Analysis of Exact Solutions to Some Systems Of Difference Equations

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Abstract

Some nonlinear difference equations can be sometimes solved analytically using manual iteration which begins with some given initial conditions. Obtaining next iterations always depends on the previous ones. Through this paper, we utilize the manual iteration in investigating the exact solutions of the following recursive sequences

$$x_{n+1} = \frac{y_{n-5}x_{n-8}}{y_{n-2}(-1-y_{n-5}x_{n-8})}, \quad y_{n+1} = \frac{x_{n-5}y_{n-8}}{x_{n-2}(\pm 1 \pm x_{n-5}y_{n-8})},$$

where the initial conditions x_δ, y_δ, $\delta \in \{0, 1, ..., 8\}$ are non-zero real numbers. Some numerical solutions are also presented in some figures to show the behaviour of the solutions.

Keywords: Difference Equations, Systems of Difference Equations, Exact Solution, Stability, Periodicity.
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1 Introduction

The investigation of dynamical systems of recursive equations has been rapidly increased in recent years. This can be attributed to the fact that most discrete and natural phenomena, which appear in physics, biology, economics, computer sciences, etc, can be modelled by using difference equations. Furthermore, the majority of nonlinear differential equations can be approximated into difference relations or systems. Among many existing studies, we mention the following ones. Almatrafi and Elsayed [1] discovered the theoretical and numerical solutions of the following systems of rational difference equations:

$$x_{n+1} = \frac{y_{n-1}x_{n-3}}{y_{n-1}(1 + y_{n-1}x_{n-3})}, \quad y_{n+1} = \frac{x_{n-1}y_{n-3}}{x_{n-1}(\pm 1 \pm x_{n-1}y_{n-3})}.$$

Elsayed and Alzubaidi [2] obtained the solutions' structures of the following systems of fractional recursive relations:

$$x_{n+1} = \frac{y_{n-8}}{1 + y_{n-2}x_{n-5}y_{n-8}}, \quad y_{n+1} = \frac{x_{n-8}}{\pm 1 \pm x_{n-2}y_{n-5}x_{n-8}}.$$

Din [3] explored the local asymptotic behaviour and global stability of the equilibrium points of the following discrete Lotka-Volterra model:

$$x_{n+1} = \frac{\alpha x_n - \beta x_n y_n}{1 + \gamma x_n}, \quad y_{n+1} = \frac{\delta y_n + \epsilon x_n y_n}{1 + \eta y_n}.$$

In [4], Gűműş and Őcalan analysed the behaviour of the positive solutions of the following system of fractional recursive equations:

$$u_{n+1} = \frac{\alpha u_{n-1}}{\beta + \gamma v_n^p v_{n-2}^q}, \quad v_{n+1} = \frac{\alpha_1 v_{n-1}}{\beta_1 + \gamma_1 u_n^{p_1} u_{n-2}^{q_1}}.$$

Moreover, Haddad et al. [5] dealt with the solution form of the following dynamical systems of recursive equations:

$$x_{n+1} = \frac{a x_n y_{n-1}}{y_n - \alpha} + \beta, \quad y_{n+1} = \frac{b x_{n-1} y_n}{x_n - \beta} + \alpha.$$

Kurbanli et al. [6] examined the solutions the following dynamical system of recursive equations:

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}, \quad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}, \quad z_{n+1} = \frac{x_n}{y_n z_{n-1}}.$$

For more results on the qualitative behaviour of difference equations, one can see refs. [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22].

In this article, we will analyse the expressions of the solutions of the following nonlinear systems:

$$x_{n+1} = \frac{y_{n-5} x_{n-8}}{y_{n-2}(-1 - y_{n-5} x_{n-8})}, \quad y_{n+1} = \frac{x_{n-5} y_{n-8}}{x_{n-2}\left(\pm 1 \pm x_{n-5} y_{n-8}\right)},$$

where the initial conditions x_δ, y_δ, $\delta \in \{0, 1, ..., 8\}$ are required to be non-zero real numbers. We also aim to present the numerical solutions of these systems.

2 Main Results

2.1 The First System $x_{n+1} = \frac{y_{n-5} x_{n-8}}{y_{n-2}(-1 - y_{n-5} x_{n-8})}$, $y_{n+1} = \frac{x_{n-5} y_{n-8}}{x_{n-2}(1 + x_{n-5} y_{n-8})}$

The major duty of this subsection is to provide the solutions of the system on the form:

$$x_{n+1} = \frac{y_{n-5} x_{n-8}}{y_{n-2}\left(-1 - y_{n-5} x_{n-8}\right)}, \quad y_{n+1} = \frac{x_{n-5} y_{n-8}}{x_{n-2}\left(1 + x_{n-5} y_{n-8}\right)}, \tag{1}$$

where the initial data as described previously and all denominators are assumed to be non-zero.

Theorem 1 *Let $\{x_n, y_n\}$ be a solution to system (1) and assume that $x_{-8} = a$, $x_{-7} = b$, $x_{-6} = c$, $x_{-5} = d$, $x_{-4} = e$, $x_{-3} = f$, $x_{-2} = g$, $x_{-1} = h$, $x_0 = k$, $y_{-8} = l$, $y_{-7} = m$, $y_{-6} = p$, $y_{-5} = q$, $y_{-4} = r$, $y_{-3} = s$, $y_{-2} = t$, $y_{-1} = u$, and $y_0 = v$. Then, for $n = 0, 1, ...,$ we have*

$$x_{24n-8} = (-1)^n \left(\frac{gt}{l}\right)^{2n} \frac{(1+dl)^{2n}}{a^{2n-1}(1+2gq)^n(-1+dt)^n(1+dt)^n},$$
$$x_{24n-7} = (-1)^n \left(\frac{hu}{m}\right)^{2n} \frac{(1+em)^{2n}}{b^{2n-1}(1+2hr)^n(-1+eu)^n(1+eu)^n},$$
$$x_{24n-6} = (-1)^n \left(\frac{kv}{p}\right)^{2n} \frac{(1+fp)^{2n}}{c^{2n-1}(1+2ks)^n(-1+fv)^n(1+fv)^n},$$
$$x_{24n-5} = (-1)^n d \left(\frac{al}{gt}\right)^{2n} \frac{(1+gq)^{2n}}{(1+2dl)^n(-1+aq)^n(1+aq)^n},$$
$$x_{24n-4} = (-1)^n e \left(\frac{bm}{hu}\right)^{2n} \frac{(1+hr)^{2n}}{(1+2em)^n(-1+br)^n(1+br)^n},$$
$$x_{24n-3} = (-1)^n f \left(\frac{cp}{kv}\right)^{2n} \frac{(1+ks)^{2n}}{(1+2fp)^n(-1+cs)^n(1+cs)^n},$$
$$x_{24n-2} = (-1)^n \left(\frac{t}{al}\right)^{2n} \frac{g^{2n+1}(1+dl)^{2n}}{(1+2gq)^n(-1+dt)^n(1+dt)^n},$$
$$x_{24n-1} = (-1)^n \left(\frac{u}{bm}\right)^{2n} \frac{h^{2n+1}(1+em)^{2n}}{(1+2hr)^n(-1+eu)^n(1+eu)^n},$$

$$x_{24n} = (-1)^n \left(\frac{v}{cp}\right)^{2n} \frac{k^{2n+1}(1+fp)^{2n}}{(1+2ks)^n(-1+fv)^n(1+fv)^n},$$

$$x_{24n+1} = (-1)^n q \left(\frac{a}{t}\right)^{2n+1} \left(\frac{l}{g}\right)^{2n} \frac{(1+gq)^{2n}}{(1+2dl)^n(-1+aq)^n(1+aq)^{n+1}},$$

$$x_{24n+2} = (-1)^n r \left(\frac{b}{u}\right)^{2n+1} \left(\frac{m}{h}\right)^{2n} \frac{(1+hr)^{2n}}{(1+2em)^n(-1+br)^n(1+br)^{n+1}},$$

$$x_{24n+3} = (-1)^n s \left(\frac{c}{v}\right)^{2n+1} \left(\frac{p}{k}\right)^{2n} \frac{(1+ks)^{2n}}{(1+2fp)^n(-1+cs)^n(1+cs)^{n+1}},$$

$$x_{24n+4} = (-1)^{n+1} \left(\frac{gt}{l}\right)^{2n+1} \frac{(1+dl)^{2n+1}}{a^{2n}(1+2gq)^n(-1+dt)^n(1+dt)^{n+1}},$$

$$x_{24n+5} = (-1)^{n+1} \left(\frac{hu}{m}\right)^{2n+1} \frac{(1+em)^{2n+1}}{b^{2n}(1+2hr)^n(-1+eu)^n(1+eu)^{n+1}},$$

$$x_{24n+6} = (-1)^{n+1} \left(\frac{kv}{p}\right)^{2n+1} \frac{(1+fp)^{2n+1}}{c^{2n}(1+2ks)^n(-1+fv)^n(1+fv)^{n+1}},$$

$$x_{24n+7} = (-1)^n d \left(\frac{al}{gt}\right)^{2n+1} \frac{(1+gq)^{2n+1}}{(1+2dl)^{n+1}(-1+aq)^n(1+aq)^{n+1}},$$

$$x_{24n+8} = (-1)^n e \left(\frac{bm}{hu}\right)^{2n+1} \frac{(1+hr)^{2n+1}}{(1+2em)^{n+1}(-1+br)^n(1+br)^{n+1}},$$

$$x_{24n+9} = (-1)^n f \left(\frac{cp}{kv}\right)^{2n+1} \frac{(1+ks)^{2n+1}}{(1+2fp)^{n+1}(-1+cs)^n(1+cs)^{n+1}},$$

$$x_{24n+10} = (-1)^{n+1} g^{2n+2} \left(\frac{t}{al}\right)^{2n+1} \frac{(1+dl)^{2n+1}}{(1+2gq)^{n+1}(-1+dt)^n(1+dt)^{n+1}},$$

$$x_{24n+11} = (-1)^{n+1} h^{2n+2} \left(\frac{u}{bm}\right)^{2n+1} \frac{(1+em)^{2n+1}}{(1+2hr)^{n+1}(-1+eu)^n(1+eu)^{n+1}},$$

$$x_{24n+12} = (-1)^{n+1} k^{2n+2} \left(\frac{v}{cp}\right)^{2n+1} \frac{(1+fp)^{2n+1}}{(1+2ks)^{n+1}(-1+fv)^n(1+fv)^{n+1}},$$

$$x_{24n+13} = (-1)^n q \left(\frac{a}{t}\right)^{2n+2} \left(\frac{l}{g}\right)^{2n+1} \frac{(1+gq)^{2n+1}}{(1+2dl)^{n+1}(-1+aq)^{n+1}(1+aq)^{n+1}},$$

$$x_{24n+14} = (-1)^n r \left(\frac{b}{u}\right)^{2n+2} \left(\frac{m}{h}\right)^{2n+1} \frac{(1+hr)^{2n+1}}{(1+2em)^{n+1}(-1+br)^{n+1}(1+br)^{n+1}},$$

$$x_{24n+15} = (-1)^n s \left(\frac{c}{v}\right)^{2n+2} \left(\frac{p}{k}\right)^{2n+1} \frac{(1+ks)^{2n+1}}{(1+2fp)^{n+1}(1-+cs)^{n+1}(1+cs)^{n+1}}.$$

And

$$y_{24n-8}=(-1)^{n}\left(\frac{gt}{a}\right)^{2n}\frac{(1+2dl)^{n}(-1+aq)^{n}(1+aq)^{n}}{l^{2n-1}(1+gq)^{2n}},$$

$$y_{24n-7}=(-1)^{n}\left(\frac{hu}{b}\right)^{2n}\frac{(1+2em)^{n}(-1+br)^{n}(1+br)^{n}}{m^{2n-1}(1+hr)^{2n}},$$

$$y_{24n-6}=(-1)^{n}\left(\frac{kv}{c}\right)^{2n}\frac{(1+2fp)^{n}(-1+cs)^{n}(1+cs)^{n}}{p^{2n-1}(1+ks)^{2n}},$$

$$y_{24n-5}=(-1)^{n}q\left(\frac{al}{gt}\right)^{2n}\frac{(1+2gq)^{n}(-1+dt)^{n}(1+dt)^{n}}{(1+dl)^{2n}},$$

$$y_{24n-4}=(-1)^{n}r\left(\frac{bm}{hu}\right)^{2n}\frac{(1+2hr)^{n}(-1+eu)^{n}(1+eu)^{n}}{(1+em)^{2n}},$$

$$y_{24n-3}=(-1)^{n}s\left(\frac{cp}{kv}\right)^{2n}\frac{(1+2ks)^{n}(-1+fv)^{n}(1+fv)^{n}}{(1+fp)^{2n}},$$

$$y_{24n-2}=(-1)^{n}\left(\frac{g}{al}\right)^{2n}\frac{t^{2n+1}(1+2dl)^{n}(-1+aq)^{n}(1+aq)^{n}}{(1+gq)^{2n}},$$

$$y_{24n-1}=(-1)^{n}\left(\frac{h}{bm}\right)^{2n}\frac{u^{2n+1}(1+2em)^{n}(-1+br)^{n}(1+br)^{n}}{(1+hr)^{2n}},$$

$$y_{24n}=(-1)^{n}\left(\frac{k}{cp}\right)^{2n}\frac{v^{2n+1}(1+2fp)^{n}(-1+cs)^{n}(1+cs)^{n}}{(1+ks)^{2n}},$$

$$y_{24n+1}=(-1)^{n}d\left(\frac{a}{t}\right)^{2n}\left(\frac{l}{g}\right)^{2n+1}\frac{(1+2gq)^{n}(-1+dt)^{n}(1+dt)^{n}}{(1+dl)^{2n+1}},$$

$$y_{24n+2}=(-1)^{n}e\left(\frac{b}{u}\right)^{2n}\left(\frac{m}{h}\right)^{2n+1}\frac{(1+2hr)^{n}(-1+eu)^{n}(1+eu)^{n}}{(1+em)^{2n+1}},$$

$$y_{24n+3}=(-1)^{n}f\left(\frac{c}{v}\right)^{2n}\left(\frac{p}{k}\right)^{2n+1}\frac{(1+2ks)^{n}(-1+fv)^{n}(1+fv)^{n}}{(1+fp)^{2n+1}},$$

$$y_{24n+4}=(-1)^{n+1}\left(\frac{gt}{a}\right)^{2n+1}\frac{(1+2dl)^{n}(-1+aq)^{n}(1+aq)^{n+1}}{l^{2n}(1+gq)^{2n+1}},$$

$$y_{24n+5}=(-1)^{n+1}\left(\frac{hu}{b}\right)^{2n+1}\frac{(1+2em)^{n}(-1+br)^{n}(1+br)^{n+1}}{m^{2n}(1+hr)^{2n+1}},$$

$$y_{24n+6}=(-1)^{n+1}\left(\frac{kv}{c}\right)^{2n+1}\frac{(1+2fp)^{n}(-1+cs)^{n}(1+cs)^{n+1}}{p^{2n}(1+ks)^{2n+1}},$$

$$y_{24n+7}=(-1)^{n}q\left(\frac{al}{gt}\right)^{2n+1}\frac{(1+2gq)^{n}(-1+dt)^{n}(1+dt)^{n+1}}{(1+dl)^{2n+1}},$$

$$y_{24n+8} = (-1)^n r\left(\frac{bm}{hu}\right)^{2n+1} \frac{(1+2hr)^n(-1+eu)^n(1+eu)^{n+1}}{(1+em)^{2n+1}},$$
$$y_{24n+9} = (-1)^n s\left(\frac{cp}{kv}\right)^{2n+1} \frac{(1+2ks)^n(-1+fv)^n(1+fv)^{n+1}}{(1+fp)^{2n+1}},$$
$$y_{24n+10} = (-1)^{n+1} t^{2n+2}\left(\frac{g}{al}\right)^{2n+1} \frac{(1+2dl)^{n+1}(-1+aq)^n(1+aq)^{n+1}}{(1+gq)^{2n+1}},$$
$$y_{24n+11} = (-1)^{n+1} u^{2n+2}\left(\frac{h}{bm}\right)^{2n+1} \frac{(1+2em)^{n+1}(-1+br)^n(1+br)^{n+1}}{(1+hr)^{2n+1}},$$
$$y_{24n+12} = (-1)^{n+1} v^{2n+2}\left(\frac{k}{cp}\right)^{2n+1} \frac{(1+2fp)^{n+1}(-1+cs)^n(1+cs)^{n+1}}{(1+ks)^{2n+1}},$$
$$y_{24n+13} = (-1)^n d\left(\frac{a}{t}\right)^{2n+1}\left(\frac{l}{g}\right)^{2n+2} \frac{(1+2gq)^{n+1}(-1+dt)^n(1+dt)^{n+1}}{(1+dl)^{2n+2}},$$
$$y_{24n+14} = (-1)^n e\left(\frac{b}{u}\right)^{2n+1}\left(\frac{m}{h}\right)^{2n+2} \frac{(1+2hr)^{n+1}(-1+eu)^n(1+eu)^{n+1}}{(1+em)^{2n+2}},$$
$$y_{24n+15} = (-1)^n f\left(\frac{c}{v}\right)^{2n+1}\left(\frac{p}{k}\right)^{2n+2} \frac{(1+2ks)^{n+1}(-1+fv)^n(1+fv)^{n+1}}{(1+fp)^{2n+2}}.$$

Proof. For $n=0$ the solution is true. Now, we assume that $n>0$ and that our solution holds for $n-1$. that is,

$$x_{24n-17} = (-1)^{n-1} d\left(\frac{al}{gt}\right)^{2n-1} \frac{(1+gq)^{2n-1}}{(1+2dl)^n(-1+aq)^{n-1}(1+aq)^n},$$
$$x_{24n-16} = (-1)^{n-1} e\left(\frac{bm}{hu}\right)^{2n-1} \frac{(1+hr)^{2n-1}}{(1+2em)^n(-1+br)^{n-1}(1+br)^n},$$
$$x_{24n-15} = (-1)^{n-1} f\left(\frac{cp}{kv}\right)^{2n-1} \frac{(1+ks)^{2n-1}}{(1+2fp)^n(-1+cs)^{n-1}(1+cs)^n},$$
$$x_{24n-14} = (-1)^n g^{2n}\left(\frac{t}{al}\right)^{2n-1} \frac{(1+dl)^{2n-1}}{(1+2gq)^n(-1+dt)^{n-1}(1+dt)^n},$$
$$x_{24n-13} = (-1)^n h^{2n}\left(\frac{u}{bm}\right)^{2n-1} \frac{(1+em)^{2n-1}}{(1+2hr)^n(-1+eu)^{n-1}(1+eu)^n},$$
$$x_{24n-12} = (-1)^n k^{2n}\left(\frac{v}{cp}\right)^{2n-1} \frac{(1+fp)^{2n-1}}{(1+2ks)^n(-1+fv)^{n-1}(1+fv)^n},$$
$$x_{24n-11} = (-1)^{n-1} q\left(\frac{a}{t}\right)^{2n}\left(\frac{l}{g}\right)^{2n-1} \frac{(1+gq)^{2n-1}}{(1+2dl)^n(-1+aq)^n(1+aq)^n},$$

$$x_{24n-10} = (-1)^{n-1} r\left(\frac{b}{u}\right)^{2n}\left(\frac{m}{h}\right)^{2n-1} \frac{(1+hr)^{2n-1}}{(1+2em)^n(-1+br)^n(1+br)^n},$$
$$x_{24n-9} = (-1)^{n-1} s\left(\frac{c}{v}\right)^{2n}\left(\frac{p}{k}\right)^{2n-1} \frac{(1+ks)^{2n-1}}{(1+2fp)^n(1-+cs)^n(1+cs)^n}.$$

And

$$y_{24n-17} = (-1)^{n-1} q \left(\frac{al}{gt}\right)^{2n-1} \frac{(1+2gq)^{n-1}(-1+dt)^{n-1}(1+dt)^{n}}{(1+dl)^{2n-1}},$$
$$y_{24n-16} = (-1)^{n-1} r \left(\frac{bm}{hu}\right)^{2n-1} \frac{(1+2hr)^{n-1}(-1+eu)^{n-1}(1+eu)^{n}}{(1+em)^{2n-1}},$$
$$y_{24n-15} = (-1)^{n-1} s \left(\frac{cp}{kv}\right)^{2n-1} \frac{(1+2ks)^{n-1}(-1+fv)^{n-1}(1+fv)^{n}}{(1+fp)^{2n-1}},$$
$$y_{24n-14} = (-1)^{n} t^{2n} \left(\frac{g}{al}\right)^{2n-1} \frac{(1+2dl)^{n}(-1+aq)^{n-1}(1+aq)^{n}}{(1+gq)^{2n-1}},$$
$$y_{24n-13} = (-1)^{n} u^{2n} \left(\frac{h}{bm}\right)^{2n-1} \frac{(1+2em)^{n}(-1+br)^{n-1}(1+br)^{n}}{(1+hr)^{2n-1}},$$
$$y_{24n-12} = (-1)^{n} v^{2n} \left(\frac{k}{cp}\right)^{2n-1} \frac{(1+2fp)^{n}(-1+cs)^{n-1}(1+cs)^{n}}{(1+ks)^{2n-1}},$$
$$y_{24n-11} = (-1)^{n-1} d \left(\frac{a}{t}\right)^{2n-1} \left(\frac{l}{g}\right)^{2n} \frac{(1+2gq)^{n}(-1+dt)^{n-1}(1+dt)^{n}}{(1+dl)^{2n}},$$
$$y_{24n-10} = (-1)^{n-1} e \left(\frac{b}{u}\right)^{2n-1} \left(\frac{m}{h}\right)^{2n} \frac{(1+2hr)^{n}(-1+eu)^{n-1}(1+eu)^{n}}{(1+em)^{2n}},$$
$$y_{24n-9} = (-1)^{n-1} f \left(\frac{c}{v}\right)^{2n-1} \left(\frac{p}{k}\right)^{2n} \frac{(1+2ks)^{n}(-1+fv)^{n-1}(1+fv)^{n}}{(1+fp)^{2n}}.$$

Next, it can be obviously observed from system.(1) that

$$x_{24n-8} = \frac{y_{24n-14}x_{24n-17}}{y_{24n-11}(-1-y_{24n-14}x_{24n-17})}$$
$$= \frac{(-1)^{n} t^{2n}\left(\frac{g}{al}\right)^{2n-1}\frac{(1+2dl)^{n}(-1+aq)^{n-1}(1+aq)^{n}}{(1+gq)^{2n-1}}(-1)^{n-1} d\left(\frac{al}{gt}\right)^{2n-1}\frac{(1+gq)^{2n-1}}{(1+2dl)^{n}(-1+aq)^{n-1}(1+aq)^{n}}}{(-1)^{n-1} d\left(\frac{a}{t}\right)^{2n-1}\left(\frac{l}{g}\right)^{2n}\frac{(1+2gq)^{n}(-1+dt)^{n-1}(1+dt)^{n}}{(1+dl)^{2n}}\left[-1-y_{24n-14}x_{24n-17}\right]}$$

$$= \frac{-td}{(-1)^{n-1} d\left(\frac{a}{t}\right)^{2n-1}\left(\frac{l}{g}\right)^{2n}\frac{(1+2gq)^{n}(-1+dt)^{n-1}(1+dt)^{n}}{(1+dl)^{2n}}\left[-1+td\right]}$$
$$= \frac{-(-1)^{n+1} t(1+dl)^{2n}}{\left(\frac{a}{t}\right)^{2n-1}\left(\frac{l}{g}\right)^{2n}(1+2gq)^{n}(-1+dt)^{n}(1+dt)^{n}}$$
$$= (-1)^{n}\left(\frac{gt}{l}\right)^{2n}\frac{(1+dl)^{2n}}{a^{2n-1}(1+2gq)^{n}(-1+dt)^{n}(1+dt)^{n}}.$$

$$x_{24n-7} = \frac{y_{24n-13}x_{24n-16}}{y_{24n-10}\left(-1-y_{24n-13}x_{24n-16}\right)}$$
$$= \frac{(-1)^n u^{2n}\left(\frac{h}{bm}\right)^{2n-1} \frac{(1+2em)^n(-1+br)^{n-1}(1+br)^n}{(1+hr)^{2n-1}} (-1)^{n-1} e\left(\frac{bm}{hu}\right)^{2n-1} \frac{(1+hr)^{2n-1}}{(1+2em)^n(-1+br)^{n-1}(1+br)^n}}{(-1)^{n-1} e\left(\frac{b}{u}\right)^{2n-1}\left(\frac{m}{h}\right)^{2n} \frac{(1+2hr)^n(-1+eu)^{n-1}(1+eu)^n}{(1+em)^{2n}}\left[-1-y_{24n-13}x_{24n-16}\right]}$$
$$= \frac{-u}{(-1)^{n-1}\left(\frac{b}{u}\right)^{2n-1}\left(\frac{m}{h}\right)^{2n} \frac{(1+2hr)^n(-1+eu)^{n-1}(1+eu)^n}{(1+em)^{2n}}\left[-1+ue\right]}$$
$$= (-1)^n\left(\frac{hu}{m}\right)^{2n} \frac{(1+em)^{2n}}{b^{2n-1}(1+2hr)^n(-1+eu)^n(1+eu)^n}.$$

$$y_{24n-8} = \frac{x_{24n-14}y_{24n-17}}{x_{24n-11}\left(1+x_{24n-14}y_{24n-17}\right)}$$
$$= \frac{(-1)^n g^{2n}\left(\frac{t}{al}\right)^{2n-1} \frac{(1+dl)^{2n-1}}{(1+2gq)^n(-1+dt)^{n-1}(1+dt)^n} (-1)^{n-1} q\left(\frac{al}{gt}\right)^{2n-1} \frac{(1+2gq)^{n-1}(-1+dt)^{n-1}(1+dt)^n}{(1+dl)^{2n-1}}}{(-1)^{n-1} q\left(\frac{a}{t}\right)^{2n}\left(\frac{l}{g}\right)^{2n-1} \frac{(1+gq)^{2n-1}}{(1+2dl)^n(-1+aq)^n(1+aq)^n}\left[1+x_{24n-14}y_{24n-17}\right]}$$
$$= \frac{-(-1)^{-n+1}\frac{gq}{1+2gq}}{q\left(\frac{a}{t}\right)^{2n}\left(\frac{l}{g}\right)^{2n-1} \frac{(1+gq)^{2n-1}}{(1+2dl)^n(-1+aq)^n(1+aq)^n}\left[1-\frac{gq}{1+2gq}\right]}$$
$$= (-1)^n\left(\frac{gt}{a}\right)^{2n} \frac{(1+2dl)^n(-1+aq)^n(1+aq)^n}{l^{2n-1}(1+gq)^{2n}}.$$

$$y_{24n-7} = \frac{x_{24n-13}y_{24n-16}}{x_{24n-10}\left(1+x_{24n-13}y_{24n-16}\right)}$$
$$= \frac{(-1)^n h^{2n}\left(\frac{u}{bm}\right)^{2n-1} \frac{(1+em)^{2n-1}}{(1+2hr)^n(-1+eu)^{n-1}(1+eu)^n} (-1)^{n-1} r\left(\frac{bm}{hu}\right)^{2n-1} \frac{(1+2hr)^{n-1}(-1+eu)^{n-1}(1+eu)^n}{(1+em)^{2n-1}}}{(-1)^{n-1} r\left(\frac{b}{u}\right)^{2n}\left(\frac{m}{h}\right)^{2n-1} \frac{(1+hr)^{2n-1}}{(1+2em)^n(-1+br)^n(1+br)^n}\left[1+x_{24n-13}y_{24n-16}\right]}$$
$$= \frac{-(-1)^{-n+1}\frac{hr}{1+2hr}}{r\left(\frac{b}{u}\right)^{2n}\left(\frac{m}{h}\right)^{2n-1} \frac{(1+hr)^{2n-1}}{(1+2em)^n(-1+br)^n(1+br)^n}\left[1-\frac{hr}{1+2hr}\right]}$$
$$= (-1)^n\left(\frac{hu}{b}\right)^{2n} \frac{(1+2em)^n(-1+br)^n(1+br)^n}{m^{2n-1}(1+hr)^{2n}}.$$

Hence, the other relations can be proved in a similar way. Thus, they are omitted.

2.2 The Second System $x_{n+1} = \frac{y_{n-5}x_{n-8}}{y_{n-2}(-1-y_{n-5}x_{n-8})}$, $y_{n+1} = \frac{x_{n-5}y_{n-8}}{x_{n-2}(-1-x_{n-5}y_{n-8})}$

In this part, we investigate the existence of the solutions for the following system of difference equations:

$$x_{n+1} = \frac{y_{n-5}x_{n-8}}{y_{n-2}\left(-1-y_{n-5}x_{n-8}\right)}, \quad y_{n+1} = \frac{x_{n-5}y_{n-8}}{x_{n-2}\left(-1-x_{n-5}y_{n-8}\right)}, \tag{2}$$

with non-zero real initial conditions and $y_i x_{i-3} \neq -1$ and $x_i y_{i-3} \neq -1$ for all $i = -5, -4, \ldots$.

Theorem 2 *Assume that $\{x_n, y_n\}$ is a solution to system (2) and let $x_{-8} = a$, $x_{-7} = b$, $x_{-6} = c$, $x_{-5} = d$, $x_{-4} = e$, $x_{-3} = f$, $x_{-2} = g$, $x_{-1} = h$, $x_0 = k$, $y_{-8} = l$, $y_{-7} = m$, $y_{-6} = p$, $y_{-5} = q$, $y_{-4} = r$, $y_{-3} = s$, $y_{-2} = t$, $y_{-1} = u$, and $y_0 = v$. Then, for $n = 0, 1, \ldots$, we have*

$$x_{12n-8} = \left(\frac{gt}{l}\right)^n \frac{(1+dl)^n}{a^{n-1}(1+dt)^n}, \qquad x_{12n-7} = \left(\frac{hu}{m}\right)^n \frac{(1+em)^n}{b^{n-1}(1+eu)^n},$$

$$x_{12n-6} = \left(\frac{kv}{p}\right)^n \frac{(1+fp)^n}{c^{n-1}(1+fv)^n}, \qquad x_{12n-5} = d\left(\frac{al}{gt}\right)^n \frac{(1+gq)^n}{(1+aq)^n},$$

$$x_{12n-4} = e\left(\frac{bm}{hu}\right)^n \frac{(1+hr)^n}{(1+br)^n}, \qquad x_{12n-3} = f\left(\frac{cp}{kv}\right)^n \frac{(1+ks)^n}{(1+cs)^n},$$

$$x_{12n-2} = g^{n+1}\left(\frac{t}{al}\right)^n \frac{(1+dl)^n}{(1+dt)^n}, \qquad x_{12n-1} = h^{n+1}\left(\frac{u}{bm}\right)^n \frac{(1+em)^n}{(1+eu)^n},$$

$$x_{12n} = k^{n+1}\left(\frac{v}{cp}\right)^n \frac{(1+fp)^n}{(1+fv)^n}, \qquad x_{12n+1} = -q\left(\frac{l}{g}\right)^n \left(\frac{a}{t}\right)^{n+1} \frac{(1+gq)^n}{(1+aq)^{n+1}},$$

$$x_{12n+2} = -r\left(\frac{m}{h}\right)^n \left(\frac{b}{u}\right)^{n+1} \frac{(1+hr)^n}{(1+br)^{n+1}}, \qquad x_{12n+3} = -s\left(\frac{p}{k}\right)^n \left(\frac{c}{v}\right)^{n+1} \frac{(1+ks)^n}{(1+cs)^{n+1}},$$

$$y_{12n-8} = \left(\frac{gt}{a}\right)^n \frac{(1+aq)^n}{l^{n-1}(1+gq)^n}, \qquad y_{12n-7} = \left(\frac{hu}{b}\right)^n \frac{(1+br)^n}{m^{n-1}(1+hr)^n},$$

$$y_{12n-6} = \left(\frac{kv}{c}\right)^n \frac{(1+cs)^n}{p^{n-1}(1+ks)^n}, \qquad y_{12n-5} = q\left(\frac{al}{gt}\right)^n \left(\frac{1+dt}{1+dl}\right)^n,$$

$$y_{12n-4} = r\left(\frac{bm}{hu}\right)^n \left(\frac{1+eu}{1+em}\right)^n, \qquad y_{12n-3} = s\left(\frac{cp}{kv}\right)^n \left(\frac{1+fv}{1+fp}\right)^n,$$

$$y_{12n-2} = t^{n+1}\left(\frac{g}{al}\right)^n \left(\frac{1+aq}{1+gq}\right)^n, \qquad y_{12n-1} = u^{n+1}\left(\frac{h}{bm}\right)^n \left(\frac{1+br}{1+hr}\right)^n,$$

$$y_{12n} = v^{n+1}\left(\frac{k}{cp}\right)^n \left(\frac{1+cs}{1+ks}\right)^n, \qquad y_{12n+1} = -d\left(\frac{a}{t}\right)^n \left(\frac{l}{g}\right)^{n+1} \frac{(1+dt)^n}{(1+dl)^{n+1}},$$

$$y_{12n+2} = -e\left(\frac{b}{u}\right)^n \left(\frac{m}{h}\right)^{n+1} \frac{(1+eu)^n}{(1+em)^{n+1}}, \qquad y_{12n+3} = -f\left(\frac{c}{v}\right)^n \left(\frac{p}{k}\right)^{n+1} \frac{(1+fv)^n}{(1+fp)^{n+1}}.$$

Proof. it can be easily seen that the results hold for $n = 0$. Now, we suppose that $n > 0$ and that our assumption holds for $n-1$. That is,

$$x_{12n-20}=\left(\frac{gt}{l}\right)^{n-1}\frac{(1+dl)^{n-1}}{a^{n-2}(1+dt)^{n-1}},\qquad x_{12n-19}=\left(\frac{hu}{m}\right)^{n-1}\frac{(1+em)^{n-1}}{b^{n-2}(1+eu)^{n-1}},$$

$$x_{12n-18}=\left(\frac{kv}{p}\right)^{n-1}\frac{(1+fp)^{n-1}}{c^{n-2}(1+fv)^{n-1}},\qquad x_{12n-17}=d\left(\frac{al}{gt}\right)^{n-1}\frac{(1+gq)^{n-1}}{(1+aq)^{n-1}},$$

$$x_{12n-16}=e\left(\frac{bm}{hu}\right)^{n-1}\frac{(1+hr)^{n-1}}{(1+br)^{n-1}},\qquad x_{12n-15}=f\left(\frac{cp}{kv}\right)^{n-1}\frac{(1+ks)^{n-1}}{(1+cs)^{n-1}},$$

$$x_{12n-14}=g^n\left(\frac{t}{al}\right)^{n-1}\frac{(1+dl)^{n-1}}{(1+dt)^{n-1}},\qquad x_{12n-13}=h^n\left(\frac{u}{bm}\right)^{n-1}\frac{(1+em)^{n-1}}{(1+eu)^{n-1}},$$

$$x_{12n-12}=k^n\left(\frac{v}{cp}\right)^{n-1}\frac{(1+fp)^{n-1}}{(1+fv)^{n-1}},\qquad x_{12n-11}=-q\left(\frac{l}{g}\right)^{n-1}\left(\frac{a}{t}\right)^{n}\frac{(1+gq)^{n-1}}{(1+aq)^{n}},$$

$$x_{12n-10}=-r\left(\frac{m}{h}\right)^{n-1}\left(\frac{b}{u}\right)^{n}\frac{(1+hr)^{n-1}}{(1+br)^{n}},\qquad x_{12n-9}=-s\left(\frac{p}{k}\right)^{n-1}\left(\frac{c}{v}\right)^{n}\frac{(1+ks)^{n-1}}{(1+cs)^{n}},$$

$$y_{12n-20}=\left(\frac{gt}{a}\right)^{n-1}\frac{(1+aq)^{n-1}}{l^{n-2}(1+gq)^{n-1}},\qquad y_{12n-19}=\left(\frac{hu}{b}\right)^{n-1}\frac{(1+br)^{n-1}}{m^{n-2}(1+hr)^{n-1}},$$

$$y_{12n-18}=\left(\frac{kv}{c}\right)^{n-1}\frac{(1+cs)^{n-1}}{p^{n-2}(1+ks)^{n-1}},\qquad y_{12n-17}=q\left(\frac{al}{gt}\right)^{n-1}\left(\frac{1+dt}{1+dl}\right)^{n-1},$$

$$y_{12n-16}=r\left(\frac{bm}{hu}\right)^{n-1}\left(\frac{1+eu}{1+em}\right)^{n-1},\qquad y_{12n-15}=s\left(\frac{cp}{kv}\right)^{n-1}\left(\frac{1+fv}{1+fp}\right)^{n-1},$$

$$y_{12n-14}=t^n\left(\frac{g}{al}\right)^{n-1}\left(\frac{1+aq}{1+gq}\right)^{n-1},\qquad y_{12n-13}=u^n\left(\frac{h}{bm}\right)^{n-1}\left(\frac{1+br}{1+hr}\right)^{n-1},$$

$$y_{12n-12}=v^n\left(\frac{k}{cp}\right)^{n-1}\left(\frac{1+cs}{1+ks}\right)^{n-1},\qquad y_{12n-11}=-d\left(\frac{a}{t}\right)^{n-1}\left(\frac{l}{g}\right)^{n}\frac{(1+dt)^{n-1}}{(1+dl)^{n}},$$

$$y_{12n-10}=-e\left(\frac{b}{u}\right)^{n-1}\left(\frac{m}{h}\right)^{n}\frac{(1+eu)^{n-1}}{(1+em)^{n}},\qquad y_{12n-9}=-f\left(\frac{c}{v}\right)^{n-1}\left(\frac{p}{k}\right)^{n}\frac{(1+fv)^{n-1}}{(1+fp)^{n}}.$$

Next, it can be deduced from system.(2) that

$$\begin{aligned}x_{12n-8}&=\frac{y_{12n-14}x_{12n-17}}{y_{12n-11}\left[-1-y_{12n-14}x_{12n-17}\right]}\\&=\frac{t^n\left(\frac{g}{al}\right)^{n-1}\left(\frac{1+aq}{1+gq}\right)^{n-1}d\left(\frac{al}{gt}\right)^{n-1}\frac{(1+gq)^{n-1}}{(1+aq)^{n-1}}}{-d\left(\frac{a}{t}\right)^{n-1}\left(\frac{l}{g}\right)^{n}\frac{(1+dt)^{n-1}}{(1+dl)^{n}}\left[-1-t^n\left(\frac{g}{al}\right)^{n-1}\left(\frac{1+aq}{1+gq}\right)^{n-1}d\left(\frac{al}{gt}\right)^{n-1}\frac{(1+gq)^{n-1}}{(1+aq)^{n-1}}\right]}\\&=\frac{td}{d\left(\frac{a}{t}\right)^{n-1}\left(\frac{l}{g}\right)^{n}\frac{(1+dt)^{n-1}}{(1+dl)^{n}}\left[1+td\right]}=\left(\frac{gt}{l}\right)^{n}\frac{(1+dl)^{n}}{a^{n-1}(1+dt)^{n}}.\end{aligned}$$

$$
\begin{aligned}
x_{12n-7} &= \frac{y_{12n-13}x_{12n-16}}{y_{12n-10}\left[-1-y_{12n-13}x_{12n-16}\right]} \\
&= \frac{u^n\left(\frac{h}{bm}\right)^{n-1}\left(\frac{1+br}{1+hr}\right)^{n-1}e\left(\frac{bm}{hu}\right)^{n-1}\frac{(1+hr)^{n-1}}{(1+br)^{n-1}}}{-e\left(\frac{b}{u}\right)^{n-1}\left(\frac{m}{h}\right)^{n}\frac{(1+eu)^{n-1}}{(1+em)^{n}}\left[-1-u^n\left(\frac{h}{bm}\right)^{n-1}\left(\frac{1+br}{1+hr}\right)^{n-1}e\left(\frac{bm}{hu}\right)^{n-1}\frac{(1+hr)^{n-1}}{(1+br)^{n-1}}\right]} \\
&= \frac{ue}{e\left(\frac{b}{u}\right)^{n-1}\left(\frac{m}{h}\right)^{n}\frac{(1+eu)^{n-1}}{(1+em)^{n}}\left[1+ue\right]} = \left(\frac{hu}{m}\right)^{n}\frac{(1+em)^{n}}{b^{n-1}(1+eu)^{n}}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
y_{12n-8} &= \frac{x_{12n-14}y_{12n-17}}{x_{12n-11}\left[-1-x_{12n-14}y_{12n-17}\right]} \\
&= \frac{g^n\left(\frac{t}{al}\right)^{n-1}\frac{(1+dl)^{n-1}}{(1+dt)^{n-1}}q\left(\frac{al}{gt}\right)^{n-1}\left(\frac{1+dt}{1+dl}\right)^{n-1}}{-q\left(\frac{l}{g}\right)^{n-1}\left(\frac{a}{t}\right)^{n}\frac{(1+gq)^{n-1}}{(1+aq)^{n}}\left[-1-g^n\left(\frac{t}{al}\right)^{n-1}\frac{(1+dl)^{n-1}}{(1+dt)^{n-1}}q\left(\frac{al}{gt}\right)^{n-1}\left(\frac{1+dt}{1+dl}\right)^{n-1}\right]} \\
&= \frac{gq}{q\left(\frac{l}{g}\right)^{n-1}\left(\frac{a}{t}\right)^{n}\frac{(1+gq)^{n-1}}{(1+aq)^{n}}\left[1+gq\right]} = \left(\frac{gt}{a}\right)^{n}\frac{(1+aq)^{n}}{l^{n-1}(1+gq)^{n}}.
\end{aligned}
$$

$$
\begin{aligned}
y_{12n-7} &= \frac{x_{12n-13}y_{12n-16}}{x_{12n-10}\left[-1-x_{12n-13}y_{12n-16}\right]} \\
&= \frac{h^n\left(\frac{u}{bm}\right)^{n-1}\frac{(1+em)^{n-1}}{(1+eu)^{n-1}}r\left(\frac{bm}{hu}\right)^{n-1}\left(\frac{1+eu}{1+em}\right)^{n-1}}{-r\left(\frac{m}{h}\right)^{n-1}\left(\frac{b}{u}\right)^{n}\frac{(1+hr)^{n-1}}{(1+br)^{n}}\left[-1-h^n\left(\frac{u}{bm}\right)^{n-1}\frac{(1+em)^{n-1}}{(1+eu)^{n-1}}r\left(\frac{bm}{hu}\right)^{n-1}\left(\frac{1+eu}{1+em}\right)^{n-1}\right]} \\
&= \frac{hr}{r\left(\frac{m}{h}\right)^{n-1}\left(\frac{b}{u}\right)^{n}\frac{(1+hr)^{n-1}}{(1+br)^{n}}\left[1+hr\right]} = \left(\frac{hu}{b}\right)^{n}\frac{(1+br)^{n}}{m^{n-1}(1+hr)^{n}}.
\end{aligned}
$$

Similarly, one can show other formulae.

2.3 The Third System $x_{n+1} = \frac{y_{n-5}x_{n-8}}{y_{n-2}(-1-y_{n-5}x_{n-8})}$, $y_{n+1} = \frac{x_{n-5}y_{n-8}}{x_{n-2}(-1+x_{n-5}y_{n-8})}$

In this subsection, we obtain the solutions of the system of higher order difference equations in the form

$$
x_{n+1} = \frac{y_{n-5}x_{n-8}}{y_{n-2}\left(-1-y_{n-5}x_{n-8}\right)}, \quad y_{n+1} = \frac{x_{n-5}y_{n-8}}{x_{n-2}\left(-1+x_{n-5}y_{n-8}\right)}, \tag{3}
$$

with non-zero real initial conditions $x_\delta, y_\delta, \delta \in \{0, 1, 2, \ldots 8\}$.

Theorem 3 *Suppose that $\{x_n, y_n\}$ is a solution to system (3) and assume that $x_{-8} = a,\ x_{-7} = b,\ x_{-6} = c,\ x_{-5} = d,\ x_{-4} = e,\ x_{-3} = f,\ x_{-2} = g,\ x_{-1} = h,\ x_0 = k,\ y_{-8} = l,\ y_{-7} = m,\ y_{-6} = p,\ y_{-5} = q,\ y_{-4} = r,\ y_{-3} = s,\ y_{-2} = t,\ y_{-1} = u,$ and $y_0 = v$. Then, for $n = 0, 1, \ldots,$ we have*

$$x_{12n+1} = \frac{-q}{1+aq}\left(\frac{a}{t}\right)^{n+1}\left(\frac{l}{g}\right)^{n}\prod_{i=0}^{n-1}\frac{(-1+(2i+1)gq)(1+(2i+2)dt)}{(-1+(2i+2)dl)(1+(2i+3)aq)},$$

$$x_{12n+2} = \frac{-r}{1+br}\left(\frac{b}{u}\right)^{n+1}\left(\frac{m}{h}\right)^{n}\prod_{i=0}^{n-1}\frac{(-1+(2i+1)hr)(1+(2i+2)eu)}{(-1+(2i+2)em)(1+(2i+3)br)},$$

$$x_{12n+3} = \frac{-s}{1+cs}\left(\frac{c}{v}\right)^{n+1}\left(\frac{p}{k}\right)^{n}\prod_{i=0}^{n-1}\frac{(-1+(2i+1)ks)(1+(2i+2)fv)}{(-1+(2i+2)fp)(1+(2i+3)cs)},$$

$$x_{12n+4} = -\frac{(-1+dl)}{a^{n}(1+dt)}\left(\frac{gt}{l}\right)^{n+1}\prod_{i=0}^{n-1}\frac{(-1+(2i+3)dl)(1+(2i+2)aq)}{(-1+(2i+2)gq)(1+(2i+3)dt)},$$

$$x_{12n+5} = -\frac{(-1+em)}{b^{n}(1+eu)}\left(\frac{hu}{m}\right)^{n+1}\prod_{i=0}^{n-1}\frac{(-1+(2i+3)em)(1+(2i+2)br)}{(-1+(2i+2)hr)(1+(2i+3)eu)},$$

$$x_{12n+6} = -\frac{(-1+fp)}{c^{n}(1+fv)}\left(\frac{kv}{p}\right)^{n+1}\prod_{i=0}^{n-1}\frac{(-1+(2i+3)fp)(1+(2i+2)cs)}{(-1+(2i+2)ks)(1+(2i+3)fv)},$$

$$x_{12n+7}=d\frac{-1+gq}{(-1+2dl)(1+aq)}\left(\frac{al}{gt}\right)^{n+1}\prod_{i=0}^{n-1}\frac{(-1+(2i+3)gq)(1+(2i+2)dt)}{(-1+(2i+4)dl)(1+(2i+3)aq)},$$

$$x_{12n+8}=e\frac{-1+hr}{(-1+2em)(1+br)}\left(\frac{bm}{hu}\right)^{n+1}\prod_{i=0}^{n-1}\frac{(-1+(2i+3)hr)(1+(2i+2)eu)}{(-1+(2i+4)em)(1+(2i+3)br)},$$

$$x_{12n+9}=f\frac{-1+ks}{(-1+2fp)(1+cs)}\left(\frac{cp}{kv}\right)^{n+1}\prod_{i=0}^{n-1}\frac{(-1+(2i+3)ks)(1+(2i+2)fv)}{(-1+(2i+4)fp)(1+(2i+3)cs)},$$

$$x_{12n+10}=g^{n+2}\frac{(-1+dl)(1+2aq)}{(-1+2gq)(1+dt)}\left(\frac{t}{al}\right)^{n+1}\prod_{i=0}^{n-1}\frac{(-1+(2i+3)dl)(1+(2i+4)aq)}{(-1+(2i+4)gq)(1+(2i+3)dt)},$$

$$x_{12n+11}=h^{n+2}\frac{(-1+em)(1+2br)}{(-1+2hr)(1+eu)}\left(\frac{u}{bm}\right)^{n+1}\prod_{i=0}^{n-1}\frac{(-1+(2i+3)em)(1+(2i+4)br)}{(-1+(2i+4)hr)(1+(2i+3)eu)},$$

$$x_{12n+12}=k^{n+2}\frac{(-1+fp)(1+2cs)}{(-1+2ks)(1+fv)}\left(\frac{v}{cp}\right)^{n+1}\prod_{i=0}^{n-1}\frac{(-1+(2i+3)fp)(1+(2i+4)cs)}{(-1+(2i+4)ks)(1+(2i+3)fv)},$$

$$y_{12n+1}=\frac{d}{-1+dl}\left(\frac{l}{g}\right)^{n+1}\left(\frac{a}{t}\right)^{n}\prod_{i=0}^{n-1}\frac{(-1+(2i+2)gq)(1+(2i+1)dt)}{(-1+(2i+3)dl)(1+(2i+2)aq)},$$

$$y_{12n+2}=\frac{e}{-1+em}\left(\frac{m}{h}\right)^{n+1}\left(\frac{b}{u}\right)^{n}\prod_{i=0}^{n-1}\frac{(-1+(2i+2)hr)(1+(2i+1)eu)}{(-1+(2i+3)em)(1+(2i+2)br)},$$

$$y_{12n+3}=\frac{f}{-1+fp}\left(\frac{p}{k}\right)^{n+1}\left(\frac{c}{v}\right)^{n}\prod_{i=0}^{n-1}\frac{(-1+(2i+2)ks)(1+(2i+1)fv)}{(-1+(2i+3)fp)(1+(2i+2)cs)},$$

$$y_{12n+4}=-\frac{(1+aq)}{l^{n}(-1+gq)}\left(\frac{gt}{a}\right)^{n+1}\prod_{i=0}^{n-1}\frac{(-1+(2i+2)dl)(1+(2i+3)aq)}{(-1+(2i+3)gq)(1+(2i+2)dt)},$$

$$y_{12n+5}=-\frac{(1+br)}{l^{n}(-1+hr)}\left(\frac{hu}{b}\right)^{n+1}\prod_{i=0}^{n-1}\frac{(-1+(2i+2)em)(1+(2i+3)br)}{(-1+(2i+3)hr)(1+(2i+2)eu)},$$

$$y_{12n+6}=-\frac{(1+cs)}{p^{n}(-1+ks)}\left(\frac{kv}{c}\right)^{n+1}\prod_{i=0}^{n-1}\frac{(-1+(2i+2)fp)(1+(2i+3)cs)}{(-1+(2i+3)ks)(1+(2i+2)fv)},$$

$$y_{12n+7}=-q\frac{1+dt}{(-1+dl)(1+2aq)}\left(\frac{al}{gt}\right)^{n+1}\prod_{i=0}^{n-1}\frac{(-1+(2i+2)gq)(1+(2i+3)dt)}{(-1+(2i+3)dl)(1+(2i+4)aq)},$$

$$y_{12n+8}=-r\frac{1+eu}{(-1+em)(1+2br)}\left(\frac{bm}{hu}\right)^{n+1}\prod_{i=0}^{n-1}\frac{(-1+(2i+2)hr)(1+(2i+3)eu)}{(-1+(2i+3)em)(1+(2i+4)br)},$$

$$y_{12n+9}=-s\frac{1+fv}{(-1+fp)(1+2cs)}\left(\frac{cp}{kv}\right)^{n+1}\prod_{i=0}^{n-1}\frac{(-1+(2i+2)ks)(1+(2i+3)fv)}{(-1+(2i+3)fp)(1+(2i+4)cs)},$$

$$y_{12n+10}=t^{n+2}\frac{(-1+2dl)(1+aq)}{(-1+gq)(1+2dt)}\left(\frac{g}{al}\right)^{n+1}\prod_{i=0}^{n-1}\frac{(-1+(2i+4)dl)(1+(2i+3)aq)}{(-1+(2i+3)gq)(1+(2i+4)dt)},$$

$$y_{12n+11}=u^{n+2}\frac{(-1+2em)(1+br)}{(-1+hr)(1+2eu)}\left(\frac{h}{bm}\right)^{n+1}\prod_{i=0}^{n-1}\frac{(-1+(2i+4)em)(1+(2i+3)br)}{(-1+(2i+3)hr)(1+(2i+4)eu)},$$

$$y_{12n+12}=v^{n+2}\frac{(-1+2fp)(1+cs)}{(-1+ks)(1+2fv)}\left(\frac{k}{cp}\right)^{n+1}\prod_{i=0}^{n-1}\frac{(-1+(2i+4)fp)(1+(2i+3)cs)}{(-1+(2i+3)ks)(1+(2i+3)fv)}.$$

Proof. This proof is achieved via induction. The solutions are true for $n = 0$. Now, let $n > 0$ and assume that the results are true for $n-1$. That is,

$$x_{12n-11} = \frac{-q}{1+aq}\left(\frac{a}{t}\right)^{n}\left(\frac{l}{g}\right)^{n-1}\prod_{i=0}^{n-2}\frac{(-1+(2i+1)gq)(1+(2i+2)dt)}{(-1+(2i+2)dl)(1+(2i+3)aq)},$$

$$x_{12n-10} = \frac{-r}{1+br}\left(\frac{b}{u}\right)^{n}\left(\frac{m}{h}\right)^{n-1}\prod_{i=0}^{n-2}\frac{(-1+(2i+1)hr)(1+(2i+2)eu)}{(-1+(2i+2)em)(1+(2i+3)br)},$$

$$x_{12n-9} = \frac{-s}{1+cs}\left(\frac{c}{v}\right)^{n}\left(\frac{p}{k}\right)^{n-1}\prod_{i=0}^{n-2}\frac{(-1+(2i+1)ks)(1+(2i+2)fv)}{(-1+(2i+2)fp)(1+(2i+3)cs)},$$

$$x_{12n-8} = -\frac{(-1+dl)}{a^{n}(1+dt)}\left(\frac{gt}{l}\right)^{n}\prod_{i=0}^{n-2}\frac{(-1+(2i+3)dl)(1+(2i+2)aq)}{(-1+(2i+2)gq)(1+(2i+3)dt)},$$

$$x_{12n-7} = -\frac{(-1+em)}{b^{n}(1+eu)}\left(\frac{hu}{m}\right)^{n}\prod_{i=0}^{n-2}\frac{(-1+(2i+3)em)(1+(2i+2)br)}{(-1+(2i+2)hr)(1+(2i+3)eu)},$$

$$x_{12n-6} = -\frac{(-1+fp)}{c^{n}(1+fv)}\left(\frac{kv}{p}\right)^{n}\prod_{i=0}^{n-2}\frac{(-1+(2i+3)fp)(1+(2i+2)cs)}{(-1+(2i+2)ks)(1+(2i+3)fv)},$$

$$x_{12n-5} = d\frac{-1+gq}{(-1+2dl)(1+aq)}\left(\frac{al}{gt}\right)^{n}\prod_{i=0}^{n-2}\frac{(-1+(2i+3)gq)(1+(2i+2)dt)}{(-1+(2i+4)dl)(1+(2i+3)aq)},$$

$$x_{12n-4} = e\frac{-1+hr}{(-1+2em)(1+br)}\left(\frac{bm}{hu}\right)^{n}\prod_{i=0}^{n-2}\frac{(-1+(2i+3)hr)(1+(2i+2)eu)}{(-1+(2i+4)em)(1+(2i+3)br)},$$

$$x_{12n-3} = f\frac{-1+ks}{(-1+2fp)(1+cs)}\left(\frac{cp}{kv}\right)^{n}\prod_{i=0}^{n-2}\frac{(-1+(2i+3)ks)(1+(2i+2)fv)}{(-1+(2i+4)fp)(1+(2i+3)cs)},$$

$$x_{12n-2} = g^{n+1}\frac{(-1+dl)(1+2aq)}{(-1+2gq)(1+dt)}\left(\frac{t}{al}\right)^{n}\prod_{i=0}^{n-2}\frac{(-1+(2i+3)dl)(1+(2i+4)aq)}{(-1+(2i+4)gq)(1+(2i+3)dt)},$$

$$x_{12n-1} = h^{n+1}\frac{(-1+em)(1+2br)}{(-1+2hr)(1+eu)}\left(\frac{u}{bm}\right)^{n}\prod_{i=0}^{n-2}\frac{(-1+(2i+3)em)(1+(2i+4)br)}{(-1+(2i+4)hr)(1+(2i+3)eu)},$$

$$x_{12n} = k^{n+1}\frac{(-1+fp)(1+2cs)}{(-1+2ks)(1+fv)}\left(\frac{v}{cp}\right)^{n}\prod_{i=0}^{n-2}\frac{(-1+(2i+3)fp)(1+(2i+4)cs)}{(-1+(2i+4)ks)(1+(2i+3)fv)}.$$

And

$$y_{12n-11} = \frac{d}{-1+dl}\left(\frac{l}{g}\right)^n\left(\frac{a}{t}\right)^{n-1}\prod_{i=0}^{n-2}\frac{(-1+(2i+2)gq)(1+(2i+1)dt)}{(-1+(2i+3)dl)(1+(2i+2)aq)},$$

$$y_{12n-10} = \frac{e}{-1+em}\left(\frac{m}{h}\right)^n\left(\frac{b}{u}\right)^{n-1}\prod_{i=0}^{n-2}\frac{(-1+(2i+2)hr)(1+(2i+1)eu)}{(-1+(2i+3)em)(1+(2i+2)br)},$$

$$y_{12n-9} = \frac{f}{-1+fp}\left(\frac{p}{k}\right)^n\left(\frac{c}{v}\right)^{n-1}\prod_{i=0}^{n-2}\frac{(-1+(2i+2)ks)(1+(2i+1)fv)}{(-1+(2i+3)fp)(1+(2i+2)cs)},$$

$$y_{12n-8} = -\frac{(1+aq)}{l^{n-1}(-1+gq)}\left(\frac{gt}{a}\right)^n\prod_{i=0}^{n-2}\frac{(-1+(2i+2)dl)(1+(2i+3)aq)}{(-1+(2i+3)gq)(1+(2i+2)dt)},$$

$$y_{12n-7} = -\frac{(1+br)}{l^{n-1}(-1+hr)}\left(\frac{hu}{b}\right)^n\prod_{i=0}^{n-2}\frac{(-1+(2i+2)em)(1+(2i+3)br)}{(-1+(2i+3)hr)(1+(2i+2)eu)},$$

$$y_{12n-6} = -\frac{(1+cs)}{p^{n-1}(-1+ks)}\left(\frac{kv}{c}\right)^n\prod_{i=0}^{n-2}\frac{(-1+(2i+2)fp)(1+(2i+3)cs)}{(-1+(2i+3)ks)(1+(2i+2)fv)},$$

$$y_{12n-5} = -q\frac{1+dt}{(-1+dl)(1+2aq)}\left(\frac{al}{gt}\right)^n\prod_{i=0}^{n-2}\frac{(-1+(2i+2)gq)(1+(2i+3)dt)}{(-1+(2i+3)dl)(1+(2i+4)aq)},$$

$$y_{12n-4} = -r\frac{1+eu}{(-1+em)(1+2br)}\left(\frac{bm}{hu}\right)^n\prod_{i=0}^{n-2}\frac{(-1+(2i+2)hr)(1+(2i+3)eu)}{(-1+(2i+3)em)(1+(2i+4)br)},$$

$$y_{12n-3} = -s\frac{1+fv}{(-1+fp)(1+2cs)}\left(\frac{cp}{kv}\right)^n\prod_{i=0}^{n-2}\frac{(-1+(2i+2)ks)(1+(2i+3)fv)}{(-1+(2i+3)fp)(1+(2i+4)cs)},$$

$$y_{12n-2} = t^{n+1}\frac{(-1+2dl)(1+aq)}{(-1+gq)(1+2dt)}\left(\frac{g}{al}\right)^n\prod_{i=0}^{n-2}\frac{(-1+(2i+4)dl)(1+(2i+3)aq)}{(-1+(2i+3)gq)(1+(2i+4)dt)},$$

$$y_{12n-1} = u^{n+1}\frac{(-1+2em)(1+br)}{(-1+hr)(1+2eu)}\left(\frac{h}{bm}\right)^n\prod_{i=0}^{n-2}\frac{(-1+(2i+4)em)(1+(2i+3)br)}{(-1+(2i+3)hr)(1+(2i+4)eu)},$$

$$y_{12n} = v^{n+1}\frac{(-1+2fp)(1+cs)}{(-1+ks)(1+2fv)}\left(\frac{k}{cp}\right)^n\prod_{i=0}^{n-2}\frac{(-1+(2i+4)fp)(1+(2i+3)cs)}{(-1+(2i+3)ks)(1+(2i+3)fv)}.$$

Next, system.(3) gives us that

$$x_{12n+1} = \frac{y_{12n-5}x_{12n-8}}{y_{12n-2}[-1-y_{12n-5}x_{12n-8}]}$$

$$= \frac{\frac{aq}{1+2aq}\prod_{i=0}^{n-2}\frac{(1+(2i+2)aq}{(1+(2i+4)aq}}{t^{n+1}\frac{(-1+2dl)(1+aq)}{(-1+gq)(1+2dt)}\left(\frac{g}{al}\right)^{n}\prod_{i=0}^{n-2}\frac{(-1+(2i+4)dl)(1+(2i+3)aq)}{(-1+(2i+3)gq)(1+(2i+4)dt)}\left[-1-\frac{aq}{1+2aq}\prod_{i=0}^{n-2}\frac{(1+(2i+2)aq}{(1+(2i+4)aq}\right]}$$

$$= \frac{-aq}{t^{n+1}\frac{(-1+2dl)(1+aq)}{(-1+gq)(1+2dt)}\left(\frac{g}{al}\right)^{n}\prod_{i=0}^{n-2}\frac{(-1+(2i+4)dl)(1+(2i+3)aq)}{(-1+(2i+3)gq)(1+(2i+4)dt)}\left(-\frac{(1+(2n+1)aq}{1+2naq}\right)}$$

$$= \left(\frac{a}{t}\right)^{n+1}\left(\frac{l}{g}\right)^{n}\frac{-q}{1+(2n+1)aq}\prod_{i=0}^{n-1}\frac{(-1+(2i+1)gq)(1+(2i+2)dt)}{(-1+(2i+2)dl)(1+(2i+1)aq)}$$

$$= \frac{-q}{1+aq}\left(\frac{a}{t}\right)^{n+1}\left(\frac{l}{g}\right)^{n}\prod_{i=0}^{n-1}\frac{(-1+(2i+1)gq)(1+(2i+2)dt)}{(-1+(2i+2)dl)(1+(2i+3)aq)}.$$

The proofs of other relations are done in a similar way. Hence, they are omitted.

2.4 The Fourth System $x_{n+1} = \frac{y_{n-5}x_{n-8}}{y_{n-2}(-1-y_{n-5}x_{n-8})}$, $y_{n+1} = \frac{x_{n-5}y_{n-8}}{x_{n-2}(1-x_{n-5}y_{n-8})}$

This subsection is devoted to obtain the solutions of the system of higher order difference equations in the form

$$x_{n+1} = \frac{y_{n-5}x_{n-8}}{y_{n-2}(-1-y_{n-5}x_{n-8})},\quad y_{n+1} = \frac{x_{n-5}y_{n-8}}{x_{n-2}(1-x_{n-5}y_{n-8})}, \tag{4}$$

with non-zero real initial conditions $x_\delta, y_\delta, \delta \in \{0,1,2,...8\}$.

Theorem 4 *Assume that* $\{x_n, y_n\}$ *is a solution to system (4) and let* $x_{-8} = a,\ x_{-7} = b,\ x_{-6} = c,\ x_{-5} = d,\ x_{-4} = e,\ x_{-3} = f,\ x_{-2} = g,\ x_{-1} = h,\ x_0 = k,\ y_{-8} = l,\ y_{-7} = m,\ y_{-6} = p,\ y_{-5} = q,\ y_{-4} = r,\ y_{-3} = s,\ y_{-2} = t,\ y_{-1} = u,$ *and* $y_0 = v$. *Then, for* $n = 0,1,...,$ *we have*

$$x_{24n-8} = (-1)^n\left(\frac{gt}{l}\right)^{2n}\frac{(-1+dl)^n(1+dl)^n(1+2aq)^n}{(1+dt)^{2n}},$$

$$x_{24n-7} = (-1)^n\left(\frac{hu}{m}\right)^{2n}\frac{(-1+em)^n(1+em)^n(1+2br)^n}{(1+dt)^{2n}},$$

$$x_{24n-6} = (-1)^n\left(\frac{kv}{p}\right)^{2n}\frac{(-1+fp)^n(1+fp)^n(1+2cs)^n}{(1+fv)^{2n}},$$

$$x_{24n-5} = (-1)^n d\left(\frac{al}{gt}\right)^{2n}\frac{(-1+gp)^n(1+gp)^n(1+2dt)^n}{(1+aq)^{2n}},$$

$$x_{24n-4}=(-1)^n e\left(\frac{bm}{hu}\right)^{2n}\frac{(-1+hr)^n(1+hr)^n(1+2eu)^n}{(1+br)^{2n}},$$
$$x_{24n-3}=(-1)^n f\left(\frac{cp}{kv}\right)^{2n}\frac{(-1+ks)^n(1+ks)^n(1+2fv)^n}{(1+cs)^{2n}},$$
$$x_{24n-2}=(-1)^n\left(\frac{t}{al}\right)^{2n}\frac{g^{2n+1}-1+dl)^n(1+dl)^n(1+2aq)^n}{(1+dt)^{2n}},$$
$$x_{24n-1}=(-1)^n\left(\frac{u}{bm}\right)^{2n}\frac{h^{2n+1}(-1+em)^n(1+em)^n(1+2br)^n}{(1+eu)^{2n}},$$
$$x_{24n}=(-1)^n\left(\frac{v}{cp}\right)^{2n}\frac{k^{2n+1}(-1+fp)^n(1+fp)^n(1+2cs)^n}{(1+fv)^{2n}},$$
$$x_{24n+1}=(-1)^{n+1} q\left(\frac{a}{t}\right)^{2n+1}\left(\frac{l}{g}\right)^{2n}\frac{(-1+gq)^n(1+gq)^n(1+2dt)^n}{(1+aq)^{2n+1}},$$
$$x_{24n+2}=(-1)^{n+1} r\left(\frac{b}{u}\right)^{2n+1}\left(\frac{m}{h}\right)^{2n}\frac{(-1+hr)^n(1+hr)^n(1+2eu)^n}{(1+br)^{2n+1}},$$
$$x_{24n+3}=(-1)^{n+1} s\left(\frac{c}{v}\right)^{2n+1}\left(\frac{p}{k}\right)^{2n}\frac{(-1+ks)^n(1+ks)^n(1+2fv)^n}{(1+cs)^{2n+1}},$$
$$x_{24n+4}=(-1)^n\left(\frac{gt}{l}\right)^{2n+1}\frac{(-1+dl)^{n+1}(1+dl)^n(1+2aq)^n}{a^{2n}(1+dt)^{2n+1}},$$
$$x_{24n+5}=(-1)^n\left(\frac{hu}{m}\right)^{2n+1}\frac{(-1+em)^{n+1}(1+em)^n(1+2br)^n}{b^{2n}(1+eu)^{2n+1}},$$
$$x_{24n+6}=(-1)^n\left(\frac{kv}{p}\right)^{2n+1}\frac{(-1+fp)^{n+1}(1+fp)^n(1+2cs)^n}{c^{2n}(1+fv)^{2n+1}},$$
$$x_{24n+7}=(-1)^{n+1} d\left(\frac{al}{gt}\right)^{2n+1}\frac{(-1+gq)^{n+1}(1+gq)^n(1+2dt)^n}{(1+aq)^{2n+1}},$$
$$x_{24n+8}=(-1)^{n+1} e\left(\frac{bm}{hu}\right)^{2n+1}\frac{(-1+hr)^{n+1}(1+hr)^n(1+2eu)^n}{(1+br)^{2n+1}},$$
$$x_{24n+9}=(-1)^{n+1} f\left(\frac{cp}{kv}\right)^{2n+1}\frac{(-1+ks)^{n+1}(1+ks)^n(1+2fv)^n}{(1+cs)^{2n+1}},$$
$$x_{24n+10}=(-1)^n g^{2n+2}\left(\frac{t}{al}\right)^{2n+1}\frac{(-1+dl)^{n+1}(1+dl)^n(1+2aq)^{n+1}}{(1+dt)^{2n+1}},$$
$$x_{24n+11}=(-1)^n h^{2n+2}\left(\frac{u}{bm}\right)^{2n+1}\frac{(-1+em)^{n+1}(1+em)^n(1+2br)^{n+1}}{(1+eu)^{2n+1}},$$
$$x_{24n+12}=(-1)^n k^{2n+2}\left(\frac{v}{cp}\right)^{2n+1}\frac{(-1+fp)^{n+1}(1+fp)^n(1+2cs)^n}{(1+fv)^{2n+1}},$$

$$x_{24n+13}=(-1)^n q\left(\frac{a}{t}\right)^{2n+2}\left(\frac{l}{g}\right)^{2n+1}\frac{(-1+gq)^{n+1}(1+gq)^n(1+2dt)^{n+1}}{(1+aq)^{2n+2}},$$
$$x_{24n+14}=(-1)^n r\left(\frac{b}{u}\right)^{2n+2}\left(\frac{m}{h}\right)^{2n+1}\frac{(-1+hr)^{n+1}(1+hr)^n(1+2eu)^{n+1}}{(1+br)^{2n+2}},$$
$$x_{24n+15}=(-1)^n s\left(\frac{c}{v}\right)^{2n+2}\left(\frac{p}{k}\right)^{2n+1}\frac{(-1+ks)^{n+1}(1+ks)^n(1+2fv)^{n+1}}{(1+cs)^{2n+1}}.$$

And

$$y_{24n-8} = (-1)^n \left(\frac{gt}{a}\right)^{2n} \frac{(1+aq)^{2n}}{l^{2n-1}(-1+gq)^n(1+gq)^n(1+2dt)^n},$$

$$y_{24n-7} = (-1)^n \left(\frac{hu}{b}\right)^{2n} \frac{(1+br)^{2n}}{m^{2n-1}(-1+hr)^n(1+hr)^n(1+2eu)^n},$$

$$y_{24n-6} = (-1)^n \left(\frac{kv}{c}\right)^{2n} \frac{(1+cs)^{2n}}{p^{2n-1}(-1+ks)^n(1+ks)^n(1+2fv)^n},$$

$$y_{24n-5} = (-1)^n q \left(\frac{al}{gt}\right)^{2n} \frac{(1+dt)^{2n}}{(-1+dl)^n(1+dl)^n(1+2aq)^n},$$

$$y_{24n-4} = (-1)^n r \left(\frac{bm}{hu}\right)^{2n} \frac{(1+eu)^{2n}}{(-1+em)^n(1+em)^n(1+2br)^n},$$

$$y_{24n-3} = (-1)^n s \left(\frac{cp}{kv}\right)^{2n} \frac{(1+fv)^{2n}}{(-1+fp)^n(1+fp)^n(1+2cs)^n},$$

$$y_{24n-2} = (-1)^n \left(\frac{g}{al}\right)^{2n} \frac{t^{2n+1}(1+aq)^{2n}}{(-1+gq)^n(1+gq)^n(1+2dt)^n},$$

$$y_{24n-1} = (-1)^n \left(\frac{h}{bm}\right)^{2n} \frac{u^{2n+1}(1+br)^{2n}}{(-1+hr)^n(1+hr)^n(1+2eu)^n},$$

$$y_{24n} = (-1)^n \left(\frac{k}{cp}\right) \frac{v^{2n+1}(1+cs)^{2n}}{(-1+ks)^n(1+ks)^n(1+2fv)^n},$$

$$y_{24n+1} = (-1)^{n+1} d \left(\frac{a}{t}\right)^{2n} \left(\frac{l}{g}\right)^{2n+1} \frac{(1+dt)^{2n}}{(-1+dl)^{n+1}(1+dl)^n(1+2aq)^n},$$

$$y_{24n+2} = (-1)^{n+1} e \left(\frac{b}{u}\right)^{2n} \left(\frac{m}{h}\right)^{2n+1} \frac{(1+eu)^{2n}}{(-1+em)^{n+1}(1+em)^n(1+2br)^n},$$

$$y_{24n+3} = (-1)^{n+1} f \left(\frac{c}{v}\right)^{2n} \left(\frac{p}{k}\right)^{2n+1} \frac{(1+fv)^{2n}}{(-1+fp)^{n+1}(1+fp)^n(1+2cs)^n},$$

$$y_{24n+4} = (-1)^n \left(\frac{gt}{a}\right)^{2n+1} \frac{(1+aq)^{2n+1}}{l^{2n}(-1+gq)^{n+1}(1+gq)^n(1+2dt)^n},$$

$$
\begin{aligned}
y_{24n+5} &= (-1)^n \left(\frac{hu}{b}\right)^{2n+1} \frac{(1+br)^{2n+1}}{m^{2n}(-1+hr)^{n+1}(1+hr)^n(1+2eu)^n},\\
y_{24n+6} &= (-1)^n \left(\frac{kv}{c}\right)^{2n+1} \frac{(1+cs)^{2n+1}}{p^{2n}(-1+ks)^{n+1}(1+ks)^n(1+2fv)^n},\\
y_{24n+7} &= (-1)^{n+1} q \left(\frac{al}{gt}\right)^{2n+1} \frac{(1+dt)^{2n+1}}{(-1+dl)^{n+1}(1+dl)^n(1+2aq)^{n+1}},\\
y_{24n+8} &= (-1)^{n+1} r \left(\frac{bm}{hu}\right)^{2n+1} \frac{(1+eu)^{2n+1}}{(-1+em)^{n+1}(1+em)^n(1+2br)^{n+1}},\\
y_{24n+9} &= (-1)^{n+1} s \left(\frac{cp}{kv}\right)^{2n+1} \frac{(1+fv)^{2n+1}}{(-1+fp)^{n+1}(1+fp)^n(1+2cs)^{n+1}},\\
y_{24n+10} &= (-1)^n t^{2n+2} \left(\frac{g}{al}\right)^{2n+1} \frac{(1+aq)^{2n+1}}{(-1+gq)^{n+1}(1+gq)^n(1+2dt)^{n+1}},\\
y_{24n+11} &= (-1)^n u^{2n+2} \left(\frac{h}{bm}\right)^{2n+1} \frac{(1+br)^{2n+1}}{(-1+hr)^{n+1}(1+hr)^n(1+2eu)^{n+1}},\\
y_{24n+12} &= (-1)^n v^{2n+2} \left(\frac{k}{cp}\right)^{2n+1} \frac{(1+cs)^{2n+1}}{(-1+ks)^{n+1}(1+ks)^n(1+2fv)^{n+1}},\\
y_{24n+13} &= (-1)^{n+1} d \left(\frac{a}{t}\right)^{2n+1} \left(\frac{l}{g}\right)^{2n+2} \frac{(1+dt)^{2n+1}}{(-1+dl)^{n+1}(1+dl)^{n+1}(1+2aq)^{n+1}},\\
y_{24n+14} &= (-1)^{n+1} e \left(\frac{b}{u}\right)^{2n+1} \left(\frac{m}{h}\right)^{2n+2} \frac{(1+eu)^{2n+1}}{(-1+em)^{n+1}(1+em)^{n+1}(1+2br)^{n+1}},\\
y_{24n+15} &= (-1)^{n+1} f \left(\frac{c}{v}\right)^{2n+1} \left(\frac{p}{k}\right)^{2n+2} \frac{(1+fv)^{2n+1}}{(-1+fp)^{n+1}(1+fp)^{n+1}(1+2cs)^{n+1}}.
\end{aligned}
$$

Proof. The proof can be achieved by induction. Thus, it is omitted.

2.5 Numerical Examples

In order to confirm our theoretical results, we will consider in this subsection some numerical examples under some specific initial conditions.

Example 1. Figure 1 presents the solutions of system (1) with the following random initial conditions $x_{-8} = -1.1,\ x_{-7} = -3.6,\ x_{-6} = -8.6,\ x_{-5} = -0.5,\ x_{-4} = -1.2,\ x_{-3} = -3.5,\ x_{-2} = -4.3,\ x_{-1} = -6.3,\ x_0 = -2.4,\ y_{-8} = -1.3,\ y_{-7} = -1.5,\ y_{-6} = -5.4,\ y_{-5} = -12.4,\ y_{-4} = -1.2,\ y_{-3} = -1.5,\ y_{-2} = -17.4,\ y_{-1} = -1.3$, and $y_0 = -2$.

Example 2. The solutions of system (2) with the random initial conditions $x_{-8} = 2.3,\ x_{-7} = 3.1,\ x_{-6} = 2,\ x_{-5} = 11,\ x_{-4} = 2.1,\ x_{-3} = 5,\ x_{-2} = 9,\ x_{-1} = 10.2,\ x_0 = 12.1,\ y_{-8} = 3,\ y_{-7} = -11,\ y_{-6} = 2.1,\ y_{-5} = 4,\ y_{-4} = 2.6,\ y_{-3} = 3.4,\ y_{-2} = 5,\ y_{-1} = 11.3$, and $y_0 = 1.2$, are pictured in Figure 2.

Example 3. This example is included to plot the solutions of system (3) with the initial data $x_{-8} = 3.1,\ x_{-7} = 2.6,\ x_{-6} = 1.6,\ x_{-5} = 1.5,\ x_{-4} = 10.2,\ x_{-3} = -9,\ x_{-2} = 1.3,\ x_{-1} = 6.3,\ x_0 = 15.4,\ y_{-8} = 3.3,\ y_{-7} = -1.5,\ y_{-6} = 5.9,\ y_{-5} = 12.4,\ y_{-4} = 3.2,\ y_{-3} = 9,\ y_{-2} = 2.4,\ y_{-1} = 1.3$, and $y_0 = -2$. The solutions are shown in Figure 3.

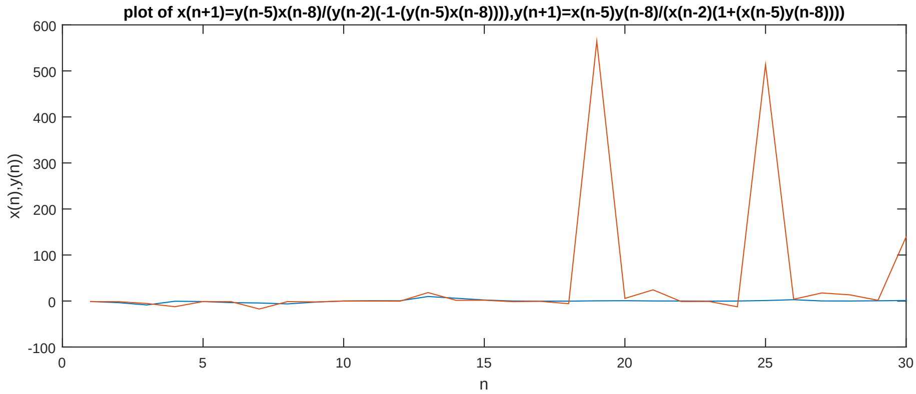

Figure 1: Solutions of System 1.

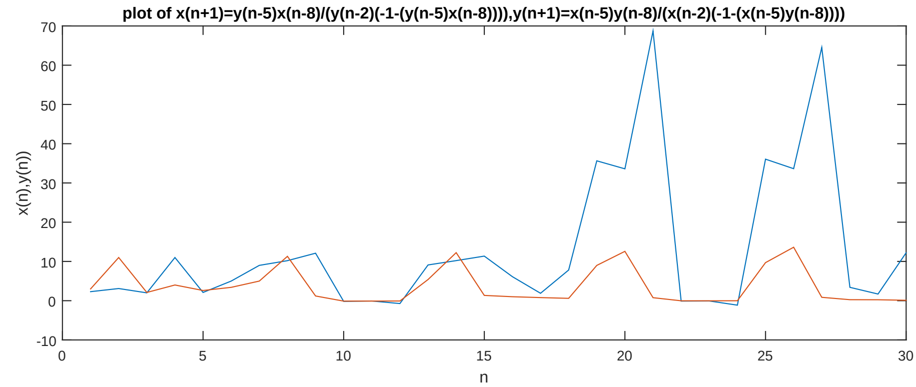

Figure 2: Solutions of System 2.

Example 4. In Figure 4, we illustrate the behaviour of the solutions of system (4) under the the initial conditions $x_{-8} = 0.3$, $x_{-7} = 4.1$, $x_{-6} = 2.4$, $x_{-5} = 7.1$, $x_{-4} = 3.5$, $x_{-3} = 5.2$, $x_{-2} = 3.3$, $x_{-1} = 1.2$, $x_0 = -2.1$, $y_{-8} = 3.2$, $y_{-7} = -4.3$, $y_{-6} = -3.1$, $y_{-5} = 6.5$, $y_{-4} = 3.2$, $y_{-3} = 3.4$, $y_{-2} = 4.5$, $y_{-1} = -9.3$, and $y_0 = 10.2$.

3 Conclusion

This article has shown the exact solutions for four different systems of difference equations. For example, Subsection 2.1 has clearly demonstrated the solutions of system 1 and presented them by some sophisticated expressions. The fractional solutions of system 2 are proved in Theorem 2. Example 3 with Figure 3 are devoted to illustrate the behaviour of the product solutions of system 3 under specific parameters given above. We have confirmed the analytical results by depicting four different figures. This work can be easily achieved for more difference equations with higher orders.

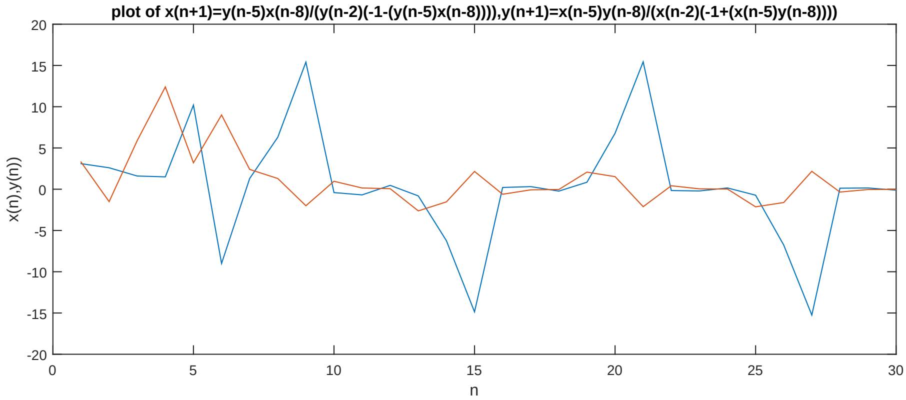

Figure 3: Solutions of System 3.

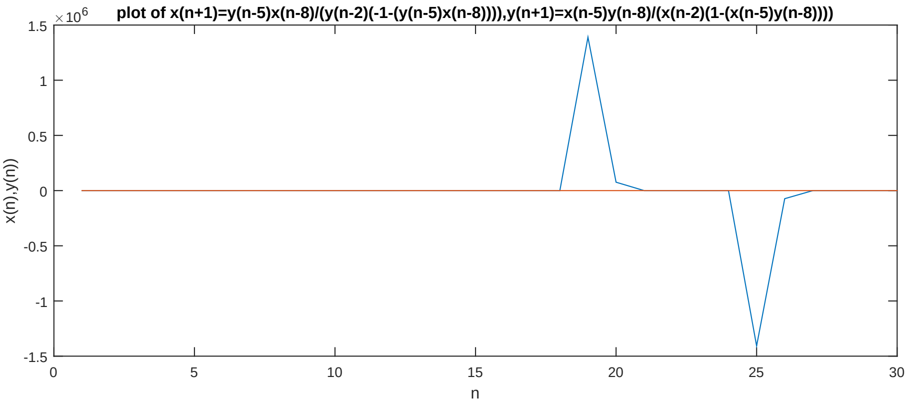

Figure 4: Solutions of System 4.

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