A UNIFYING FRAMEWORK FOR STUDYING CONTINUITY, INCREASINGNESS, AND GALOIS CONNECTIONS

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Abstract. By using relators (families of relations) and some natural operations for relators, we offer some general definitions for continuity, increasingness, and Galois connections.

1. Introduction

In this paper, by continuing our unifying investigations on continuity properties of relations in relator spaces [18, 19, 24, 16, 25, 31, 32], some general definitions for *continuity, increasingness*, and *Galois connections* are motivated, clarified, and offered for detailed investigations.

A family $\mathcal{R}$ of relations on one set X to another Y is called a *relator* on X to Y. Moreover, the ordered pair $(X, Y)(\mathcal{R}) = \big((X, Y), \mathcal{R}\big)$ is called a *relator space*. Thus, relator spaces are common generalizations of *ordered sets* [3], *formal contexts* [7], and *uniform spaces* [6].

For any two relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z, we may naturally define

$$\mathcal{R}^{-1} = \big\{ R^{-1} : \quad R \in \mathcal{R} \big\} \qquad \text{and} \qquad \mathcal{S} \circ \mathcal{R} = \big\{ S \circ R : \quad R \in \mathcal{R}, \ S \in \mathcal{S} \big\}.$$

A function $\square$ of the class of all relator spaces to the class of all relators is called a *direct unary operation for relators* if, for any relator $\mathcal{R}$ on X to Y, the value $\mathcal{R}^{\square} = \mathcal{R}^{\square_{XY}} = \square\big((X, Y)(\mathcal{R})\big)$ is again a relator on X to Y.

Unfortunately, in the present generality, the *inversion* -1 is already not a direct unary operation for relators. However, for instance, the *uniform, proximal, topological and paratopological refinements* $*$, $\#$, $\wedge$, and $\triangle$, defined by

$$\mathcal{R}^{*} = \big\{ S \subseteq X \times Y : \quad \exists \ R \in \mathcal{R} : \quad R \subseteq S \big\},$$
$$\mathcal{R}^{\#} = \big\{ S \subseteq X \times Y : \quad \forall \ A \subseteq X : \quad \exists \ R \in \mathcal{R} : \quad R[A] \subseteq S[A] \big\},$$
$$\mathcal{R}^{\wedge} = \big\{ S \subseteq X \times Y : \quad \forall \ x \in X : \quad \exists \ R \in \mathcal{R} : \quad R(x) \subseteq S(x) \big\},$$

and

$$\mathcal{R}^{\triangle} = \big\{ S \subseteq X \times Y : \quad \forall \ x \in X : \quad \exists \ u \in X : \quad \exists \ R \in \mathcal{R} : \quad R(u) \subseteq S(x) \big\}$$

for any relator $\mathcal{R}$ on X to Y, are important direct unary operations for relators. It can be easily seen that they are actually *algebraic closure operations* [1, p. 111].

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Now, under the assumptions that

(a) $\Box$ is a direct unary operation for relators,

(b) $(X, Y)(\mathcal{R})$ and $(Z, W)(\mathcal{S})$ are relator spaces,

(c) $\mathcal{F}$ is a relator on X to Z and $\mathcal{G}$ is a relator on Y to W,

we say that the ordered pair

(1) $(\mathcal{F}, \mathcal{G})$ is *upper* $\Box$*–continuous*, with respect to the relators $\mathcal{R}$ and $\mathcal{S}$, if

$$\left(\mathcal{S}^{\Box} \circ \mathcal{F}^{\Box}\right)^{\Box} \subseteq \left(\mathcal{G}^{\Box} \circ \mathcal{R}^{\Box}\right)^{\Box},$$

(2) $(\mathcal{F}, \mathcal{G})$ is *mildly* $\Box$*–continuous*, with respect to the relators $\mathcal{R}$ and $\mathcal{S}$, if

$$\Big(\left(\mathcal{G}^{\Box}\right)^{-1} \circ \mathcal{S}^{\Box} \circ \mathcal{F}^{\Box}\Big)^{\Box} \subseteq \mathcal{R}^{\Box\Box},$$

(3) $(\mathcal{F}, \mathcal{G})$ is *vaguely* $\Box$*–continuous*, with respect to the relators $\mathcal{R}$ and $\mathcal{S}$, if

$$\mathcal{S}^{\Box\Box} \subseteq \Big(\mathcal{G}^{\Box} \circ \mathcal{R}^{\Box} \circ \left(\mathcal{F}^{\Box}\right)^{-1}\Big)^{\Box},$$

(4) $(\mathcal{F}, \mathcal{G})$ is *lower* $\Box$*–continuous*, with respect to the relators $\mathcal{R}$ and $\mathcal{S}$, if

$$\Big(\left(\mathcal{G}^{\Box}\right)^{-1} \circ \mathcal{S}^{\Box}\Big)^{\Box} \subseteq \Big(\mathcal{R}^{\Box} \circ \left(\mathcal{F}^{\Box}\right)^{-1}\Big)^{\Box}.$$

To keep in mind these definitions, for any $R \in \mathcal{R}$, $S \in \mathcal{S}$, $F \in \mathcal{F}$ and $G \in \mathcal{G}$, one can consider the diagram:

$$\begin{array}{ccc} X & \xrightarrow{\ F\ } & Z \\ {\scriptstyle R}\big\downarrow & & \big\downarrow{\scriptstyle S} \\ Y & \xrightarrow{\ G\ } & W \end{array}$$

It will turn out that, in the above definitions, instead of " $\Box$–continuous" we may naturally write "$\Box$–increasing". Moreover, if in particular the operation $\Box$ commutes with inversion, then we may naturally say that the relators $\mathcal{F}$ and $\mathcal{G}^{-1}$ form an *upper (lower)* $\Box$*–Galois connection* if the pair $(\mathcal{F}, \mathcal{G})$ is upper (lower) $\Box$–continuous. Thus, we can obtain some reasonable generalizations not only of the usual continuities, but also those of the usual increasingness and Galois connections.

2. Two motivating examples and a preliminary unifying definition for increasingness and continuity

The following examples were first presented in the talk [38] held by the author to motivate a simple common definition for increasingness and continuity.

Example 2.1. Suppose that $X = X(\leq_X)$ and $Y = Y(\leq_Y)$ are *generalized ordered sets* in the sense that $\leq_X$ and $\leq_Y$ are arbitrary relations on the sets X and Y, respectively.

Then, a function f of X to Y may be naturally called *increasing*, with respect to the inequalities $\leq_X$ and $\leq_Y$, if for every $u, v \in X$

$$u \leq_X v \quad\Longrightarrow\quad f(u) \leq_Y f(v).$$

Now, by using the more convenient notations $R = \leq_X$ and $S = \leq_Y$, the above implication can be reformulated in the form that

$$u\,R\,v \quad\Longrightarrow\quad f(u)\,S\,f(v)\,,$$

or equivalently

$$(u,\,v) \in R \quad\Longrightarrow\quad \big(f(u),\ f(v)\big) \in S\,.$$

Example 2.2. Suppose that $X = X(d_X)$ and $Y = Y(d_Y)$ are *generalized metric spaces* in the sense that d_X and d_Y are arbitrary functions of X^2 and Y^2 to $[0,\ +\infty]$, respectively.

Then, a function f of X to Y may be naturally called *uniformly continuous*, with respect to the distance functions d_X and d_Y, if for each $s > 0$ there exists $r > 0$ such that for every $u,\,v \in X$

$$d_X(u,\,v) < r \quad\Longrightarrow\quad d_Y\big(f(u)\ f(v)\big) < s\,.$$

Now, by using the *surroundings*

$$R = B_r^{d_X} = \big\{ x \in X^2 : \quad d_X(x_1,\,x_2) < r \big\}$$

and

$$S = B_s^{d_Y} = \big\{ y \in Y^2 : \quad d_Y(y_1,\,y_2) < s \big\}\,,$$

the above implication can be reformulated in the form that

$$(u,\,v) \in R \quad\Longrightarrow\quad \big(f(u),\ f(v)\big) \in S\,.$$

The above two examples clearly reveal that the seemingly quite different algebraic and topological notions such as "increasingness" and "uniform continuity" are actually equivalent.

Moreover, they naturally lead us to the following simple unifying definition.

Definition 2.3. Suppose that $X = X(R)$ and $Y = Y(S)$ are *relational spaces* in the sense that R are S arbitrary relations on X and Y, respectively.

Then, a function f of X to Y will be called *increasing* or *continuous*, with respect to the relations R and S, if for any $u,\,v \in X$

$$(u,\,v) \in R \quad\Longrightarrow\quad \big(f(u),\ f(v)\big) \in S\,.$$

Remark 2.4. Having in mind Example 2.2, the above property can be expressed by saying that if u and v are R*–near*, then $f(u)$ and $f(v)$ are S*–near*.

Moreover, since the above implication can also be written in the form

$$v \in R(u) \quad\Longrightarrow\quad f(v) \in S\big(f(u)\big)\,,$$

we may also say that if v is in the R*–neighbourhood* of u, then $f(v)$ is in the S*–neighbourhood* of $f(u)$.

3. Some basic facts on the box product of relations

To briefly reformulate Definition 2.3, in addition to the composition of relations, we shall also need the pointwise Cartesian product of relations.

Definition 3.1. If F is a relation on X to Z and G is a relation on Y to W, then for any $x \in X$ and $y \in Y$, we define

$$(F \boxtimes G)(x,\,y) = F(x) \times G(y)\,.$$

Remark 3.2. Thus, $F \boxtimes G$ is a relation on $X \times Y$ to $Z \times W$, which has been called the *box product* of F and G in [33].

By a letter of B. M. Schein this product was already considered by some authors much before a thesis of J. Riquet in 1951 who named it tensor product.

The importance of the box product is already apparent from the following theorem.

Theorem 3.3. *If F is a relation on X to Z and G is a relation on Y to W, then for any $R \subseteq X \times Y$ we have*

$$(F \boxtimes G)[R] = G \circ R \circ F^{-1}.$$

Proof. If $(z, w) \in (F \boxtimes G)[R]$, then there exists $(x, y) \in R$ such that

$$(z, w) \in (F \boxtimes G)(x, y) = F(x) \times G(y),$$

and thus $z \in F(x)$ and $w \in G(y)$. Hence, by noticing that $x \in F^{-1}(z)$, we can already see that

$$y \in R(x) \subset R\big[F^{-1}(z)\big] = \big(R \circ F^{-1}\big)(y),$$

and thus

$$w \in G(y) \subseteq G\big[\big(R \circ F^{-1}\big)(z)\big] = \big(G \circ \big(R \circ F^{-1}\big)\big)(z).$$

Therefore, $(z, w) \in G \circ \big(R \circ F^{-1}\big) = G \circ R \circ F^{-1}$ also holds.

Thus, we have proved that $(F \boxtimes G)[R] \subseteq G \circ R \circ F^{-1}$. The converse inclusion can be proved quite similarly.

From Theorem 3.3, by taking $R = \{(x, y)\}$, we can immediately derive the following corollary.

Corollary 3.4. *If F is a relation on X to Z and G is a relation on Y to W, then for any $x \in X$ and $y \in Y$, we have*

$$(F \boxtimes G)(x, y) = G \circ \{(x, y)\} \circ F^{-1}.$$

Moreover, by using Theorem 3.3, we can also easily prove the following corollary.

Corollary 3.5. *For any relations F on X to Y and G on Y to Z, we have*

$$G \circ F = \big(F^{-1} \boxtimes G\big)[\Delta_Y],$$

where Δ_Y is the identity function of Y.

Proof. By the corresponding definitions and Theorem 3.3, it is clear that

$$G \circ F = G \circ \Delta_Y \circ \big(F^{-1}\big)^{-1} = \big(F^{-1} \boxtimes G\big)[\Delta_Y].$$

Remark 3.6. The above corollaries show that the box and composition products of relations are actually equivalent tools.

However, in contrast to the composition product, the box product of relations can be immediately defined for an arbitrary family of relations.

Moreover, concerning the box product, we can prove a simpler inversion formula.

Theorem 3.7. *For any relations F on X to Z and G on Y to W, we have*

$$(F \boxtimes G)^{-1} = F^{-1} \boxtimes G^{-1}.$$

Proof. For any $(x, y) \in X \times Y$ and $(z, w) \in Z \times W$, we have

$$(x, y) \in (F \boxtimes G)^{-1}(z, w) \iff (z, w) \in (F \boxtimes G)(x, y) \iff$$
$$(z, w) \in F(x) \times G(y) \iff z \in F(x), \quad w \in G(y) \iff x \in F^{-1}(z), \quad y \in G^{-1}(w)$$
$$\iff (x, y) \in F^{-1}(z) \times G^{-1}(w) \iff (x, y) \in \big(F^{-1} \boxtimes G^{-1}\big)(z, w).$$

Therefore, $(F \boxtimes G)^{-1}(z, w) = \big(F^{-1} \boxtimes G^{-1}\big)(z, w)$ for all $(z, w) \in Z \times W$, and thus the required equality is also true.

Now, by using Theorems 3.3 and 3.7, we can also easily prove the following theorem.

Theorem 3.8. *If F is a relation on X to Z and G is a relation on Y to W, then for any $S \subseteq Z \times W$ we have*

$$(F \boxtimes G)^{-1}[S] = G^{-1} \circ S \circ F.$$

Proof. By Theorems 3.7 and 3.3, it is clear that

$$(F \boxtimes G)^{-1}[S] = \big(F^{-1} \boxtimes G^{-1}\big)[S] = G^{-1} \circ S \circ \big(F^{-1}\big)^{-1} = G^{-1} \circ S \circ F.$$

In the sequel, we shall also need the following definition.

Definition 3.9. If R is a relation on X to Y, then for any $x \in X$ and $B \subseteq Y$ we write

(1) $x \in \operatorname{lb}_R(B)$ if $B \subseteq R(x)$, (2) $x \in \operatorname{int}_R(B)$ if $R(x) \subseteq B$.

Remark 3.10. Thus, lb_R and int_R are relations on $\mathcal{P}(Y)$ to X, which are called the *lower bound and topological interior relations* induced by R.

These relations are not independent of each other. Namely, for any $x \in X$ and $B \subseteq Y$, we have

$$x \in \operatorname{lb}_R(B) \iff B \subseteq R(x) \iff R(x)^c \subseteq B^c$$
$$\iff R^c(x) \subseteq B^c \iff x \in \operatorname{int}_{R^c}(B^c).$$

Hence, by using the notation $\mathcal{C}_Y(B) = B^c$, we can see that $\operatorname{lb}_R = \operatorname{int}_{R^c} \circ \mathcal{C}_Y$, and thus also $\operatorname{int}_R = \operatorname{lb}_{R^c} \circ \mathcal{C}_Y$.

Now, by using the *closure formula*

$$\operatorname{cl}_R(B) = \operatorname{int}_R(B^c)^c = \big\{ x \in X : \quad R(x) \cap B \neq \emptyset \big\} = R^{-1}[B],$$

from Theorems 3.8 we can immediately derive the following corollary.

Corollary 3.11. *If F is a relation on X to Z and G is a relation on Y to W, then for any $S \subseteq Z \times W$ we have*

$$\operatorname{cl}_{F \boxtimes G}(S) = G^{-1} \circ S \circ F.$$

Remark 3.12. This corollary, together with [37, Theorem 1.5], will give us an important Galois connection [3, 30] between the power sets $\mathcal{P}(Z \times W)$ and $\mathcal{P}(X \times Y)$ which can be used to put into a proper perspective the results of [33, Section 9].

4. Some preliminary characterizations of increasingness and continuity

Now, by using a particular case of Definition 3.1, we can easily prove the following theorem.

Theorem 4.1. *For any function f of one relational space $X(R)$ to another $Y(S)$, the following assertions are equivalent:*

(1) f *is increasing (continuous),*

(2) $(f \boxtimes f)[R] \subseteq S$, *(3)* $R \subseteq (f \boxtimes f)^{-1}[S]$.

Proof. By using the function $f \boxtimes f$, the implication

$$(u, v) \in R \quad \Longrightarrow \quad \big(f(u),\ f(v)\big) \in S$$

can be written in the form that

$$(u, v) \in R \quad \Longrightarrow \quad (f \boxtimes f)(u, v) \in S.$$

However, this means, in a concise form, only that

$$(f \boxtimes f)[R] \subseteq S, \quad \text{or equivalently} \quad R \subseteq (f \boxtimes f)^{-1}[S].$$

Remark 4.2. To check the latter equivalence, note that if f is a function of X to Y, then for any $A \subseteq X$ and $B \subseteq Y$, we have

$$f[A] \subseteq B \quad \Longleftrightarrow \quad A \subseteq f^{-1}[B].$$

Therefore, the *corelations* (set-to-set functions) generated by the function f and the relation f^{-1} [34] form a *Galois connection* between the power sets $\mathcal{P}(X)$ and $\mathcal{P}(Y)$.

From Theorem 4.1, by using the corresponding particular cases of the closure formula and Definition 3.9, we can immediately derive the following corollary.

Corollary 4.3. *For any function f of one relational space $X(R)$ to another $Y(S)$, the following assertions are equivalent:*

(1) f *is increasing (continuous),*

(2) $R \subseteq \operatorname{cl}_{f \boxtimes f}(S)$, *(3)* $R \subseteq \operatorname{int}_{f \boxtimes f}(S)$.

Remark 4.4. Note that if f is a function of X to Y, then by Remark 4.2 for any $A \subseteq Y$ and $B \subseteq Y$ we have $A \subseteq \operatorname{cl}_f(B)$ if and only if $A \subseteq \operatorname{int}_f(B)$.

Hence, by taking $x \in X$ and $A = \{x\}$, we can see that $\operatorname{cl}_f(B) = \operatorname{int}_f(B)$ for all $B \subseteq Y$, and thus $\operatorname{cl}_f = \operatorname{int}_f$ also holds.

However, it is now more important to note that, by using the inverses and compositions of relations, we can also easily prove the following theorem.

Theorem 4.5. *For any function f of one relational space $X(R)$ to another $Y(S)$, the following assertions are equivalent:*

(1) f *is increasing (continuous),*

(2) $f \circ R \subseteq S \circ f$, *(3)* $R \subseteq f^{-1} \circ S \circ f$,

(4) $f \circ R \circ f^{-1} \subseteq S$, *(5)* $R \circ f^{-1} \subseteq f^{-1} \circ S$.

Proof. By the corresponding definitions on images and composition for relations, it is clear that the following assertions are equivalent:

(a) $f \circ R \subseteq S \circ f$,

(b) $\forall\ u \in X: \quad (f \circ R)(u) \subseteq (S \circ f)(u)$,

(c) $\forall\ u \in X: \quad f\,[\,R(u)\,] \subseteq S\big(f(u)\big)$,

(d) $\forall\ u \in X: \quad \forall\ v \in R(u): \ \ f(v) \in S\big(f(u)\big)$,

(e) $\forall\ u,\, v \in X: \quad \big((u, v) \in R \implies \big(f(u),\, f(v)\big) \in S\big)$.

Therefore, assertions (2) and (1) are equivalent.

Moreover, from Theorem 4.1, by using Theorems 3.3 and 3.8, we can immediately see that assertions (4) and (3) are also equivalent to assertion (1). Therefore, to complete the proof, it is enough to show only that assertions (4) and (5) are also equivalent.

For this note that if (5) holds, then the increasingness and the associativity of composition give us

$$f \circ R \circ f^{-1} \subseteq f \circ f^{-1} \circ S\,.$$

Moreover, by using that f is a function, we can easily see that $f \circ f^{-1} \subseteq \Delta_Y$. Hence, by the corresponding properties of composition, it is clear that

$$f \circ f^{-1} \circ S \subseteq \Delta_Y \circ S = S\,.$$

Therefore, (4) also holds.

While, if (4) holds, then we can quite similarly infer that

$$f^{-1} \circ f \circ R \circ f^{-1} \subseteq f^{-1} \circ S\,.$$

Moreover, by using that X is the domain of f, we can easily see that $\Delta_X \subseteq f^{-1} \circ f$. Hence, by by the corresponding properties of composition, it is clear that

$$R \circ f^{-1} = \Delta_X \circ R \circ f^{-1} \subseteq f^{-1} \circ f \circ R \circ f^{-1}\,.$$

Therefore, (5) also holds.

From this theorem, by using Corollary 3.5 and Theorem 3.7, we can immediately derive the following corollary.

Corollary 4.6. *For any function f of one relational space $X(R)$ to another $Y(S)$, the following assertions are equivalent:*

(1) *f is increasing (continuous)*,

(2) $\big(f \boxtimes R\big)\,[\,\Delta_X\,] \subseteq (\,S \boxtimes f\,)^{-1}\,[\,\Delta_Y\,]$,

(3) $\big(R^{-1} \boxtimes f\,\big)\,[\,\Delta_X\,] \subseteq \big(f^{-1} \boxtimes S\big)\,[\,\Delta_Y\,]$.

5. Two natural increasingness properties of relations

The proofs given in Section 4 indicate that some of the assertions in Theorems 4.1 and 4.5 and their corollaries do not need be equivalent for an arbitrary relation f on $X(R)$ to $Y(S)$.

Therefore, they can be naturally used to define different increasingness and continuity properties of relations with respect to other relations.

For instance, we can easily prove the following theorem which shows that the $S = \Delta_Y$ particular cases of Theorem 4.5 and its corollary may also be of some interest.

Theorem 5.1. *For a relation F on a relational space $X(R)$ to a set Y, the following assertions are equivalent:*

(1) $u\,R\,v$ *implies* $F(u) \subseteq F(v)$ *for all* $u, v \in X$,

(2) $R \circ F^{-1} \subseteq F^{-1}$, *(3)* $\big(F \boxtimes R\big)[\Delta_X] \subseteq \big(\Delta_Y \boxtimes F\big)^{-1}[\Delta_Y]$.

Proof. By the corresponding definitions, it is clear that following assertions are equivalent:

(a) $R \circ F^{-1} \subseteq F^{-1}$

(b) $\forall\ y \in Y : \quad \big(R \circ F^{-1}\big)(y) \subseteq F^{-1}(y)$,

(c) $\forall\ y \in Y : \quad R\,[\,F^{-1}(y)\,] \subseteq F^{-1}(y)$,

(d) $\forall\ y \in Y : \quad \forall\ u \in F^{-1}(y) : \quad R(u) \subseteq F^{-1}(y)$,

(e) $\forall\ y \in Y : \quad \forall\ u \in F^{-1}(y) : \quad \forall\ v \in R(u) : \quad v \in F^{-1}(y)$,

(f) $\forall\ u \in X : \quad \forall\ v \in R(u) : \quad \big(y \in F(u) \implies y \in F(v)\big)$.

Therefore, assertions (2) and (1) are also equivalent. Moreover, by using Corollary 3.5 and Theorem 3.7, we can see that assertions (2) and (3) are also equivalent.

Remark 5.2. Now, a relation F on a relational space $X(R)$ to a set Y may be naturally called *inclusion increasing* if the implication (1) holds.

Moreover, a subset A of a relational space $X(R)$ may be naturally called *open* or *ascending* if $A \subseteq \operatorname{int}_R(A)$, or equivalently $R\,[\,A\,] \subseteq A$.

And, a relation F on a set X to a relational space $Y(S)$ may be naturally called *open (ascending) valued* if the set $F(x)$ is open (ascending) for all $x \in X$.

Therefore, as an immediate consequence of the above theorem, we can also state the following corollary.

Corollary 5.3. *For a relation F on a relational space $X(R)$ to a set Y, the following assertions are equivalent:*

(1) F *is inclusion increasing*, *(2)* F^{-1} *is open (ascending) valued.*

To obtain another plausible increasingness property of relations, we can also prove the following theorem.

Theorem 5.4. *For a relation F on one relational space $X(R)$ to another $Y(S)$, the following assertions are equivalent:*

(1) $u\,R\,v$ *implies* $F(u)\,S\,F(v)$ *for all* $u, v \in X$,

(2) $\big(F \boxtimes F\big)\,[R] \subseteq S$ *(3)* $F \circ R \circ F^{-1} \subseteq S$.

Proof. In (1), the notation $u\,R\,v$ means that $(u, v) \in R$. While, the notation $F(u)\,S\,F(v)$ means that $y\,S\,z$, i.e., $(y, z) \in S$ for all $y \in F(u)$ and $z \in F(v)$. That is, $F(u) \times F(v) \subseteq S$, or equivalently $\big(F \boxtimes F\big)(u, v) \subseteq S$. Hence, it is clear that assertions (1) and (2) are equivalent. Moreover, from Theorem 3.3 we can see that assertions (2) and (3) are also equivalent.

Remark 5.5. Now, a relation F on one relational space $X(R)$ to another $Y(S)$ may be naturally called *order increasing* if the implication (1) holds.

Note that assertion $u\,R\,v$ can also be written in the form that $v \in R(u)$. While, assertion $F(u)\ S\ F(v)$ can also written in the form that $F(v) \subseteq \operatorname{ub}_S\big(F(u)\big)$.

Therefore, as an immediate consequence of the above theorem, we can also state the following corollary.

Corollary 5.6. *For a relation F on one relational space $X(R)$ to another $Y(S)$, the following assertions are equivalent:*

(1) F is order increasing, *(2) $F\,[\,R(x)\,] \subseteq \operatorname{ub}_S\big(F(x)\big)$ for all $x \in X$.*

6. Some basic facts on relators and relator spaces

In the sequel, to establish some instructive reformulations of Theorem 4.5, we shall need some basic facts about relators and relator spaces [18, 22, 26].

Definition 6.1. If $\mathcal{R}$ is a family of relations on one set X to another Y, then $\mathcal{R}$ is called a *relator* on X to Y.

Moreover, the ordered pair $(X, Y)(\mathcal{R}) = \big((X, Y), \mathcal{R}\big)$ is called a *relator space.* (The more general, but less flexible concept of *corelator spaces* can be defined quite similarly by using the ideas of [34].)

Remark 6.2. If in particular $\mathcal{R}$ is a relator on X to itself, then we may simply say that $\mathcal{R}$ is a relator on X.

In this case, by identifying singletons with their elements, we may naturally write $X(\mathcal{R})$ in place of $(X, X)(\mathcal{R})$. Namely, $(X, X) = \{\{X\}, \{X, X\}\} = \{\{X\}\}$.

Thus, relator spaces are straightforward generalizations of *ordered sets* [3], *formal contexts* [7], and *uniform spaces* [6]. Their definition can primarily be motivated by the following two examples.

Example 6.3. If d is a function of $X \times Y$ to $[\,0, +\infty\,]$ and

$$B_r^d = \big\{(x, y) \in X \times Y : \quad d(x, y) < r\,\big\}$$

for all $r > 0$, then the family $\mathcal{R}_d = \{\,B_r^d : \quad r > 0\,\}$ is a natural relator on X to Y whose particular cases were already considered by Weil [41]

Note that if in particular d is a pseudo-metric on X, then $\mathcal{R}_d$ is already a *tolerance relator* on X in the sense that each member of $\mathcal{R}_d$ is a tolerance (reflexive and symmetric) relation on X.

Moreover, it also noteworthy that in this case $\mathcal{R}_d$ has the strong enough transitivity property that $B_r^d \circ B_s^d \subseteq B_{r+s}^d$ for all $r > 0$ and $s > 0$.

Example 6.4. If $\mathfrak{R}$ is a relation on $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ and

$$R_{(A,\,B)} = A \times B \,\cup\, A^c \times Y$$

for all $(A, B) \in \mathfrak{R}$, then the family $\mathcal{R}_{\mathfrak{R}} = \{\,R_{(A,\,B)} : \quad (A, B) \in \mathfrak{R}\,\}$ is a natural relator on X to Y whose particular cases were already considered by Császár [2, p. 42], Davis [4], Pervin [15], and Hunsaker and Lindgren [9].

Note that if in particular $\mathcal{A}$ is a family of subsets of X, then the identity function $\Delta_{\mathcal{A}}$ is a relation on $\mathcal{P}(X)$ such that $\mathcal{R}_{\mathcal{A}} = \mathcal{R}_{\Delta_{\mathcal{A}}} = \{\,R_A : \quad A \in \mathcal{A}\,\}$, with $R_A = R_{(A,\,A)} = A^2 \cup A^c \times X$, is a preorder relator on X in the sense that each member of $\mathcal{R}_\mathcal{A}$ is a preorder (reflexive and transitive) relation on X.

Moreover, it is noteworthy that $R_A^{-1} = R_{A^c}$ for all $A \subseteq X$. Therefore, the relator $\mathcal{R}_\mathcal{A}$ does not, in general, have any reasonable symmetry property. This is one the reasons why $\mathcal{R}_d$ with a metric d, is usually a more convenient relator than $\mathcal{R}_\mathcal{A}$ with a topology or filter $\mathcal{A}$.

Remark 6.5. Note that the above two examples can be naturally generalized to families of such *distance functions* d and *hyper-relations* $\mathfrak{R}$.

Namely, if $\mathcal{R}_i$ is a relator on X to Y for all $i \in I$, then it is clear that $\mathcal{R} = \bigcup_{i \in I} \mathcal{R}_i$ is also a reasonable relator on X to Y.

In this respect it is also worth noticing that, by using the above two examples, we can define several natural relators on the real line $\mathbb{R}$. However, the most immediate one is $\mathcal{R} = \{ \leq \}$, where $\leq$ is the usual ordering on $\mathbb{R}$.

Definition 6.6. A relator $\mathcal{R}$ on X to Y, or a relator space $(X, Y)(\mathcal{R})$, is called *simple* if there exists a relation R on X to Y such that $\mathcal{R} = \{R\}$.

In this, case, by identifying singletons with their elements, we shall simply write $(X, Y)(R)$ in place of $(X, Y)\big(\{R\}\big)$.

Remark 6.7. Some less simple relator spaces have mainly been studied by Pataki [12] by using the ideas of [21].

Note that, for any relator $\mathcal{R}$ on X to Y, we have the trivial, but important decomposition $\mathcal{R} = \bigcup_{R \in \mathcal{R}} \{R\}$.

Therefore, the study of the most general relator spaces can frequently traced back to that of the simple ones.

That is, to that of generalized ordered sets and context spaces which have also been called relational spaces by the present author.

Remark 6.8. For instance, if $\mathcal{R}$ is a relator on X to Y, and for any $A \subseteq X$ and $B \subseteq Y$ we write:

(1) $A \in \mathrm{Lb}_\mathcal{R}(B)$ if $A \times B \subseteq R$ for some $R \in \mathcal{R}$,

(2) $A \in \mathrm{Int}_\mathcal{R}(B)$ if $R[A] \subseteq B$ for some $R \in \mathcal{R}$,

then it can be easily seen that $\mathrm{Lb}_\mathcal{R}$ and $\mathrm{Int}_\mathcal{R}$ are relations on $\mathcal{P}(Y)$ to $\mathcal{P}(X)$ such that

$$\mathrm{Lb}_\mathcal{R} = \bigcup_{R \in \mathcal{R}} \mathrm{Lb}_R \qquad \text{and} \qquad \mathrm{Int}_\mathcal{R} = \bigcup_{R \in \mathcal{R}} \mathrm{Int}_R\,,$$

Hence, for instance, by defining

$$\mathrm{int}_\mathcal{R}(B) = \big\{ x \in X : \ \{x\} \in \mathrm{Int}_\mathcal{R}(B) \big\} \quad \text{and} \quad \mathcal{E}_\mathcal{R} = \big\{ B \subseteq Y : \ \mathrm{int}_\mathcal{R}(B) \neq \emptyset \big\},$$

we can at once see that

$$\mathrm{int}_\mathcal{R} = \bigcup_{R \in \mathcal{R}} \mathrm{int}_R \qquad \text{and} \qquad \mathcal{E}_\mathcal{R} = \bigcup_{R \in \mathcal{R}} \mathcal{E}_R\,.$$

Moreover, if in particular $\mathcal{R}$ is a relator on X and

$$\tau_\mathcal{R} = \big\{ A \subseteq Y : \ A \in \mathrm{Int}_\mathcal{R}(A) \big\},$$

then we can also easily see that $\tau_\mathcal{R} = \bigcup_{R \in \mathcal{R}} \tau_R$. However, if

$$\mathcal{T}_\mathcal{R} = \big\{ A \subseteq Y : \ A \subseteq \mathrm{int}_\mathcal{R}(A) \big\},$$

then we can only prove that $\mathcal{T}_\mathcal{R} = \bigcup_{R \in \mathcal{R}^\wedge} \mathcal{T}_R$ whenever $\mathcal{R} \neq \emptyset$.

This is the most serious disadvantage of the family $\mathcal{T}_{\mathcal{R}}$ of the *topologically open sets* to the families $\mathcal{E}_{\mathcal{R}}$ and $\tau_{\mathcal{R}}$ of the *fat sets* and *proximally open sets.* In the relator space $X(\mathcal{R})$, the fat and dense sets are usually more important tools than the open and closed sets. Their duality was first revealed in [20], and later applied in [27, 32].

7. Some important involution and modification operations for relators

Definition 7.1. A function $\square$ of the class of all relator spaces to the class of all relators is called a *direct (inderect) unary operation for relators* if, for any relator $\mathcal{R}$ on X to Y, the value

$$\mathcal{R}^{\square} = \mathcal{R}^{\square_{XY}} = \square\big((X, Y)(\mathcal{R})\big)$$

is a relator on X to Y (on Y to X).

Remark 7.2. A unary operation $\square$ for relators is called *increasing* if for any two relators $\mathcal{R}$ and $\mathcal{S}$, with $\mathcal{R} \subseteq S$, we also have $\mathcal{R}^{\square} \subseteq \mathcal{S}^{\square}$.

Moreover, the operation is called *extensive, intensive, involutive*, and *idempotent* if for any relator $\mathcal{R}$ on X to Y we have $\mathcal{R} \subseteq \mathcal{R}^{\square}$, $\mathcal{R}^{\square} \subseteq \mathcal{R}$, $\mathcal{R}^{\square\square} = \mathcal{R}$, and $\mathcal{R}^{\square\square} = \mathcal{R}^{\square}$, respectively.

In particular, an increasing idempotent operation for relators is called a *modification operation.* While, an extensive (intensive) modification operation for relators is called a *closure (interior) operation.*

Example 7.3. For instance, if for any relator $\mathcal{R}$ on X to Y we define

$$\mathcal{R}^{c} = \big\{ R^{c} : \quad R \in \mathcal{R} \big\} \qquad \text{and} \qquad \mathcal{R}^{-1} = \big\{ R^{-1} : \quad R \in \mathcal{R} \big\},$$

then c is a direct and -1 is an indirect unary operation for relators. (Note that here R^{c} means again the complement of R with respect to $X \times Y$.)

Moreover, it can be easily seen the above operations are involutions which are *compatible (commuting)* in the sense that $\big(\mathcal{R}^{c}\big)^{-1} = \big(\mathcal{R}^{-1}\big)^{c}$ for any relator $\mathcal{R}$ on X to Y.

Of course, if we restrict ourself to the particular case $X = Y$, then -1 is also a direct unary operation for relators.

Example 7.4. Moreover, if in particular, for any relator $\mathcal{R}$ on X we define

$$\mathcal{R}^{\infty} = \big\{ R^{\infty} : \quad R \in \mathcal{R} \big\} \qquad \text{and} \qquad \mathcal{R}^{\partial} = \big\{ S \subseteq X^{2} : \quad S^{\infty} \in \mathcal{R} \big\},$$

then ∞ and ∂ are also direct unary operations for relators. (Note that here $R^{\infty} = \bigcup_{n=0}^{\infty} R^{n}$ is the smallest preorder relation on X containing R which was mainly studied in [8].)

Moreover, it can be easily seen that the above two operations are modification operations for relators.

Remark 7.5. The importance of the operations ∞ and ∂ is also apparent from the fact that, for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X, we have

$$\mathcal{R}^{\infty} \subseteq \mathcal{S} \quad \Longleftrightarrow \quad \mathcal{R} \subseteq S^{\partial}.$$

Therefore, the operations ∞ and ∂ form a *Galois connection* between $\mathcal{P}(X^2)$ and itself. Thus, in particular $\infty\,\partial$ is already a closure operation for relators such that $\infty = \infty\,\partial\,\infty$.

By using the corresponding definitions, one can easily prove the following theorem, whose origin goes back to R. Dedekind by a remark of Erné [5, p. 50].

Theorem 7.6. *For a unary operation* $\Box$ *for relators, the following assertions are equivalent:*

(1) $\Box$ *is a closure operation,*

(2) for any two relators $\mathcal{R}$ *and* $\mathcal{S}$ *on* X *to* Y*, we have*

$$\mathcal{R}^{\Box} \subseteq \mathcal{S}^{\Box} \quad \Longleftrightarrow \quad \mathcal{R} \subseteq \mathcal{S}^{\Box}.$$

Proof. To prove the implication $(2) \Longrightarrow (1)$, it is convenient to follow the arguments given in [30] by using the ideas of [13].

Analogously to this theorem, we can also easily prove the following theorem.

Theorem 7.7. *For a unary operation* $\Box$ *for relators, the following assertions are equivalent:*

(1) $\Box$ *is an increasing involution,*

(2) for any two relators $\mathcal{R}$ *and* $\mathcal{S}$ *on* X *to* Y*, we have*

$$\mathcal{R}^{\Box} \subseteq \mathcal{S} \quad \Longleftrightarrow \quad \mathcal{R} \subseteq \mathcal{S}^{\Box}.$$

Proof. To prove the implication $(2) \Longrightarrow (1)$, note that if (2) holds, then for any relator $\mathcal{R}$ on X to Y

$$\mathcal{R}^{\Box} \subseteq \mathcal{R}^{\Box} \quad \Longrightarrow \quad \mathcal{R} \subseteq \mathcal{R}^{\Box\Box}, \quad \mathcal{R}^{\Box\Box} \subseteq \mathcal{R} \quad \Longrightarrow \quad \mathcal{R} = \mathcal{R}^{\Box\Box}.$$

Therefore, $\Box$ is involutive. Thus, for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y

$$\mathcal{R} \subseteq \mathcal{S} \quad \Longrightarrow \quad \mathcal{R}^{\Box\Box} \subseteq \mathcal{S}^{\Box\Box} \quad \Longrightarrow \quad \mathcal{R}^{\Box} \subseteq \mathcal{S}^{\Box\Box\Box} \quad \Longrightarrow \quad \mathcal{R}^{\Box} \subseteq \mathcal{S}^{\Box}.$$

Therefore, $\Box$ is increasing, and thus (1) also holds.

Remark 7.8. By Theorem 7.7, a unary operation $\Box$ for relators is an increasing involution if and only if $\Box$ with itself form a Galois connection.

Moreover, by Theorem 7.6, a unary operation $\Box$ for relators is a closure operation if and only if $\Box$ with itself form a *Pataki connection* [36].

8. Some important closure operations for relators

In addition to Theorems 7.6 and 7.7, it is also worth proving the the following theorem.

Theorem 8.1. *If* $\Box$ *is a closure (modification) and* $\diamond$ *is an increasing involution operation for relators, then* $\Diamond = \diamond\Box\diamond$ *is also a closure (modification) operation for relators.*

Proof. To prove the idempotency of $\Diamond$, note that by the associativity of composition, the involutiveness of $\diamond$, and the idempontency of $\Box$, we have

$$\begin{aligned} \Diamond\,\Diamond &= (\diamond\Box\diamond)(\diamond\Box\diamond) = (\diamond\Box)\big((\diamond\diamond)(\Box\diamond)\big) \\ &= (\diamond\Box)\big(\Delta(\Box\diamond)\big) = (\diamond\Box)(\Box\diamond) = \diamond\big((\Box\Box)\diamond\big) = \diamond(\Box\diamond) = \Diamond, \end{aligned}$$

where Δ is the identity operation for relators.

Because of this theorem, we may also naturally introduce the following definition.

Definition 8.2. For any unary operation $\square$ for relators, we write

$$\circledcirc = c \,\square\, c \qquad \text{and} \qquad \boxdot = -1 \,\square\, -1\,.$$

Example 8.3. By defining

$$\mathcal{R}^{*} = \big\{ S \subseteq X \times Y : \quad \exists \; R \in \mathcal{R} : \quad R \subseteq S \big\}$$

for any relator $\mathcal{R}$ on X to Y, it can be easily seen that $*$ is a closure operation for relators such that:

(1) $\mathcal{R}^{\boxast} = \mathcal{R}^{*}$, (2) $\mathcal{R}^{\circledast} = \bigcup_{R \in \mathcal{R}} \mathcal{P}(R)$.

Namely, if for instance $S \in \mathcal{R}^{\circledast}$, then $S \in \mathcal{R}^{c*c}$, and thus $S^{c} \in \mathcal{R}^{c*}$. Therefore, there exists $R \in \mathcal{R}$ such that $R^{c} \subseteq S^{c}$. Hence, it follows that $S \subseteq R$, and thus $S \in \mathcal{P}(R)$. Therefore, $S \in \bigcup_{R \in \mathcal{R}} \mathcal{P}(R)$ also holds.

Remark 8.4. Moreover, the operation $*$ can also be easily seen to be *inversion and composition compatible* in the sense that:

(1) $\big(\mathcal{R}^{*}\big)^{-1} = \big(\mathcal{R}^{-1}\big)^{*}$ for any relator $\mathcal{R}$ on X to Y,

(2) $\big(\mathcal{S} \circ \mathcal{R}\big)^{*} = \big(\mathcal{S} \circ \mathcal{R}^{*}\big)^{*} = \big(\mathcal{S}^{*} \circ \mathcal{R}\big)^{*}$ for any relators $\mathcal{R}$ on X to Y and $\mathcal{S}$ on Y to Z.

Note that here, analogously $\mathcal{R}^{-1} = \{ R^{-1} : \; R \in \mathcal{R} \}$, we may also naturally define $\mathcal{S} \circ \mathcal{R} = \{ S \circ R : \; R \in \mathcal{R}, \; S \in \mathcal{S} \}$.

Remark 8.5. In addition to $*$, the operations $\#$, $\wedge$ and $\vartriangle$, defined by

$$\mathcal{R}^{\#} = \big\{ S \subseteq X \times Y : \quad \forall \; A \subseteq X : \quad A \in \operatorname{Int}_{\mathcal{R}}\big(S[A] \big) \big\},$$
$$\mathcal{R}^{\wedge} = \big\{ S \subseteq X \times Y : \quad \forall \; x \in X : \quad x \in \operatorname{int}_{\mathcal{R}}\big(S(x) \big) \big\},$$
$$\mathcal{R}^{\vartriangle} = \big\{ S \subseteq X \times Y : \quad \forall \; x \in X : \quad S(x) \in \mathcal{E}_{\mathcal{R}} \big\}.$$

for any relator $\mathcal{R}$ on X to Y, are also important closure operations for relators. (For a unified proof of this fact, the reader is referred to [35].)

Moreover, it can be easily seen that, for any relator $\mathcal{R}$ on X to Y, we have

$$\mathcal{R} \subseteq \mathcal{R}^{*} \subseteq \mathcal{R}^{\#} \subseteq \mathcal{R}^{\wedge} \subseteq \mathcal{R}^{\vartriangle},$$

and in particular $\mathcal{R}^{\infty} \subseteq \mathcal{R}^{*\infty} \subseteq \mathcal{R}^{\infty *} \subseteq \mathcal{R}^{*}$ whenever $X = Y$.

However, in contrast to the *uniform and proximal closures* $*$ and $\#$, the *topological and paratopological closures* $\wedge$ and $\vartriangle$ are not inversion and composition compatible. (See [10] and [35].)

Thus, in addition to $\wedge$ and $\vartriangle$, we have to consider the operations $\vee$ and $\triangledown$ defined by

$$\mathcal{R}^{\vee} = \big(\mathcal{R}^{\wedge}\big)^{-1} \qquad \text{amd} \qquad \mathcal{R}^{\triangledown} = \big(\mathcal{R}^{\vartriangle}\big)^{-1}$$

for every relator $\mathcal{R}$ on X to Y.

However, these operations already have some very curious properties. For instance, the operations $\vee\vee$ and $\triangledown\triangledown$ already coincide with the extremal closure operations $\bullet$ and $\blacklozenge$, defined for any relator $\mathcal{R}$ on X to Y such that

$$\mathcal{R}^{\bullet} = \{ \delta_{\mathcal{R}} \}^{*}, \qquad \text{where} \qquad \delta_{\mathcal{R}} = \bigcap \mathcal{R},$$

and

$$\mathcal{R}^{\blacklozenge}=\mathcal{R} \quad \text{if} \quad \mathcal{R}=\{X\times Y\} \qquad \text{and} \qquad \mathcal{R}^{\blacklozenge}=\mathcal{P}(X\times Y) \quad \text{if} \quad \mathcal{R}\neq\{X\times Y\}.$$

Note that $\square=\blacklozenge$ is the ultimate *stable unary operation for relators* in the sense that $\{X\times Y\}^{\square}=\{X\times Y\}$ for any two sets X and Y.

Unfortunately, the operation ∂ is not stable. Therefore, for instance, the modification operation $\#\partial$ is usually a less convenient mean than $\#\infty$.

9. Some general definitions for increasingness and continuity

Now, by using a particular case of the definition of the uniform closure operation $*$, Theorem 4.5 can be reformulated in the following form.

Theorem 9.1. *For any function f of one simple relator space $X(R)$ to another $Y(S)$, the following assertions are equivalent:*

(1) *f is increasing (continuous),*

(2) $\{S\circ f\}\subseteq\{f\circ R\}^*$, (3) $\{f^{-1}\circ S\circ f\}\subseteq\{R\}^*$,

(4) $\{S\}\subseteq\{f\circ R\circ f^{-1}\}^*$, (5) $\{f^{-1}\circ S\}\subseteq\{R\circ f^{-1}\}^*$.

Proof. To check the equivalence of the assertions (3) of Theorems 4.5 and 9.1, note that by the definition of the operation $*$ we have

$$R\subseteq f^{-1}\circ S\circ f \iff f^{-1}\circ S\circ f\in\{R\}^* \iff \{f^{-1}\circ S\circ f\}\subseteq\{R\}^*.$$

Moreover, by using a particular case of the definitions of the elementwise inverse and composition of relators, we can also prove the following less simple, but more instructive theorem.

Theorem 9.2. *If f is a function of one simple relator space $X(R)$ to another $Y(S)$, then under the notations*

$$\mathcal{F}=\{f\}, \qquad \mathcal{R}=\{R\} \qquad \textit{and} \qquad \mathcal{S}=\{S\}$$

the following assertions are equivalent:

(1) *f is increasing (continuous),*

(2) $\big(\mathcal{S}^*\circ\mathcal{F}^*\big)^*\subseteq\big(\mathcal{F}^*\circ\mathcal{R}^*\big)^*$, (3) $\Big(\big(\mathcal{F}^*\big)^{-1}\circ\mathcal{S}^*\circ\mathcal{F}^*\Big)^*\subseteq\mathcal{R}^{**}$,

(4) $\mathcal{S}^{**}\subseteq\Big(\mathcal{F}^*\circ\mathcal{R}^*\circ\big(\mathcal{F}^*\big)^{-1}\Big)^*$, (5) $\Big(\big(\mathcal{F}^*\big)^{-1}\circ\mathcal{S}^*\Big)^*\subseteq\Big(\mathcal{R}^*\circ\big(\mathcal{F}^*\big)^{-1}\Big)^*$.

Proof. To prove the equivalence of the assertions (3) of Theorems 9.1 and 9.2, note that by the inversion and composition compatibility of the operation $*$, we have

$$\Big(\big(\mathcal{G}^*\big)^{-1}\circ\mathcal{S}^*\circ\mathcal{F}^*\Big)^*=\Big(\big(\mathcal{G}^{-1}\big)^*\circ\mathcal{S}^*\circ\mathcal{F}^*\Big)^*=\big(\mathcal{G}^{-1}\circ\mathcal{S}\circ\mathcal{F}\big)^*.$$

Moreover, since $*$ is a closure operation for relators, for any two relators $\mathcal{U}$ and $\mathcal{V}$ on X to Y we also have

$$\mathcal{U}^*\subseteq\mathcal{V}^{**} \iff \mathcal{U}^*\subseteq\mathcal{V}^* \iff \mathcal{U}\subseteq\mathcal{V}^*.$$

Now, the Pexiderizations of the inclusions in Theorem 9.2, and a former abstraction of the operation $*$, naturally lead us to the following substantial extension of Definition 2.3 whose particular cases were already considered in [25].

Definition 9.3. Suppose that

(a) $\Box$ *is a direct unary operation for relators,*

(b) $(X, Y)(\mathcal{R})$ *and* $(Z, W)(\mathcal{S})$ *are relator spaces,*

(c) $\mathcal{F}$ *is a relator on* X *to* Z *and* $\mathcal{G}$ *is a relator on* Y *to* W.

Then, we say that the ordered pair

(1) $(\mathcal{F}, \mathcal{G})$ is *upper* $\Box$*–continuous*, with respect to the relators $\mathcal{R}$ and $\mathcal{S}$, if

$$\big(\mathcal{S}^{\Box}\circ\mathcal{F}^{\Box}\big)^{\Box}\subseteq\big(\mathcal{G}^{\Box}\circ\mathcal{R}^{\Box}\big)^{\Box},$$

(2) $(\mathcal{F}, \mathcal{G})$ is *mildly* $\Box$*–continuous*, with respect to the relators $\mathcal{R}$ and $\mathcal{S}$, if

$$\Big(\big(\mathcal{G}^{\Box}\big)^{-1}\circ\mathcal{S}^{\Box}\circ\mathcal{F}^{\Box}\Big)^{\Box}\subseteq\mathcal{R}^{\Box\Box},$$

(3) $(\mathcal{F}, \mathcal{G})$ is *vaguely* $\Box$*–continuous*, with respect to the relators $\mathcal{R}$ and $\mathcal{S}$, if

$$\mathcal{S}^{\Box\Box}\subseteq\Big(\mathcal{G}^{\Box}\circ\mathcal{R}^{\Box}\circ\big(\mathcal{F}^{\Box}\big)^{-1}\Big)^{\Box},$$

(4) $(\mathcal{F}, \mathcal{G})$ is *lower* $\Box$*–continuous*, with respect to the relators $\mathcal{R}$ and $\mathcal{S}$, if

$$\Big(\big(\mathcal{G}^{\Box}\big)^{-1}\circ\mathcal{S}^{\Box}\Big)^{\Box}\subseteq\Big(\mathcal{R}^{\Box}\circ\big(\mathcal{F}^{\Box}\big)^{-1}\Big)^{\Box}.$$

Now, to keep in mind the above assumptions, we can use the diagram mentioned in the Introduction.

Moreover, to derive the definition of lower $\Box$-continuity from that of the upper $\Box$–continuity, we can prove the following theorem.

Theorem 9.4. *If in particular the operation* $\Box$ *is inversion compatible, then the following assertions are equivalent:*

(1) $(\mathcal{F}, \mathcal{G})$ *is lower* $\Box$*–continuous with respect to the relators* $\mathcal{R}$ *and* $\mathcal{S}$

(2) $(\mathcal{G}, \mathcal{F})$ *is upper* $\Box$*–continuous with respect to the relators* $\mathcal{R}^{-1}$ *and* $\mathcal{S}^{-1}$.

Proof. By using Definition 9.3, the inversion compatibility of $\Box$, and inversion property of composition, we can easily see that

$$\begin{aligned}
(1)\iff&\Big(\big(\mathcal{G}^{\Box}\big)^{-1}\circ\mathcal{S}^{\Box}\Big)^{\Box}\subseteq\Big(\mathcal{R}^{\Box}\circ\big(\mathcal{F}^{\Box}\big)^{-1}\Big)^{\Box}\\
\iff&\bigg(\Big(\big(\mathcal{G}^{\Box}\big)^{-1}\circ\mathcal{S}^{\Box}\Big)^{\Box}\bigg)^{-1}\subseteq\bigg(\Big(\mathcal{R}^{\Box}\circ\big(\mathcal{F}^{\Box}\big)^{-1}\Big)^{\Box}\bigg)^{-1}\\
\iff&\bigg(\Big(\big(\mathcal{G}^{\Box}\big)^{-1}\circ\mathcal{S}^{\Box}\Big)^{-1}\bigg)^{\Box}\subseteq\bigg(\Big(\mathcal{R}^{\Box}\circ\big(\mathcal{F}^{\Box}\big)^{-1}\Big)^{-1}\bigg)^{\Box}\\
\iff&\Big(\big(\mathcal{S}^{\Box}\big)^{-1}\circ\mathcal{G}^{\Box}\Big)^{\Box}\subseteq\Big(\mathcal{F}^{\Box}\circ\big(\mathcal{R}^{\Box}\big)^{-1}\Big)^{\Box}\\
\iff&\Big(\big(\mathcal{S}^{-1}\big)^{\Box}\circ\mathcal{G}^{\Box}\Big)^{\Box}\subseteq\Big(\mathcal{F}^{\Box}\circ\big(\mathcal{R}^{-1}\big)^{\Box}\Big)^{\Box}\iff(2).
\end{aligned}$$

Remark 9.5. Concerning mild $\Box$–continuity, we can quite similarly prove that $(\mathcal{F}, \mathcal{G})$ is mildly $\Box$–continuous with respect to the relators $\mathcal{R}$ and $\mathcal{S}$ if and only if $(\mathcal{G}, \mathcal{F})$ is mildly $\Box$–continuous with respect to the relators $\mathcal{R}^{-1}$ and $\mathcal{S}^{-1}$.

10. Some supplementary notes to Definition 9.3

Remark 10.1. Now, the pair $(\mathcal{F}, \mathcal{G})$ may, for instance, be naturally called *properly mildly continuous* if it is mildly $\square$–continuous with $\square$ being the identity operation for relators. That is, $\mathcal{G}^{-1} \circ \mathcal{S} \circ \mathcal{F} \subseteq \mathcal{R}$.

Remark 10.2. Moreover, the pair $(\mathcal{F}, \mathcal{G})$ may, for instance, be naturally called *uniformly, proximally, topologically, and paratopologically mildly continuous* if it is mildly $\square$–continuous with $\square = *$, $\#$, $\wedge$, and $\triangle$, respectively.

Remark 10.3. And, the pair $(\mathcal{F}, \mathcal{G})$ may, for instance, be naturally called *quasi-topologically and ultra-topologically* mildly continuous if its is mildly $\square$–continuous with $\square = \wedge\infty$ and $\wedge\partial$, respectively.

Remark 10.4. Moreover, the pair $(\mathcal{F}, \mathcal{G})$ may, for instance, be naturally called *infinitesimally and ultimately mildly continuous* if it is $\square$–mildly continuous with $\square = \bullet$ and $\blacklozenge$, respectively.

Now, by specializing part (2) of Definition 9.3, we may also naturally have the following definition.

Definition 10.5. For any $F \in \mathcal{F}$ and $G \in \mathcal{G}$, the pair (F, G) of relations is called *mildly $\square$–continuous*, with respect to the relators $\mathcal{R}$ and $\mathcal{S}$, if the pair $\big(\{F\}, \{G\}\big)$ of relators has the same property.

Remark 10.6. To apply this definition, note that if in particular $\square = \#$ or $\wedge$, then for any $F \in \mathcal{F}$ and $G \in \mathcal{G}$ we have

$$\{F\}^{\square} = \{F\}^{*} \qquad \text{and} \qquad \big(\{G\}^{\square}\big)^{-1} = \big(\{G\}^{*}\big)^{-1} = \big\{G^{-1}\big\}^{*}.$$

However, in contrast to the above equalities, for instance we already have

$$\{F\}^{\triangle} = \big(F \circ X^{X}\big)^{*} \quad \text{and} \quad \big(\{G\}^{\triangle}\big)^{-1} = \Big(\big(G \circ Y^{Y}\big)^{*}\Big)^{-1} = \Big(\big(Y^{Y}\big)^{-1} \circ G^{-1}\Big)^{*}.$$

Now, by using Definition 10.5, we may also naturally introduce the following definition.

Definition 10.7. Under the assumptions of Definition 9.3, we say that the pair $(\mathcal{F}, \mathcal{G})$ of relators is *elementwise mildly $\square$–continuous*, with respect to the relators $\mathcal{R}$ and $\mathcal{S}$, if for any $F \in \mathcal{F}$ and $G \in \mathcal{G}$ the pair (F, G) of relations is mildly $\square$–continuous with respect to the relators $\mathcal{R}$ and $\mathcal{S}$.

Remark 10.8. Thus, the pair $(\mathcal{F}, \mathcal{G})$ may, for instance, be naturally called *elementwise topologically mildly continous* if it is elementwise mildly $\square$–continuous with $\square = \wedge$.

Unfortunately, in our longer paper [40], we could not prove that an elementwise topologically mildly continuous pair $(\mathcal{F}, \mathcal{G})$ of relators need not be topologically mildly continuous.

Now, as a natural extension of [25, Definition 4.6], we may also naturally have the following definition.

Definition 10.9. Under the assumptions of Definition 9.3, we say that the pair

(1) $(\mathcal{F}, \mathcal{G})$ is *lower selectionally mildly $\square$–continuous* if for any $F \in \mathcal{F}$ and $G \in \mathcal{G}$ and any selection f of F the pair (f, G) is mildly $\square$–continuous,

(2) $(\mathcal{F}, \mathcal{G})$ is *upper selectionally mildly* $\Box$*–continuous* if for any $F \in \mathcal{F}$ and $G \in \mathcal{G}$ and any selection g of G the pair (F, g) is mildly $\Box$–continuous.

Remark 10.10. Now, the pair $(\mathcal{F}, \mathcal{G})$ may also be naturally called *selectionally mildly* $\Box$*–continuous* if it is both lower and upper selectionally mildly $\Box$–continuous.

Remark 10.11. Moreover, the pair $(\mathcal{F}, \mathcal{G})$ may also be naturally called *doubly selectionally mildly* $\Box$*–continuous* if for any $F \in \mathcal{F}$ and $G \in \mathcal{G}$ and for any selections f of F and g of G, the pair (f, g) is mildly $\Box$–continuous.

Remark 10.12. Finally, we note that, in the $X = Y$ and $Z = W$ particular case, the relator $\mathcal{F}$ and a relation $F \in \mathcal{F}$ may, for instance, be naturally called mildly $\Box$–continuous if the pairs $(\mathcal{F}, \mathcal{F})$ and (F, F), respectively, have the same property.

11. An application to a generalization of Galois connections

Because of Theorem 9.2, we may naturally write "$\Box$–increasing" instead of "$\Box$–continuous" in the corresponding definitions of Sections 9 and 10. Thus, we can obtain some reasonable generalizations of the usual increasingness.

However, it is now more important to stress that, by using the ideas of our former paper [35], analogously to Definition 9.3, we may also naturally introduce several reasonable generalizations of Galois connections.

Definition 11.1. Suppose that

(a) $\Box$ *is a direct unary operation for relators,*

(b) $(X, Y)(\mathcal{R})$ *and* $(Z, W)(\mathcal{S})$ *are relator spaces,*

(c) $\mathcal{F}$ *is a relator on* X *to* Z *and* $\mathcal{G}$ *is a relator on* W *to* Y.

Then, we say that the relator

(1) $\mathcal{F}$ is *upper* $\Box$*–*$\mathcal{G}$*–normal*, with respect to the relators $\mathcal{R}$ and $\mathcal{S}$, if

$$\left(\mathcal{S}^{\Box} \circ \mathcal{F}^{\Box}\right)^{\Box} \subseteq \left(\left(\mathcal{G}^{\Box}\right)^{-1} \circ \mathcal{R}^{\Box}\right)^{\Box},$$

(2) $\mathcal{F}$ is *lower* $\Box$*–*$\mathcal{G}$*–normal*, with respect to the relators $\mathcal{R}$ and $\mathcal{S}$, if

$$\left(\left(\mathcal{G}^{\Box}\right)^{-1} \circ \mathcal{R}^{\Box}\right)^{\Box} \subseteq \left(\mathcal{S}^{\Box} \circ \mathcal{F}^{\Box}\right)^{\Box}.$$

Remark 11.2. Now, the relator $\mathcal{F}$ may also be naturally called *upper (lower)* $\Box$*–normal* if it is upper (lower) $\Box$–$\mathcal{G}$–normal for some relator $\mathcal{G}$ on W to Z.

Moreover, as an immediate consequence of the corresponding definitions, we can state the following theorem.

Theorem 11.3. *If in particular the operation* $\Box$ *is inversion compatible, then*

(1) $\mathcal{F}$ *is upper* $\Box$*–*$\mathcal{G}$*–normal if and only if* $\left(\mathcal{F}, \mathcal{G}^{-1}\right)$ *is upper* $\Box$*–continuous,*

(2) $\mathcal{F}$ *is lower* $\Box$*–*$\mathcal{G}$*–normal if and only if* $\left(\mathcal{F}^{-1}, \mathcal{G}\right)$ *is lower* $\Box$*–continuous.*

Remark 11.4. By this theorem, for an inversion compatible operation $\Box$, several properties of upper and lower $\Box$–normal relators can, in principle, be immediately derived from those of the upper and lower $\Box$–continuous ones.

However, for instance, to prove the following theorem and its corollary it is more convenient to apply some direct arguments based upon only the corresponding definitions.

Theorem 11.5. *In in particular $\square$ is an inversion and composition compatible operation, then*

(1) $\mathcal{F}$ *is upper* $\square$*–*$\mathcal{G}$*–seminormal* $\iff$ $(\mathcal{S}\circ\mathcal{F})^{\square}\subseteq\big(\mathcal{G}^{-1}\circ\mathcal{R}\big)^{\square}$,

(2) $\mathcal{F}$ *is lower* $\square$*–*$\mathcal{G}$*–normal* $\iff$ $\big(\mathcal{G}^{-1}\circ\mathcal{R}\big)^{\square}\subseteq(\mathcal{S}\circ\mathcal{F})^{\square}$.

From this theorem, by using Theorem 7.6, we can immediately derive the following corollary.

Corollary 11.6. *In in particular $\square$ is an inversion and composition compatible closure operation, then*

(1) $\mathcal{F}$ *is upper* $\square$*–*$\mathcal{G}$*–normal* $\iff$ $\mathcal{S}\circ\mathcal{F}\subseteq\big(\mathcal{G}^{-1}\circ\mathcal{R}\big)^{\square}$,

(2) $\mathcal{F}$ *is lower* $\square$*–*$\mathcal{G}$*–normal* $\iff$ $\mathcal{G}^{-1}\circ\mathcal{R}\subseteq(\mathcal{S}\circ\mathcal{F})^{\square}$.

Now, by using the fact that the operation $\circledast = c * c$, considered in Example 8.3, is also an inversion and composition compatible closure operation for relators, for instance, we can also easily prove the following theorem.

Theorem 11.7. *Under the notations of Definition 11.1, the following assertions are equivalent:*

(1) $\mathcal{F}$ *is upper* $\circledast$*–*$\mathcal{G}$*–normal*, *(2)* $\mathcal{S}\circ\mathcal{F}\subseteq\big(\mathcal{G}^{-1}\circ\mathcal{R}\big)^{\circledast}$,

(3) for any $S\in\mathcal{S}$ *and* $F\in\mathcal{F}$ *there exist* $G\in\mathcal{G}$ *and* $R\in\mathcal{R}$ *such that* $S\circ F\subseteq G^{-1}\circ R$,

(4) for any $S\in\mathcal{S}$ *and* $F\in\mathcal{F}$ *there exist* $G\in\mathcal{G}$ *and* $R\in\mathcal{R}$ *such that* $F(x)\cap S^{-1}(w)\neq\emptyset$ *implies* $G(w)\cap R(x)\neq\emptyset$ *for all* $x\in X$ *and* $w\in W$.

Proof. To prove the implications (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4), not that if (2) holds, then by the corresponding definitions and assertion (2) in Example 8.3, for any $S\in\mathcal{S}$ and $F\in\mathcal{F}$, there exist $G\in\mathcal{G}$ and $R\in\mathcal{R}$ such that

$$S\circ F\subseteq G^{-1}\circ R.$$

Hence, we can infer that

$$\big(S\circ F\big)(x)\subseteq\big(G^{-1}\circ R\big)(x),\qquad\text{and thus}\qquad S\big[F(x)\big]\subseteq G^{-1}\big[R(x)\big]$$

for all $x\in X$. Therefore,

$$w\in S\big[F(x)\big]\ \Longrightarrow\ w\in G^{-1}\big[R(x)\big],$$

and thus

$$S^{-1}(w)\cap F(x)\neq\emptyset\ \Longrightarrow\ G(w)\cap R(x)\neq\emptyset$$

for all $x\in X$ and $w\in W$.

Now, as an immediate consequence of this theorem, we can also state the following corollary.

Corollary 11.8. *If in particular each member of the families* $\mathcal{F}$ *and* $\mathcal{G}$ *is a function, then the following assertions are equivalent:*

(1) $\mathcal{F}$ *is upper* $\circledast$*–*$\mathcal{G}$*–normal*,

(2) for any $S \in \mathcal{S}$ *and* $f \in \mathcal{F}$ *there exist* $g \in \mathcal{G}$ *and* $R \in \mathcal{R}$ *such that* $f(x)\,S\,w$ *implies* $x\,R\,g(w)$ *for all* $x \in X$ *and* $w \in W$.

Hence, by using an analogue of Definition 10.5, we can immediately derive the following corollary.

Corollary 11.9. *For any functions* $f \in \mathcal{F}$ *and* $g \in \mathcal{G}$, *then the following assertions are equivalent:*

(1) f *is upper* $\circledast$*–*g*–normal,*

(2) for any $S \in \mathcal{S}$ *there exists* $R \in \mathcal{R}$ *such that* $f(x)Sw$ *implies* $x\,R\,g(w)$ *for all* $x \in X$ *and* $w \in W$.

Remark 11.10. Because of the above two corollaries, the relators $\mathcal{F}$ and $\mathcal{G}$, considered in Definition 11.1, may be naturally said to form an *upper* $\square$*–Galois connection* between the relator spaces $(X, Y)(\mathcal{R})$ and $(Z, W)(\mathcal{S})$ if the relator $\mathcal{F}$ is upper $\square$–G–normal with respect to the relators $\mathcal{R}$ and $\mathcal{S}$.

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References

1. G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloq. Publ. **25**, Providence, RI, 1967.
2. Á. Császár, *Foundations of General Topology*, Pergamon Press, London, 1963.
3. B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, Cambridge, 2002.
4. A. S. Davis, *Indexed systems of neighborhoods for general topological spaces*, Amer. Math. Monthly **68** (1961), 886–893.
5. K. Denecke, M. Erné and S. L. Wismath (Eds.) *Galois Connections and Applications*, Kluwer Academic Publisher, Dordrecht, 2004.
6. P. Fletcher and W. F. Lindgren, *Quasi-Uniform Spaces*, Marcel Dekker, New York, 1982.
7. B. Ganter and R. Wille, *Formal Concept Analysis*, Springer-Verlag, Berlin, 1999.
8. T. Glavosits, *Generated preorders and equivalences*, Acta Acad. Paed. Agrienses, Sect. Math. **29** (2002), 95–103.
9. W. N. Hunsaker and W. Lindgren, *Constructions of quasi-uniformities*, Math. Ann. **188** (1970), 39–42.
10. J. Mala and Á. Száz, *Properly topologically conjugated relators*, Pure Math. Appl., Ser. B **3** (1992), 119–136.
11. J. Mala and Á. Száz, *Modifications of relators*, Acta Math. Hungar. **77** (1997), 69–81.
12. G. Pataki, *Supplementary notes to the theory of simple relators*, Radovi Mat. **9** (1999), 101–118.
13. G. Pataki, *On the extensions, refinements and modifications of relators*, Math. Balk. **15** (2001), 155–186.
14. G. Pataki and Á. Száz, *A unified treatment of well-chainedness and connectedness properties*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **19** (2003), 101–165.
15. W. J. Pervin, *Quasi-uniformization of topological spaces*, Math. Ann. **147** (1962), 316–317.
16. Cs. Rakaczki and Á. Száz, *Semicontinuity and closedness properties of relations in relator spaces*, Mathematica (Cluj) **45** (2003), 73–92.
17. M. J. Riguet, *Quelques propriétés des relations difonctionnelles*, C. R. Acad. Sci. Paris **230** (1950), 1999–2000.
18. Á. Száz, *Basic tools and mild continuities in relator spaces*, Acta Math. Hungar. **50** (1987), 177–201.
19. Á. Száz, *Projective and inductive generations of relator spaces*, Acta Math. Hungar. **53** (1989), 407–430.

20. Á. Száz, *The fat and dense sets are more important than the open and closed ones*, Abstracts of the Seventh Prague Topological Symposium, Inst. Math. Czechoslovak Acad. Sci., 1991, p. 106.
21. Á. Száz, *Relators, Nets and Integrals*, Unfinished doctoral thesis, Debrecen, 1991, 126 pp.
22. Á. Száz, *Structures derivable from relators*, Singularité **3** (1992), 14–30.
23. Á. Száz, *Refinements of relators*, Tech. Rep., Inst. Math., Univ. Debrecen **76** (1993), 19 pp.
24. Á. Száz, *An extension of Kelley's closed relation theorem to relator spaces*, Filomat **14** (2000), 49–71.
25. Á. Száz, *Somewhat continuity in a unified framework for continuities of relations*, Tatra Mt. Math. Publ. **24** (2002), 41–56.
26. Á. Száz, *Upper and lower bounds in relator spaces*, Serdica Math. J. **29** (2003), 239–270.
27. Á. Száz, *Rare and meager sets in relator spaces*, Tatra Mt. Math. Publ. **28** (2004), 75–95.
28. Á. Száz, *Galois-type connections on power sets and their applications to relators*, Tech. Rep., Inst. Math., Univ. Debrecen **2005/2**, 38 pp.
29. Á. Száz, *Minimal structures, generalized topologies, and ascending systems should not be studied without generalized uniformities*, Filomat **21** (2007), 87–97.
30. Á. Száz, *Galois type connections and closure operations on preordered sets*, Acta Math. Univ. Comen. **78** (2009), 1–21.
31. Á. Száz, *Galois-type connections and continuities of pairs of relations*, J. Int. Math. Virt. Inst. **2** (2012), 39–66.
32. Á. Száz, *Lower semicontinuity properties of relations in relator spaces*, Adv. Stud. Contemp. Math. (Kyungshang) **23** (2013), 107–158.
33. Á. Száz, *Inclusions for compositions and box products of relations*, J. Int. Math. Virt. Inst. **3** (2013), 97–125.
34. Á. Száz, *A particular Galois connection between relations and set functions*, Acta Univ. Sapientiae, Math. **6** (2014), 73–91.
35. Á. Száz, *Generalizations of Galois and Pataki connections to relator spaces*, J. Int. Math. Virt. Inst. **4** (2014), 43–75.
36. Á. Száz, *Galois and Pataki connections on generalized ordered sets*, Tech. Rep., Inst. Math., Univ. Debrecen **2014/3**, 27 pp.
37. Á. Száz, *Remarks and problems at the Conference on Inequalities and Applications, Hajdúszoboszló, Hungary, 2014*, Tech. Rep., Inst. Math., Univ. Debrecen, **2014/5**, 12 pp.
38. Á. Száz, *The Debrecen Program of Continuity*, Seminar on Analysis, Department of Mathematics, University of Debrecen, May 20, 2015.
39. Á. Száz, *Basic tools, increasing functions, and closure operations in generalized ordered sets*, In : P. M. Pardalos and Th. M. Rassias(Eds.), Contributions in Mathematics and Engineering: In Honor of Constantin Caratheodory, Springer, 2016, 551–616.
40. Á. Száz and A. Zakaria, *Mild continuity properties of relations and relators in relator spaces*, In: P. M. Pardalos and Th. M. Rassias (Eds.), Essays in Mathematics and its Applications: In Honor of Vladimir Arnold, Springer, 2016, 439–511.
41. A. Weil, *Sur les espaces á structure uniforme et sur la topologie générale*, Actual. Sci. Ind. 551, Herman and Cie, Paris, 1937.

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