An Approximation Theorem for A_p-weights

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Abstract The main results establishes a general approximation theorem for A_p weights $(1 \le p < \infty)$ by means of weights which are bounded away from 0 and infinity.

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1 Introduction

Let Ω be an open set in $\mathbb{R}^N$. We denote by $\mathcal{W}(\Omega)$ the set of all measurable a.e. in Ω, positive, finite and locally integrable functions $\omega = \omega(x)$, $x \in \Omega$ $(0 < \omega(x) < \infty$ a.e. in $\Omega)$. Elements of $\mathcal{W}(\Omega)$ will be called *weight* functions.

Every weight $w \in \mathcal{W}(\Omega)$ gives rise to a measure on the measurable subsets of $\mathbb{R}^N$ through integration. This measure will also denoted by ω. Thus $\omega(E) = \int_E \omega(x)\,dx$ for measurable sets $E \subset \mathbb{R}^N$.

Our main result provides a general approximation theorem for A_p weights (see Definition 1.2) $(1 \le p < \infty)$ by means of weights which are bounded away from 0 and infinity and whose A_p-constants depend only on the A_p-constant of ω.

Definition 1.1 *Let $\Omega \subset \mathbb{R}^N$ an open set and $\omega \in \mathcal{W}(\Omega)$. For $1 \le p < \infty$ we define $L^p(\Omega, \omega)$ as the set of measurable functions u on Ω such that*

$$\|u\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |u|^p \,\omega\, dx \right)^{1/p}.$$

The class of A_p weight was introduced by B. Muckenhoupt (see [3]), where he showed that the A_p weights are precisely those weights w for which the Hardy-Littlewood maximal operator is bounded from $L^p(\mathbb{R}^N, \omega)$ to $L^p(\mathbb{R}^N, \omega)$ $(1 \le p < \infty)$, that is

$$M : L^p(\mathbb{R}^N, \omega) \to L^p(\mathbb{R}^N, \omega)$$
$$(Mf)(x) = \sup_{r>0} \frac{1}{|B(x;r)|} \int_{B(x;r)} |f(y)|\, dy,$$

is bounded if and only if $\omega \in A_p$ $(1 \le p < \infty)$ (where $B(x;r) = \{z \in \mathbb{R}^N : |z - x| < r\}$ and $|.|$ denotes the N-dimensional Lebesgue measure in $\mathbb{R}^N$).

Definition 1.2 *Let $1 \le p < \infty$. A weight ω is said to be an A_p-weight, if there is a positive constant C such that, for every ball $B \subset \mathbb{R}^N$,*

$$\left(\frac{1}{|B|} \int_B \omega\, dx \right) \left(\frac{1}{|B|} \int_B \omega^{-1/(p-1)}\, dx \right)^{p-1} \le C, \text{ if } 1 < p < \infty,$$
$$\left(\frac{1}{|B|} \int_B \omega\, dx \right) \left(\operatorname{ess\,sup}_{x \in B} \frac{1}{\omega(x)} \right) \le C, \text{ if } p = 1.$$

for all balls $B \subset \mathbb{R}^N$. The infimum over all such constants C is called the A_p constant of ω and this constant will be denoted by $C_{p,\omega}$.

The union of all Muckenhoupt class A_p is denoted by A_∞, i.e., $A_\infty = \bigcup_{p \geq 1} A_p$.

If $\omega \in A_p$, $1 \leq p < \infty$, then since $\omega^{-1/(p-1)}$ is locally integrable (when $p > 1$) and ω^{-1} is locally bounded (when $p = 1$) we have

$$L^p(\Omega, \omega) \subset L^1_{\text{loc}}(\Omega)$$

for every open set $\Omega \subseteq \mathbb{R}^N$. Moreover, if $\omega \in A_p$, $p > 1$, then by writing $1 = \omega^{1/p}\omega^{-1/p}$, Hölder inequality implies that, for every ball B,

$$\begin{aligned} 1 &= \frac{1}{|B|}\int_B 1\,dx = \frac{1}{|B|}\int_B \omega^{1/p}\omega^{-1/p}\,dx \\ &\leq \frac{1}{|B|}\left(\int_B \omega\,dx\right)^{1/p}\left(\int_B \omega^{-p'/p}\,dx\right)^{1/p'} \\ &= \left(\frac{1}{|B|}\int_B \omega\,dx\right)\left(\frac{1}{|B|}\int_B \omega^{-1/(p-1)}\,dx\right)^{(p-1)/p} \\ &\leq C_{p,\omega}^{1/p}, \end{aligned}$$

i.e., the A_p constant of ω $C_{p,\omega} \geq 1$. Analogously, if $\omega \in A_1$ we also have $C_{1,\omega} \geq 1$.

Example *(i) If $x \in \mathbb{R}^N$, $\omega(x) = |x|^\alpha$ is in A_p if and only if $-N < \alpha\, N(p-1)$ (see Corollary 4.4 in [4]).*

(ii) $\omega(x) = \mathrm{e}^{\lambda\,\varphi(x)}\, A_2$, with $\varphi \in W^{1,N}(\Omega)$ (Sobolev space) and λ is sufficiently small (see Corollary 2.18 in [1]).

(iii) There is a connection between A_p and BMO (the John-Nirenberg space of functions with bounded mean oscillation). In fact, if ω is a weight, then $\ln(\omega) \in BMO$ if and only if there is a constant $\eta > 0$ such that $\omega^\eta \in A_2$ (see Proposition 6.1, Chapter IX in [4]).

Remark 1.1 *(i) The A_p condition is invariant under translations and dilations, i.e., if $\omega \in A_p$, then the weights $v_1(x) = \omega(x+a)$ and $v_2(x) = \omega(\delta\, x)$ ($a \in \mathbb{R}^N$ and $\delta > 0$ fixed), both belong to A_p with the same A_p constants as ω.*

(ii) As it sometimes is more convenient to work with cubes than balls, it is useful to notice that if one replaces the balls in the definitions of A_p with cubes, one gets the same class of weights and the different A_p constants are comparable.

The proof of Theorem 2.1 (main result) is based on the Calderon-Zygmund Decomposition performed for dyadic cubes of $\mathbb{R}^N$. Throughout this paper, however, we use balls instead of cubes in our estimates. This deviation from the standard proof causes only minor technical complications (see for example the Calderon-Zygmund decomposition for doubling weights in [2]).

Lemma 1.1 *If $1 \leq p < q < \infty$, then $A_1 \subset A_p \subset A_q$, and the A_q constant of a weight $\omega \in A_p$ equals the A_p constant of ω.*

Proof. We just need to observe that, if $1 < p < q$, since $(q-1)/(p-1) > 1$, by Hölder inequality we have that

$$\begin{aligned} \left(\frac{1}{|B|}\int_B \omega^{-1/(q-1)}\,dx\right)^{q-1} &\leq \left(\frac{1}{|B|}\int_B \omega^{-1/(p-1)}\,dx\right)^{p-1} \\ &\leq \operatorname*{ess\,sup}_{x \in B} \frac{1}{\omega(x)}. \end{aligned}$$

□

Lemma 1.2 (Strong doubling property) *If $\omega \in A_p$, then*

$$\left(\frac{|E|}{|B|}\right)^p \leq C_{p,\omega}\,\frac{\omega(E)}{\omega(B)}$$

whenever B is a ball in $\mathbb{R}^N$ and E is a measurable subset of B (where $\omega(E) = \int_E \omega(x)\,dx$).

Proof. We have, for $1 < p < \infty$,

$$\begin{aligned}
|E| &= \int_E 1\,dx = \int_E \omega^{1/p}\,\omega^{-1/p}\,dx \\
&\leq \left(\int_E \omega\,dx\right)^{1/p}\left(\int_E \omega^{-p'/p}\,dx\right)^{1/p'} \\
&= \left(\int_E \omega\,dx\right)^{1/p}\left(\int_E \omega^{1/(p-1)}\,dx\right)^{(p-1)/p} \\
&\leq [\omega(E)]^{1/p}\left(\frac{1}{|B|}\int_B \omega^{1/(p-1)}\,dx\right)^{(p-1)/p}|B|^{(p-1)/p} \\
&\leq C_{p,\omega}^{1/p}[\omega(E)]^{1/p}\left(\int_B \frac{1}{|B|}\int_B \omega\,dx\right)^{-1/p}|B|^{(p-1)/p} \\
&= C_{p,\omega}^{1/p}\,\frac{[\omega(E)]^{1/p}}{[\omega(B)]^{1/p}}\,\frac{|B|^{(p-1)/p}}{|B|^{-1/p}} \\
&= C_{p,\omega}^{1/p}\left(\frac{\omega(E)}{\omega(B)}\right)^{1/p}|B|.
\end{aligned}$$

Hence, $\left(\frac{|E|}{|B|}\right)^p \leq C_{p,\omega}\,\frac{\omega(E)}{\omega(B)}$. □

Lemma 1.3 *If $\omega \in A_p$, then $\omega(B(x,2r)) \leq C\,\omega(B(x,r))$ for all balls $B = B(x,r)$ in $\mathbb{R}^N$, and $C = 2^{Np}\,C_{p,\omega}$.*

Proof. Since $B(x,r) \subset B(x,2r)$, then (by Lemma 1.2)

$$\left(\frac{|B(x,r)|}{|B(x,2r)|}\right)^p \leq C_{p,\omega}\,\frac{\omega(B(x,r))}{\omega(B(x,2r))},$$

and we obtain $\omega(B(x,2r)) \leq 2^{Np}\,C_{p,\omega}\,\omega(B(x,r)$. Hence, if $\omega \in A_p$, then ω is a doubling measure. □

Theorem 1.1 (The Jones Factorization Theorem) *For $1 < p < \infty$. Then $\omega \in A_p$ if and only if $\omega = \omega_0\,\omega_1^{1-p}$, where ω_0 and ω_1 are A_1 - weights.*

Proof. See Corollary 5.3, Chapter IV in [1]. □

2 Main Results

For $k \in \mathbb{Z}$, we consider the lattice $\Gamma_k = 2^{-k}\mathbb{Z}^N$ formed by those points of $\mathbb{R}^N$ whose coordinate are integral multiples of 2^{-k}, i.e.,

$$\Gamma_k = \{x = (x_1, ..., x_N) \in \mathbb{R}^N : x_j = p2^{-k},\ p \in \mathbb{Z}, j = 1, ..., N\}.$$

Let D_k be a collection of cubes determined by Γ_k, that is, those cubes with side length 2^{-k} and vertices in Γ_k. The cubes belonging to $D = \bigcup_{k=-\infty}^{\infty} D_k$ are called *dyadic cubes*. Note that if $Q_1, Q_2 \in D$ and $|Q_1| \leq |Q_2|$, then either $Q_1 \subset Q_2$ or else Q_1 and Q_2 do not overlap (by which we mean that their interiors are disjoint).

Theorem 2.1 *Let $\alpha, \beta > 1$ be given and let $\omega \in A_p$ $(1 \le p < \infty)$, with A_p-constant $C(\omega, p)$. Then there exist weights $\omega_{\alpha\beta} \ge 0$ a.e. such that the following conditions are met.*

(**i**) $c_1(1/\beta) \le \omega_{\alpha\beta} \le c_2\, \alpha$ in Ω, *where c_1 and c_2 depend only on ω and Ω.*

(**ii**) *There exist weights $\tilde{\omega}_1$ and $\tilde{\omega}_2$ such that $\tilde{\omega}_1 \le \omega_{\alpha\beta} \le \tilde{\omega}_2$, where $\tilde{\omega}_i \in A_p$ and $C(\tilde{\omega}_i, p)$ depends only on $C(\omega, p)$ $(i = 1, 2)$.*

(**iii**) $\omega_{\alpha\beta} \in A_p$, *with constant $C(\omega_{\alpha\beta}, p)$ depending only on $C(\omega, p)$ uniformly on α and β.*

(**iv**) *There exists a closed set $F_{\alpha\beta}$ such that $\omega_{\alpha\beta} \equiv \omega$ in $F_{\alpha\beta}$ and $\omega_{\alpha\beta} \approx \tilde{\omega}_1 \approx \tilde{\omega}_2$ in $F_{\alpha\beta}$ with equivalence constants depending on α and β (i.e., there are positive constants $c_{\alpha\beta}$ and $C_{\alpha\beta}$ such that*

$$c_{\alpha\beta}\, \tilde{\omega}_i \le \omega_{\alpha\beta} \le C_{\alpha\beta}\, \tilde{\omega}_i,$$

$i = 1, 2$).Moreover, $F_{\alpha\beta} \subset F_{\alpha'\beta'}$ if $\alpha \le \alpha'$, $\beta \le \beta'$, and the complement of $\bigcup_{\alpha,\beta \ge 1} F_{\alpha\beta}$ has zero measure.

(**v**) $\omega_{\alpha\beta} \to \omega$ *a.e. in $\mathbb{R}^N$ as $\alpha, \beta \to \infty$.*

Proof. Case 1.: Suppose first $\omega \in A_1$. Since we are interested to approximate in Ω, we may assume, without loss of generality, that $\omega \in L^1(\mathbb{R}^N)$. For each $\alpha > 1$, we define

$$U_\alpha^+ = \{x \in \mathbb{R}^N \ : \ M(\omega)(x) > \alpha\}$$

where $M(\omega)$ is the usual Hardy-Littlewood maximal operator for ω, i.e. $(M\omega)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q \omega(y)\, dy$, where the supremum is taken over all cubes Q containing x (cube will always mean a compact cube with sides parallel to the axes and nonempty interior). By Calderon-Zygmund decomposition (see Theorem 1.12, Chapter II in [1]), there exists a family of non-overlapping cubes $\{Q_j^\alpha\}$ consisting of those maximal dyadic cubes over which the average of ω is greater than α, i.e.,

$(CZ1)$
$$U_\alpha^+ = \bigcup_{j=1}^{\infty} Q_j^\alpha,$$

$(CZ2)$
$$\alpha < \frac{1}{|Q_j^\alpha|} \int_{Q_j^\alpha} \omega(x) dx \le 2^N \alpha,$$

$(CZ3)$
$$\omega(x) \le \alpha, \ \forall x \in F_\alpha^+ = (U_\alpha^+)^c,$$

$(CZ4)$
$$|U_\alpha^+| \le \frac{c}{\alpha} \int_{\mathbb{R}^N} \omega(x) dx.$$

Then,

$$\{x \in \mathbb{R}^N \ : \ M(\omega)(x) > 4^N \alpha\} \subset \bigcup_j 3Q_j^\alpha. \tag{1}$$

We explicitly note that, if $\alpha < \beta$, then $U_\beta^+ \subset U_\alpha^+$.

Define the weights ω_α by

$$\omega_\alpha(x) = \sum_{k=1}^{\infty} \left(\frac{1}{|Q_k^\alpha|} \int_{Q_k^\alpha} \omega(y) dy \right) \chi_{Q_k^\alpha}(x) + \omega(x) \chi_{F_\alpha^+}(x),$$

where χ_E denotes the characteristic function of a measurable set $E \subset \mathbb{R}^N$.

We will show that

(I) $\omega_\alpha \in A_1$ and $C(\omega_\alpha, 1)$ depends only on $C(\omega, 1)$.

(II) $\omega_\alpha \to \omega$ a.e. in $\mathbb{R}^N$ as $\alpha \to \infty$.

(III) $\min\{1, \omega\} \in A_1$ and $C(\min\{1, \omega\}, 1)$ depends only on $C(\omega, 1)$.

Moreover, we also have $\min\{1, \omega\}(x) \le \omega_\alpha(x) \le C\,\omega(x)$, where C depends only on $C(\omega, 1)$.

(IV) $2^N \alpha \ge \omega_\alpha(x) \ge C = \dfrac{1}{|Q_0|} \displaystyle\int_{Q_0} \min\{1, \omega\}(y)dy$, $x \in \Omega$ and Q_0 is a fixed cube containing Ω ($\Omega \subset Q_0$).

First of all we note that, if $\omega \in L^1_{\text{loc}}$ is a weight function such that for any dyadic cubes Q we have $\omega(3Q) \le C_0\, \omega(Q)$ (C_0 depends only on N and $C(\omega, 1)$), then we can restrict ourselves to test the A_1 condition only on dyadic cubes. In fact, denoting by $C^*(\omega, 1)$ the *constant* A_1 *for dyadic cubes* , if $f \in L^1(\mathbb{R}^N, \omega)$ and $\{C_j^t\}$ is the Calderon-Zygmung decomposotion for f, we have

$$
\begin{aligned}
\omega(\{x \ : \ M(f)(x) > 4^N t\}) &\le \sum_j \omega(3C_j^t) \\
&\le C_0 \sum_j \omega(C_j^t) \\
&\le C_0\, C^*(\omega, 1) \sum_j |C_j^t| \operatorname{ess\,inf}_{C_j^t} \omega \\
&\le C_0 C^*(\omega, 1) \frac{1}{t} \sum_j \int_{C_j^t} |f(x)|\, \omega(x)\, dx \\
&\le C_0\, C^*(\omega, 1)\, \frac{1}{t} \int_\Omega |f(x)|\, \omega(x)\, dx,
\end{aligned}
$$

and hence $\omega \in A_1$ by Theorem 2.1, Chapter IV in [1] and $C(\omega, 1) = C_0\, C^*(\omega, 1)$.

Thus, let us prove first that $\omega_\alpha(3Q) \le C_0\, \omega_\alpha(Q)$ for any dyadic cube Q and for any $\alpha > 1$, where C_0 depends only on N and $C(\omega, 1)$. Let Q be a dyadic cube. We have either $|Q \cap U_\alpha^+| < \frac{1}{2}|Q|$ or $|Q \cap U_\alpha^+| \ge \frac{1}{2}\,|Q|$. First, let us suppose that $|Q \cap U_\alpha^+| < \frac{1}{2}|Q|$. We will prove later, in (III), that $\omega_\alpha \le \max\{1, C(\omega, 1)\}\omega = C_1 \omega$; we have

$$
\begin{aligned}
\omega_\alpha(3Q) &\le C_1\, \omega(3Q) = C_1\, \frac{|3Q|}{|3Q|} \int_{3Q} \omega(x)\, dx \\
&= 3^N C_1 |Q| \frac{1}{|3Q|} \int_{3Q} \omega(x)\, dx \\
&\le C_1\, C(\omega, 1) 3^N |Q| \operatorname{ess\,inf}_{3Q} \omega \\
&\le C_1\, C(\omega, 1) 3^N |Q| \operatorname{ess\,inf}_{Q \cap F_\alpha^+} \omega \\
&= C_1\, C(\omega, 1) 3^N \frac{|Q|}{|Q \cap F_\alpha^+|} |Q \cap F_\alpha^+| \operatorname{ess\,inf}_{Q \cap F_\alpha^+} \omega \\
&\le 2C_1\, C(\Omega, 1) 3^N |Q \cap F_\alpha^+| \operatorname{ess\,inf}_{Q \cap F_\alpha^+} \omega \\
&\le 2C_1\, C(\omega, 1) 3^N \int_{Q \cap F_\alpha^+} \omega(x)\, dx \\
&= 2C_1\, C(\omega, 1) 3^N \int_{Q \cap F_\alpha^+} \omega_\alpha(x)\, dx,
\end{aligned}
$$

since $\omega_\alpha \equiv \omega$ on F_α^+, and therefore

$$\begin{aligned}\omega_\alpha(3Q) &\leq 2C_1\, C(\omega,1)3^N \int_{Q\cap F_\alpha^+} \omega_\alpha(x)\, dx\\ &\leq 2C_1\, C(\omega,1)3^N \omega(Q)\\ &= C_0\; \omega(Q).\end{aligned}$$

and hence, in this case we are done.

Suppose now $|Q \cap U_\alpha^+| \geq \frac{1}{2}\,|Q|$. Then either $Q \subset Q_{j_0}^\alpha$ for some j_0 (which in turn is unique) or $Q_j^\alpha \subset Q$ for $j \in \mathcal{J}$ (a set of indices). By definition of ω_α, it follows from (CZ2) e (CZ3) that $\omega_\alpha(x) \leq 2^N\alpha$ a.e.. Therefore, if $Q \subset Q_{j_0}^\alpha$ we get

$$\begin{aligned}\omega_\alpha(3Q) &= \int_{3Q} \omega_\alpha(x)dx\\ &\leq |3Q|2^N\alpha\\ &= 3^N\, 2^N\, \alpha|Q|\\ &\leq 3^N\, 2^n|Q|\frac{1}{|Q_{j_0}^\alpha|}\int_{Q_{j_0}^\alpha}\omega(x)\, dx\\ &= 6^N\int_Q \omega_\alpha(x)dx = 6^N\omega_\alpha(Q),\end{aligned}$$

since $\omega_\alpha \equiv \frac{1}{|Q_{j_0}^\alpha|}\int_{Q_{j_0}^\alpha}\omega(x)\, dx$ on Q. Otherwise, we obtain

$$\begin{aligned}\omega_\alpha(3Q) &= \int_{3Q}\omega_\alpha(x)\, dx\\ &\leq 2^N\alpha|3Q|\\ &= 2^N\, 3^N\alpha|Q|\\ &\leq 6^N\, 2|Q\cap U_\alpha^+|\alpha\\ &= 2\, 6^N\int_{Q\cap U_\alpha^+}\alpha\, dx\\ &= 2\, 6^N\sum_{j\in\mathcal{J}}\int_{Q_j^\alpha}\alpha\, dx\\ &\leq 2\, 6^N\sum_{j\in\mathcal{J}}\int_{Q_j^\alpha}\omega_\alpha(x)\, dx\\ &\leq 2\, 6^N\int_{\cup Q_j^\alpha}\omega_\alpha(x)\, dx\\ &\leq 2\, 6^N\int_Q\omega_\alpha(x)\, dx = 2\, 6^N\omega_\alpha(Q).\end{aligned}$$

Therefore $\omega_\alpha(3Q) \leq C_0\, \omega_\alpha(Q)$, $\forall Q$ dyadic cube.

Now we are ready to prove (I). Let Q be a fixed dyadic cube, then one of the three cases can happen:

(I1) $Q \cap Q_j^\alpha = \emptyset,\ \forall j$;

(I2) $Q \subset Q_j^\alpha$ for one and only one j;

(I3) $Q_j^\alpha \subset Q$ for some index $j \in \mathcal{J}$.

In case (I1), $Q\subset F^+_\alpha$ and hence $\omega_\alpha\equiv\omega$ in Q and we are done.

In case (I2), we have

$$\omega_\alpha(Q)=\int_Q\omega_\alpha(x)\,dx=|Q|\bigg(\frac{1}{|Q^\alpha_j|}\int_{Q^\alpha_j}\omega\,dx\bigg)=|Q|\text{ess}\inf_{x\in Q}\omega_\alpha(x),$$

since $\omega_\alpha=\dfrac{1}{|Q^\alpha_j|}\displaystyle\int_{Q^\alpha_j}\omega(x)\,dx$ over Q^α_j.

Finally in case (I3), we have

$$\begin{aligned}\omega_\alpha(Q)&=\int_Q\omega_\alpha(x)dx=\int_{Q\cap U^+_\alpha}\omega_\alpha(x)\,dx+\int_{Q\cap F^+_\alpha}\omega_\alpha(x)\,dx\\&=\sum_{j\in\mathcal{J}}\int_{Q^\alpha_j}\omega_\alpha(x)\,dx+\omega_\alpha(Q\cap F^+_\alpha)\\&=\sum_{j\in\mathcal{J}}\bigg(\frac{1}{|Q^\alpha_j|}\int_{Q^\alpha_j}\omega(x)dx\bigg)|Q^\alpha_j|+\omega(Q\cap F^+_\alpha)\\&=\sum_{j\in\mathcal{J}}\int_{Q\cap Q^\alpha_j}\omega(x)\,dx+\omega(Q\cap F^+_\alpha)\\&\le\omega(Q)\\&\le C(\omega,1)\,|Q|\text{ess}\inf_{y\in Q}\omega(y).\end{aligned}$$

On the other hand we note that if $y\in U^+_\alpha$, by definition we have $\omega_\alpha(y)>\alpha$. Thus, if $y\in Q\cap U^+_\alpha$ e Q^α_k and Q^α_k is any cube contained in Q we have

$$\text{ess}\inf_Q\omega\le\text{ess}\inf_{Q^\alpha_k}\omega\le\frac{1}{|Q^\alpha_k|}\int_{Q^\alpha_k}\omega(x)\,dx\le 2^N\,\alpha<2^n\omega_\alpha(y).$$

In addition, if $y\in Q\cap F^+_\alpha$, $\omega_\alpha(y)=\omega(y)\geq\text{ess}\inf_Q\omega$.

Hence, $\text{ess}\inf\limits_Q\omega\le 2^N\text{ess}\inf\limits_Q\omega_\alpha(y)$.

Therefore, in both cases,

$$\omega_\alpha(Q)\le C(\omega,1)|Q|\text{ess}\inf_{y\in Q}\omega(y)\le 2^N\,C(\omega,1)|Q|\text{ess}\inf_{y\in Q}\omega_\alpha(y),$$

that is,

$$\frac{1}{|Q|}\int_Q\omega_\alpha(x)\,dx\le 2^N\,C(\omega,1)\text{ess}\inf_{y\in Q}\omega_\alpha(y).$$

Consequently, in (I1), (I2) and (I3) we have

$$\frac{1}{|Q|}\int_Q\omega_\alpha(x)\,dx\;\approx\;\text{ess}\inf_Q\omega_\alpha$$

i.e., $\omega_\alpha\in A_1$ and (I) is proved.

To prove (II), by definition we have

$$\omega_\alpha(x)=\sum_{j=1}^\infty\bigg(\frac{1}{|Q^\alpha_j|}\int_{Q^\alpha_j}\omega(x)\,dx\bigg)\chi_{Q^\alpha_j}(x)+\omega(x)\chi_{F^+_\alpha}(x).$$

Hence, $\omega_\alpha \equiv \omega$ in F_α^+ and that F_α^+ increases as α tends to infinity. Moreover, $|\cap (F_\alpha^+)^c| = 0$. Then $\omega_\alpha \to \omega$ a.e. as $\alpha \to \infty$.

Finally, to show (III) we know that, for any cube Q, either $\operatorname{ess\,inf}_Q \omega \geq 1$ or $\operatorname{ess\,inf}_Q \omega < 1$. In the first case,

$$\frac{1}{|Q|}\int_Q \min\{\omega,1\}(x)\,dx = \frac{1}{|Q|}\int_Q 1dx = 1 \leq \min\{\omega(y),1\}$$

for any $y \in Q$, whereas if $\inf_Q \omega < 1$ then

$$\frac{1}{|Q|}\int_Q \min\{\omega,1\}(x)\,dx \leq \frac{1}{|Q|}\int_Q \omega(x)\,dx \leq C(\omega,1)\operatorname{ess\,inf}_Q \omega,$$

since $\omega \in A_1$. Put $\lambda = \operatorname{ess\,inf}_Q \omega < 1$ and assume by contradiction that $\inf_Q(\min\{\omega,1\}) < \lambda$; then there exists $E \subset Q$, $|E| > 0$ such that $\min\{\omega,1\} < \lambda' < \lambda$ in E and hence , since $\lambda < 1$, we obtain $\omega < \lambda'$ in E which is a contradiction. Thus we have proved the first part of (III), that is, $\min\{\omega,1\} \in A_1$.

To prove the second part we note that if $x \in F_\alpha^+$ then

$$\omega_\alpha(x) = \omega(x) \geq \min\{\omega,1\}(x).$$

If $x \in Q_j^\alpha$, for some j, then by definition and (CZ2), we get

$$\omega_\alpha(x) = \frac{1}{|Q_j^\alpha|}\int_{Q_j^\alpha} \omega(y)dy > \alpha \geq \min\{\omega,1\}(x).$$

Analogously, if $x \in Q_j^\alpha$, for some j, then

$$\omega_\alpha(x) \leq C(\omega,1)\operatorname{ess\,inf}_{Q_j^\alpha} \omega \leq C(\omega,1)\omega(x).$$

Therefore, $\min\{\omega,1\} \leq \omega_\alpha(x) \leq C(\omega,1)\omega(x)$.

Finally, assertion (IV) follows straightforwardly from (III) by using (CZ1) and (CZ3).

Case 2. Suppose now $\omega \in A_p$, for $p > 1$. Then by Jones's Factorization Theorem (Theorem 1.1) there exist ω_0, $\omega_1 \in A_1$ such that $\omega = \omega_0\omega_1^{1-p}$, and $C(\omega_i,1)$ $(i = 0,1)$ depends only on $C(\omega,p)$. Choose α, $\beta > 1$ and define $\omega_{\alpha\beta}$ by

$$\omega_{\alpha\beta} = (\omega_0)_\alpha \left[(\omega_1)_{\beta^{1/(p-1)}}\right]^{1-p}.$$

We need to show that $\omega_{\alpha\beta}$ satisfies properties (i) - (v).

Obviously (i) follows from (IV).

(ii) follows from (III) and Jones's Factorization Theorem, with

$$\tilde{\omega}_1 = \min\{\omega_0,1\}\omega_1^{1-p},$$

$$\tilde{\omega}_2 = \omega_0(\min\{\omega_1, 1\})^{1-p}.$$

(iii) follows from (I) and Jones's Factorization Theorem.

(v) follows from (II).

For (iv), we define

$$F_{\alpha\beta} = F_\alpha^0 \cap F^1_{\beta^{1/(p-1)}}$$

where $F_\alpha^0 = F_\alpha^+$ for the weight ω_0 and $F^1_{\beta^{1/(p-1)}} = \left(F_{\beta^{1/(p-1)}}\right)^+$ for the weight ω_1. By definition, $(\omega_0)_\alpha \equiv \omega_0$ and $(\omega_1)_{\beta^{1/(p-1)}} \equiv \omega_1$ on $F_{\alpha\beta}$.

Hence, to prove that $\omega_{\alpha\beta} \approx \tilde{\omega}_1$, we can replace $\omega_{\alpha\beta}$ by ω (on $F_{\alpha\beta}$). We have $\min\{\omega_0, 1\} \approx \omega_0$ on F_α^0, since $\min\{\omega_0, 1\} \leq \omega_0$ (by (III)). Moreover, if $x \in F_\alpha^0$, $\min\{\omega_0(x), 1\} = 1$. Hence, by (CZ3) we obtain

$$\omega_0(x) \leq \alpha = \alpha \min\{\omega_0(x), 1\}.$$

Therefore, $\min\{\omega_0, 1\} \leq \omega_0 \leq \alpha \min\{\omega_0, 1\}$ on F_α^0 .

An analogous argument shows that $\omega_1 \approx \min\{\omega_1, 1\}$ on $F^1_{\beta^{1/(p-1)}}$, and hence,

$$\omega_{\alpha\beta} \approx \min\left\{\omega_0, 1\right\}\left[(\omega_1)_{\beta^{1/(p-1)}}\right]^{1-p} = \tilde{\omega}_1 \quad \text{on } F_{\alpha\beta},$$

and analogously, we also have $\omega_{\alpha\beta} \approx \tilde{\omega}_2$. □

3 Conclusions

We prove a general approximation theorem for A_p weights $(1 \leq p < \infty)$ by means of weights which are bounded away from 0 and infinity and whose A_p-constants depend only on the A_p-constants of ω.

Conflicts of Interest

The author declares that there is no conflict of interest in publishing this paper.

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