+1}-3j_{n} & -2j_{n+1}+6j_{n}+2j_{n-1} & 2(j_{n+1}+j_{n}-3j_{n-1}) \\ j_{n+1}+j_{n}-3j_{n-1} & -2j_{n}+6j_{n-1}+2j_{n-2} & 2(j_{n}+j_{n-1}-3j_{n-2}) \end{pmatrix}.$$

(c) *Modified Third-Order Jacobsthal-Lucas Numbers.*

$$A_{112}^{n}=\frac{1}{147}\begin{pmatrix} 19K_{n+3}-9K_{n+2}-16K_{n+1} & 3K_{n+2}+45K_{n+1}-18K_{n} & 2(19K_{n+2}-9K_{n+1}-16K_{n}) \\ 19K_{n+2}-9K_{n+1}-16K_{n} & 3K_{n+1}+45K_{n}-18K_{n-1} & 2(19K_{n+1}-9K_{n}-16K_{n-1}) \\ (19K_{n+1}-9K_{n}-16K_{n-1}) & 3K_{n}+45K_{n-1}-18K_{n-2} & 2(19K_{n}-9K_{n-1}-16K_{n-2}) \end{pmatrix}.$$

(d) *Third-Order Jacobsthal-Perrin Numbers.*

$$A_{112}^{n}=\frac{1}{70}\begin{pmatrix} 9Q_{n+3}-Q_{n+2}-6Q_{n+1} & 3Q_{n+2}+23Q_{n+1}-2Q_{n} & 2(9Q_{n+2}-Q_{n+1}-6Q_{n}) \\ 9Q_{n+2}-Q_{n+1}-6Q_{n} & 3Q_{n+1}+23Q_{n}-2Q_{n-1} & 2(9Q_{n+1}-Q_{n}-6Q_{n-1}) \\ 9Q_{n+1}-Q_{n}-6Q_{n-1} & 3Q_{n}+23Q_{n-1}-2Q_{n-2} & 2(9Q_{n}-Q_{n-1}-6Q_{n-2}) \end{pmatrix}.$$

Proof. Take $r=1, s=1, t=2$ in Theorem 5.1 (a). Then in this case, $G_n=J_n$ with $J_0=0, J_1=1, J_2=1$.

(a) In Theorem 5.1 (a), we take $G_n=J_n$ with $J_0=0, J_1=1, J_2=1$.

(b) Take $W_n=j_n$ with $j_0=2, j_1=1, j_2=5$. Note that, from Lemma 3.4 (a), we have

$$12J_n=j_{n+2}+j_{n+1}-3j_n.$$

Using the last equation and Theorem 5.1 (a), we get required result.

(c) Take $W_n=K_n$ with $K_0=3, K_1=1, K_2=3$. Note that, from Lemma 3.4 (a), we have

$$147J_n=19K_{n+2}-9K_{n+1}-16K_n.$$

Using the last equation and Theorem 5.1 (a), we get required result.

(d) Take $W_n=Q_n$ with $Q_0=3, Q_1=0, Q_2=2$. Note that, from Lemma 3.4 (a), we have

$$70J_n=9Q_{n+2}-Q_{n+1}-6Q_n.$$

Using the last equation and Theorem 5.1 (a), we get required result. □

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