Analysis of Qualitative Behavior of Fifth Order Difference Equations

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Abstract

The main aim of this paper is to investigate the stability, global attractivity and periodic nature of the solutions of the difference equations

$$x_{n+1} = ax_{n-1} \pm \frac{bx_{n-1}x_{n-2}}{cx_{n-2} \pm dx_{n-4}}, \qquad n = 0, 1, 2, ...,$$

where the initial conditions x_{-4}, x_{-3} , x_{-2}, x_{-1} and x_0 are arbitrary positive real numbers and a, b, c, d are constants.

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1 Introduction

Recently, studying the qualitative behavior of difference equations and systems is a topic of a great interest. Applications of discrete dynamical systems and difference equations have appeared recently in many areas such as ecology, population dynamics, statistical problems, number theory, geometry,genetics in biology,and psychology. Although difference equations come into view simple in form,it is quite difficult to know thoroughly the behaviour of their solutions because some model for the development of the basic theory of the global behavior of difference equations come from the results of rational difference equations. More results on the qualitative behaivor of difference equations can be obtained in [9-21].

Almatrafi et al. [1] has explored the stability, boundedness and other properties of the following difference equation

$$x_{n+1} = ax_n + \frac{bx_n^2 + cx_nx_{n-1} + dx_{n-1}^2}{\alpha x_n^2 + \beta x_nx_{n-1} + \gamma x_{n-1}^2}.$$

In [2], Abo-Zeid studied the global behavior of higher order rational difference equation

$$x_{n+1} = \frac{Ax_{n-k}}{B - C\prod_{i=0}^{k} x_{n-i}}.$$

Andruch [3] has got the solution of the difference equation

$$x_{n+1} = \frac{\alpha x_n}{b + cx_nx_{n-1}}.$$

Din and Elsayed [4] investigated the stability analysis of ecological model

$$x_{n+1} = \alpha + \beta x_n + \gamma x_{n-1}e^{-y_n}, \qquad y_{n-1} = \delta + \varepsilon y_n + \zeta y_{n-1}e^{-x_n}.$$

Elabbasy et al. [5] studied the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

In [6] Elsayed and Abdul Khaliq studied the global attractivity and periodicity behavior of difference equation

$$x_{n+1} = a x_{n-1} + \frac{b x_{n-k} + c x_{n-s}}{d + e x_{n-t}}.$$

Gibbons et al. [7] investigated the qualitative behavior of solution of the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}}{\alpha + x_n}.$$

Hamza et al. [8] studied the asymptotic stability of the nonnegative equilibrium point of the difference equation

$$x_{n+1} = \frac{A x_{n-1}}{B + C \prod_{i=l}^{k} x_{n-2i}}.$$

In this paper, we investigated the dynamics and the form of the solutions of some nonlinear difference equations of order five as follows:

$$x_{n+1} = a x_{n-1} \pm \frac{b x_{n-1} x_{n-2}}{c x_{n-2} \pm d x_{n-4}}, \qquad n = 0, 1, 2, ...,$$

with initial conditions x_{-4}, x_{-3} ,x_{-2}, x_{-1} and x_0 are arbitrary positive real numbers and a, b, c, d are constants.

Here, we will review some of the definitions and theorems used in solving special cases of difference equations.

Definition 1.1. Let I be some interval of real numbers and let

$$F : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial condition $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, x_{n-2}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{1}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1.2. A point $\overline{x} \in I$ is called an equilibrium point of Eq.(1) if

$$\overline{x} = F(\overline{x}),$$

that is,

$$x_n = \overline{x} \ for \ all \quad n \geq -k.$$

is a solution of Eq.(1), or equivalently, $\overline{x}$ is **a fixed point** of F.

Definition 1.3. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

Definition 1.4. (**Fibonacci sequence**)

The sequence $\{F_m\}_{m=0}^{\infty} = \{1, 1, 2, 3, 5, 8, ...\}$ that is $F_m = F_{m-1} + F_{m-2}$, $m \geq 0$, $F_{-2} = -1$, $F_{-1} = 1$ is called Fibonacci sequence.

Definition 1.5. (**Stability**)

Let x^* be an equilibrium point of Eq.(1).

(i) The equilibrium point $\overline{x}$ of Eq.(1) is called **locally stable** if for every $\epsilon > 0$, there exists

$\delta > 0$ such that for all $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(1) with

$$|x_{-k} - \overline{x}| + |x_{1-k} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

then

$$|x_n - \overline{x}| < \epsilon \quad for\ all \quad n \geq 0.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(1) is called **locally asymptotically stable** if it is locally stable, and if there exists $\gamma > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(1) with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

then

$$\lim_{n \to \to \infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Eq.(1) is called **a global attractor** if for every solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1) we have

$$\lim_{n \to \to \infty} x_n = \overline{x}$$

(iv) The equilibrium point $\overline{x}$ of Eq.(1) is called **globally asymptotically stable** if it is locally stable and global attractor of Eq.(1).

(v) The equilibrium point $\overline{x}$ of Eq.(1) is called **unstable** if $\overline{x}$ is not **locally stable**.

Linearized Stability Analysis

Suppose that the function F is continuously differentiable in some open neighborhood of an equilibrium point x^*. Let

$$p_i = \frac{\partial F}{\partial u_i}(\overline{x}, \overline{x}, ..., \overline{x}) \quad for \quad i = 0, 1, ..., k,$$

denote the partial derivatives of $F(u_0, u_1, ...u_k)$ evaluated at the equilibrium $\overline{x}$ of Eq.(1).
Then the equation

$$y_{n+1} = p_0 y_n + p_1 y_{n-1} + ... + p_k y_{n-k} \ , \qquad n = 0, 1, ..., \tag{2}$$

is called **the linearized equation associated** of Eq.(1) about the equilibrium point $\overline{x}$ and the equation

$$\lambda^{k+1} - p_0 \lambda^k - ... - p_{k-1}\lambda - p_k = 0, \tag{3}$$

is called the characteristic equation of Eq.(2) about $\overline{x}$.

The following result known as the Linear Stability Theorem is very useful in determining the local stability character of the equilibrium point $\overline{x}$ of Eq.(1).

Theorem A. [33] Assume that $p_0, p_2, ..., p_k$ are real numbers such that

$$|p_0| + |p_1| + ... + |p_k| < 1,$$

or

$$\sum_{i=1}^{k} |p_i| < 1.$$

Then all roots of Eq.(3) lie inside the unit disk.

Theorem B [29]: Let $g : [p,q]^{k+1} \to [p,q]$ be a continuous function, where p and q are real numbers with $p < q$, and consider the following equation:

$$x_{n+1} = g(x_n, x_{n-1}, ...x_{n-k}) \qquad n = 0, 1, 2... . \tag{4}$$

Suppose that g satisfies the following conditions:

(a) $g(x_1, x_2, ..., x_{k+1})$ is non-increasing in one component (for example x_σ) for each $x_r(r \neq \sigma)$ in $[\alpha, \beta]$, and is non-increasing in the remaining components for each $x_\sigma \in [\alpha, \beta]$;

(b) If $(m, M) \in [\alpha, \beta] \times [\alpha, \beta]$ is a solution of the system

$$M = g(m, m, ..., m, M, m, ...m, m) \;\; and \;\; m = g(M, M, ..., M, m, M, ..., M, M),$$

then

$$m = M.$$

Then Eq.(4) has a unique equilibrium $\overline{x} \in [\alpha, \beta]$ and every solution of Eq.(4) converges to $\overline{x}$.

2 Dynamics of the Equation $x_{n+1} = ax_{n-1} + \frac{bx_{n-1}x_{n-2}}{cx_{n-2}+dx_{n-4}}$

In this section, we study the some qualitative behavior properties for the recursive equation in the form:

$$x_{n+1} = ax_{n-1} + \frac{bx_{n-1}x_{n-2}}{cx_{n-2} + dx_{n-4}}, \tag{5}$$

where the initial values $x_{-4},\ x_{-3},\ x_{-2},\ x_{-1}\ and\ x_0$ are arbitrary positive real numbers. Also, $a,\ b,\ c,\ d$ are constants.

2.1 Local Stability of the Equilibrium Point

In this part we obtain the local stability character of the solution of Eq.(5) when $a,\ b,\ c$ and d are positive real numbers.

Equation (5) has a unique equilibrium point and is given by

$$\overline{x} = a\overline{x} + \frac{b\overline{x}^2}{c\overline{x} + d\overline{x}},$$

or

$$\overline{x}^2(1-a)(c+d) = b\overline{x},$$

if $(1-a)(c+d) \neq b$, then the unique equilibrium point is $\overline{x} = 0$.

Let $f : (0, \infty)^3 \to (0, \infty)$ be a function defined by

$$F(u, v, w) = au + \frac{buv}{cv + dw}.$$

Then it follows that,

$$F_u(u, v, w) = a + \frac{bv}{cv + dw},\ F_v(u, v, w) = \frac{bduw}{(cv + dw)^2},\ F_w(u, v, w) = \frac{-bduv}{(cv + dw)^2},$$

we see that

$$F_u(\overline{x}, \overline{x}, \overline{x}) = a + \frac{b}{c+d},\ F_v(\overline{x}, \overline{x}, \overline{x}) = \frac{bd}{(c+d)^2},\ F_w(\overline{x}, \overline{x}, \overline{x}) = \frac{-bd}{(c+d)^2}.$$

The linearized equation about $\overline{x}$ is

$$y_{n+1} - \left(a + \frac{b}{c+d}\right) y_{n-1} - \left(\frac{bd}{(c+d)^2}\right) y_{n-2} - \left(\frac{-bd}{(c+d)^2}\right) y_{n-4} = 0. \tag{6}$$

Theorem 1. Suppose that

$$b\left(c+3d\right)<\left(1-a\right)\left(c+d\right)^{2}.$$

Then the equilibrium point of Eq.(5) is locally asymptotically stable.

Proof. It is following by Theorem A that Eq.(6) is asymptotically stable if

$$\left|a+\frac{b}{c+d}\right|+\left|\frac{bd}{\left(c+d\right)^{2}}\right|+\left|\frac{-bd}{(c+d)^{2}}\right|<1,$$

or

$$\frac{bc+3bd}{\left(c+d\right)^{2}}<1,$$

and

$$b\left(c+3d\right)<(1-a)(c+d)^{2}.$$

The proof is complete.

2.2 Global Attractor of the Equilibrium Point of Eq.(5)

Theorem 2. The equilibrium point $\overline{x}$ of Eq.(5) is global attractor if $(1-a)\left(c+d\right)\neq b$.

Proof. Let p,q are real numbers and suppose that $g:[p,q]^{3}\rightarrow[p,q]$ be function defined by $g(u,v,w)=au+\dfrac{buv}{cv+dw}$, then we can easily see that the function $g(u,v,w)$ increasing in u,v and decreasing in w.

Suppose that (m,M) is a solution of the system

$$M=g\left(M,M,m\right)\qquad and\qquad m=g\left(m,m,M\right).$$

Then from Eq. (5) , we see that

$$M=aM+\frac{bM^{2}}{cM+dm},\quad m=am+\frac{bm^{2}}{cm+dM},$$

or

$$M^{2}c(1-a)+Mmd(1-a)=bM^{2},\quad m^{2}c(1-a)+Mmd(1-a)=bm^{2}.$$

Subtracting we have

$$(M^{2}-m^{2})\left\{c(1-a)-b\right\}=0,\qquad c(1-a)\neq b.$$

Then

$$M=m.$$

Therefore by Theorem B that $\overline{x}$ is a global attractor of Eq.(5) and then the proof is complete.

2.3 Boundedness of Solution of Eq.(5)

Theorem 3. Every solution of Eq.(5) is bounded if $\left(a+\dfrac{b}{c}\right)<1$.

Proof. Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of Eq.(5). Then from Eq.(5) we see that

$$x_{n+1}=ax_{n-1}+\frac{bx_{n-1}x_{n-2}}{cx_{n-2}+dx_{n-4}}\leq ax_{n-1}+\frac{bx_{n-1}x_{n-2}}{cx_{n-2}}=\left(a+\frac{b}{c}\right)x_{n-1}.$$

Then if $\left(a+\dfrac{b}{c}\right)<1$, we get

$$x_{n+1}\leq x_{n-1},\quad for\ all\ n\geq 0.$$

Then the sequence $\{x_n\}_{n=1}^{\infty}$ is decreasing and so are bounded from above by $M=\max\{x_{-4},x_{-3},x_{-2},x_{-1},x_0\}$.

2.4 Solutions of the Equation $x_{n+1} = x_{n-1} + \frac{x_{n-1}x_{n-2}}{x_{n-2}+x_{n-4}}$

In this part, we obtain the solution of the recursive equation in the form:

$$x_{n+1} = x_{n-1} + \frac{x_{n-1}x_{n-2}}{x_{n-2}+x_{n-4}}, \tag{7}$$

where the initial values $x_{-4},\ x_{-3},\ x_{-2},\ x_{-1}$ *and* x_0 are arbitrary non zero real numbers.

Theorem 4. Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of difference equation (7). Then for $n = 0, 1, ...$

$$x_{6n-4} = s\prod_{i=0}^{n-1}\frac{(F_{4i+3}k+F_{4i+2}p)(F_{4i+1}r+F_{4i}s)(F_{4i+1}h+F_{4i}r)}{(F_{4i+2}k+F_{4i+1}p)(F_{4i}r+F_{4i-1}s)(F_{4i}h+F_{4i-1}r)},$$

$$x_{6n-3} = p\prod_{i=0}^{n-1}\frac{(F_{4i+1}k+F_{4i}p)(F_{4i+3}r+F_{4i+2}s)(F_{4i+3}h+F_{4i+2}r)}{(F_{4i}k+F_{4i-1}p)(F_{4i+2}r+F_{4i+1}s)(F_{4i+2}h+F_{4i+1}r)},$$

$$x_{6n-2} = r\prod_{i=0}^{n-1}\frac{(F_{4i+3}k+F_{4i+2}p)(F_{4i+5}r+F_{4i+4}s)(F_{4i+1}h+F_{4i}r)}{(F_{4i+2}k+F_{4i+1}p)(F_{4i+4}r+F_{4i+3}s)(F_{4i}h+F_{4i-1}r)},$$

$$x_{6n-1} = k\prod_{i=0}^{n-1}\frac{(F_{4i+5}k+F_{4i+4}p)(F_{4i+3}r+F_{4i+2}s)(F_{4i+3}h+F_{4i+2}r)}{(F_{4i+4}k+F_{4i+3}p)(F_{4i+2}r+F_{4i+1}s)(F_{4i+2}h+F_{4i-+1}r)},$$

$$x_{6n} = h\prod_{i=0}^{n-1}\frac{(F_{4i+3}k+F_{4i+2}p)(F_{4i+5}r+F_{4i+4}s)(F_{4i+5}h+F_{4i+4}r)}{(F_{4i+2}k+F_{4i+1}p)(F_{4i+4}r+F_{4i+3}s)(F_{4i+4}h+F_{4i+3}r)},$$

$$x_{6n+1} = \frac{k(2r-s)}{r+s}\prod_{i=0}^{n-1}\frac{(F_{4i+5}k+F_{4i+4}p)(F_{4i+7}r+F_{4i+6}s)(F_{4i+3}h+F_{4i+2}r)}{(F_{4i+4}k+F_{4i+3}p)(F_{4i+6}r+F_{4i+5}s)(F_{4i+2}h+F_{4i+1}r)},$$

where $x_{-4} = s$, $x_{-3} = p$, $x_{-2} = r$, $x_{-1} = k$ and $x_0 = h$.

Proof. For $n = 0$, the result holds. Now, assume that $n > 0$ and that our assumption holds for $n-1$. That is,

$$x_{6n-9} = p\prod_{i=0}^{n-2}\frac{(F_{4i+1}k+F_{4i}p)(F_{4i+3}r+F_{4i+2}s)(F_{4i+3}h+F_{4i+2}r)}{(F_{4i}k+F_{4i-1}p)(F_{4i+2}r+F_{4i+1}s)(F_{4i+2}h+F_{4i+1}r)},$$

$$x_{6n-8} = r\prod_{i=0}^{n-2}\frac{(F_{4i+3}k+F_{4i+2}p)(F_{4i+5}r+F_{4i+4}s)(F_{4i+1}h+F_{4i}r)}{(F_{4i+2}k+F_{4i+1}p)(F_{4i+4}r+F_{4i+3}s)(F_{4i}h+F_{4i-1}r)},$$

$$x_{6n-7} = k\prod_{i=0}^{n-2}\frac{(F_{4i+5}k+F_{4i+4}p)(F_{4i+3}r+F_{4i+2}s)(F_{4i+3}h+F_{4i+2}r)}{(F_{4i+4}k+F_{4i+3}p)(F_{4i+2}r+F_{4i+1}s)(F_{4i+2}h+F_{4i-+1}r)},$$

$$x_{6n-6} = h\prod_{i=0}^{n-2}\frac{(F_{4i+3}k+F_{4i+2}p)(F_{4i+5}r+F_{4i+4}s)(F_{4i+5}h+F_{4i+4}r)}{(F_{4i+2}k+F_{4i+1}p)(F_{4i+4}r+F_{4i+3}s)(F_{4i+4}h+F_{4i+3}r)},$$

$$x_{6n-5} = \frac{k(2r+s)}{r+s}\prod_{i=0}^{n-2}\frac{(F_{4i+5}k+F_{4i+4}p)(F_{4i+7}r+F_{4i+6}s)(F_{4i+3}h+F_{4i+2}r)}{(F_{4i+4}k+F_{4i+3}p)(F_{4i+6}r+F_{4i+5}s)(F_{4i+2}h+F_{4i+1}r)}.$$

Now, it follows from Eq. (7) that

$$
\begin{aligned}
x_{6n-4} &= x_{6n-6} + \frac{x_{6n-6}x_{6n-7}}{x_{6n-7}+x_{6n-9}} \\
&= x_{6n-6}\left(1+\frac{x_{6n-7}}{x_{6n-7}+x_{6n-9}}\right) \\
&= x_{6n-6}\left(1+\frac{k\prod_{i=0}^{n-2}\frac{(F_{4i+5}k+F_{4i+4}p)(F_{4i+3}r+F_{4i+2}s)(F_{4i+3}h+F_{4i+2}r)}{(F_{4i+4}k+F_{4i+3}p)(F_{4i+2}r+F_{4i+1}s)(F_{4i+2}h+F_{4i-+1}r)}}{k\prod_{i=0}^{n-2}\frac{(F_{4i+5}k+F_{4i+4}p)(F_{4i+3}r+F_{4i+2}s)(F_{4i+3}h+F_{4i+2}r)}{(F_{4i+4}k+F_{4i+3}p)(F_{4i+2}r+F_{4i+1}s)(F_{4i+2}h+F_{4i-+1}r)} + p\prod_{i=0}^{n-2}\frac{(F_{4i+1}k+F_{4i}p)(F_{4i+3}r+F_{4i+2}s)(F_{4i+3}h+F_{4i+2}r)}{(F_{4i}k+F_{4i-1}p)(F_{4i+2}r+F_{4i+1}s)(F_{4i+2}h+F_{4i+1}r)}}\right) \\
&= x_{6n-6}\left(1+\frac{k\prod_{i=0}^{n-2}\frac{(F_{4i+5}k+F_{4i+4}p)}{(F_{4i+4}k+F_{4i+3}p)}}{k\prod_{i=0}^{n-2}\frac{(F_{4i+5}k+F_{4i+4}p)}{(F_{4i+4}k+F_{4i+3}p)} + p\prod_{i=0}^{n-2}\frac{(F_{4i+1}k+F_{4i}p)}{(F_{4i}k+F_{4i-1}p)}}\right) \\
&= x_{6n-6}\left(1+\frac{k}{k+p\prod_{i=0}^{n-2}\frac{(F_{4i+1}k+F_{4i}p)}{(F_{4i+5}k+F_{4i+4}p)}\frac{(F_{4i+4}k+F_{4i+3}p)}{(F_{4i}k+F_{4i-1}p)}}\right) \\
&= x_{6n-6}\left(1+\frac{1}{1+\frac{F_{4n-4}k+F_{4n-5}p}{F_{4n-3}k+F_{4n-4}p}}\right) = x_{6n-6}\left(1+\frac{F_{4n-3}k+F_{4n-4}p}{F_{4n-2}k+F_{4n-3}p}\right) \\
&= x_{6n-6}\left(\frac{F_{4n-1}k+F_{4n-2}p}{F_{4n-2}k+F_{4n-3}p}\right) \\
&= h\prod_{i=0}^{n-2}\frac{(F_{4i+3}k+F_{4i+2}p)(F_{4i+5}r+F_{4i+4}s)(F_{4i+5}h+F_{4i+4}r)}{(F_{4i+2}k+F_{4i+1}p)(F_{4i+4}r+F_{4i+3}s)(F_{4i+4}h+F_{4i+3}r)}\left(\frac{F_{4n-1}k+F_{4n-2}p}{F_{4n-2}k+F_{4n-3}p}\right),
\end{aligned}
$$

Consequently, we have

$$
x_{6n-4} = s\prod_{i=0}^{n-1}\frac{(F_{4i+3}k+F_{4i+2}p)(F_{4i+1}r+F_{4i}s)(F_{4i+1}h+F_{4i}r)}{(F_{4i+2}k+F_{4i+1}p)(F_{4i}r+F_{4i-1}s)(F_{4i}h+F_{4i-1}r)}.
$$

Similarly,

$$
\begin{aligned}
x_{6n-3} &= x_{6n-5} + \frac{x_{6n-5}x_{6n-6}}{x_{6n-6}+x_{6n-8}} \\
&= x_{6n-5}\left(1+\frac{x_{6n-6}}{x_{6n-6}+x_{6n-8}}\right) \\
&= x_{6n-5}\left(1+\frac{h\prod_{i=0}^{n-2}\frac{(F_{4i+5}h+F_{4i+4}r)}{(F_{4i+4}h+F_{4i+3}r)}}{h\prod_{i=0}^{n-2}\frac{(F_{4i+5}h+F_{4i+4}r)}{(F_{4i+4}h+F_{4i+3}r)}+r\prod_{i=0}^{n-2}\frac{(F_{4i+1}h+F_{4i}r)}{(F_{4i}h+F_{4i-1}r)}}\right) \\
&= x_{6n-5}\left(1+\frac{h}{h+r\prod_{i=0}^{n-2}\frac{(F_{4i+1}h+F_{4i}r)}{(F_{4i+5}h+F_{4i+4}r)}\frac{(F_{4i+4}h+F_{4i+3}r)}{(F_{4i}h+F_{4i-1}r)}}\right) \\
&= x_{6n-5}\left(1+\frac{1}{1+\frac{F_{4n-4}h+F4_{n-5}r}{F_{4n-3}h+F_{4n-4}r}}\right) \\
&= x_{6n-5}\left(1+\frac{F_{4n-3}h+F4_{n-4}r}{F_{4n-2}h+F_{4n-3}r}\right) \\
&= x_{6n-5}\left(\frac{F_{4n-1}h+F4_{n-2}r}{F_{4n-2}h+F_{4n-3}r}\right) \\
&= \frac{k\,(2r+s)}{r+s}\prod_{i=0}^{n-2}\frac{(F_{4i+5}k+F_{4i+4}p)(F_{4i+7}r+F_{4i+6}s)(F_{4i+3}h+F_{4i+2}r)}{(F_{4i+4}k+F_{4i+3}p)(F_{4i+6}r+F_{4i+5}s)(F_{4i+2}h+F_{4i+1}r)}\left(\frac{F_{4n-1}h+F4_{n-2}r}{F_{4n-2}h+F_{4n-3}r}\right)
\end{aligned}
$$

Then

$$x_{6n-3} = p\prod_{i=0}^{n-1}\frac{(F_{4i+1}k+F_{4i}p)\,(F_{4i+3}r+F_{4i+2}s)\,(F_{4i+3}h+F_{4i+2}r)}{(F_{4i}k+F_{4i-1}p)\,(F_{4i+2}r+F_{4i+1}s)\,(F_{4i+2}h+F_{4i+1}r)}.$$

Similarly, other relations can be obtain and thus, the proof has been proved.

3 Dynamics of the Equation $x_{n+1} = ax_{n-1} + \frac{bx_{n-1}x_{n-2}}{cx_{n-2}-dx_{n-4}}$

In this part, we examine the following equation

$$x_{n+1} = ax_{n-1} + \frac{bx_{n-1}x_{n-2}}{cx_{n-2}-dx_{n-4}}, \tag{8}$$

where the initial conditions $x_{-4}, x_{-3},\ x_{-2},\ x_{-1}$ *and* x_0 are arbitrary positive real numbers and $a,\ b,\ c,\ d$ are constants.

3.1 Local Stability of the Equilibrium Point

In this part we obtain the local stability character of the solution of Eq.(8) when $a,\ b,\ c$ and d are positive real numbers.

Equation (8) has a unique equilibrium point and is given by

$$\overline{x} = a\overline{x} + \frac{b\overline{x}^2}{c\overline{x}-d\overline{x}},$$

or

$$\overline{x}^2(1-a)(c-d) = b\overline{x},$$

if $(1-a)(c-d) \neq b$, then the unique equilibrium point is $\overline{x}=0$.

Let $F:(0,\infty)^3 \to (0,\infty)$ be a function defined by

$$F(u,v,w)=au+\frac{buv}{cv-dw}.$$

Then it follows that,

$$F_u(u,v,w)=a+\frac{bv}{cv-dw},\ F_v(u,v,w)=\frac{-bduw}{(cv-dw)^2},\ F_w(u,v,w)=\frac{bduv}{(cv-dw)^2},$$

we see that

$$F_u(\overline{x},\overline{x},\overline{x})=a+\frac{b}{c-d},\ F_v(\overline{x},\overline{x},\overline{x})=\frac{-bd}{(c-d)^2},\ F_w(\overline{x},\overline{x},\overline{x})=\frac{bd}{(c-d)^2}.$$

The linearized equation about $\overline{x}$ is

$$y_{n+1}-\left(a+\frac{b}{c-d}\right)y_{n-1}-\left(\frac{-bd}{(c-d)^2}\right)y_{n-2}-\left(\frac{bd}{(c-d)^2}\right)y_{n-4}=0. \tag{9}$$

Theorem 5. Suppose that

$$b\,(c+d)<(1-a)(c-d)^2, \quad \text{when } \ c>d.$$

$$b\,(3d-c)<(1-a)(d-c)^2, \quad \text{when } \ c<d.$$

Then the equilibrium point of Eq.(8) is locally asymptotically stable.

Proof. It is followed by Theorem A that Eq.(9) is asymptotically stable if

$$\left|a+\frac{b}{c-d}\right|+\left|\frac{-bd}{(c-d)^2}\right|+\left|\frac{bd}{(c-d)^2}\right|<1,$$

or

$$\left|a+\frac{b}{c-d}\right|+\left|\frac{2bd}{(c-d)^2}\right|<1,$$

and when $c>d$, we see that

$$\frac{b}{c-d}+\frac{2bd}{(c-d)^2}<1-a \qquad \Rightarrow b\,(c+d)<(1-a)(c-d)^2.$$

and when $c<d$, we see that

$$\frac{b}{d-c}+\frac{2bd}{(d-c)^2}<1-a \quad \Rightarrow b\,(3d-c)<(1-a)(d-c)^2.$$

The proof is completed.

3.2 Global Attractor of the Equilibrium Point of Eq.(8)

Theorem 6. The equilibrium point $\overline{x}$ of Eq.(8) is global attractor if $(1-a)\,(c-d)\neq b$.

Proof. Let $p.q$ are real numbers and suppose that $g:[p,q]^3 \to [p,q]$ be function defined by $g(u,v,w)=au+\frac{buv}{cv-dw}$, then we can easily see that the function $g(u,v,w)$ has two cases

(i) The function $g(u,v,w)$ increasing in u,w and decreasing in v.

Suppose that (m,M) is a solution of the system

$$M=g\,(M,m,M) \quad and \quad m=g\,(m,M,m)\,.$$

Then from Eq.(8) , we see that

$$M = aM + \frac{bMm}{cm - dM}, \quad m = am + \frac{bmM}{cM - dm},$$

or

$$Mmc(1-a) - M^2 d(1-a) = bMm, \quad Mmc(1-a) - m^2 d(1-a) = bmM.$$

Subtracting we have

$$d(1-a)(M^2 - m^2) = 0, \qquad d(1-a) \neq 0.$$

Then

$$M = m.$$

(ii) The function $g(u,v,w)$ increasing in w and decreasing in u, v. Suppose that (m, M) is a solution of the system

$$M = g(m,m,M) \quad and \quad m = g(M,M,m).$$

Then from Eq.(8), we see that

$$M = am + \frac{bm^2}{cm - dM}, \quad m = aM + \frac{bM^2}{cM - dm},$$

or

$$Mm(c+ad) - M^2 d - m^2 ac = bm^2, \quad Mm(c+ad) - m^2 d - M^2 ac = bM^2.$$

Subtracting we have

$$\left(m^2 - M^2\right)(d - (ac+b)) = 0, \qquad d - (ac+b) \neq 0$$

Then

$$M = m.$$

Therefore by Theorem B that $\overline{x}$ is a global attractor of Eq.(8) and then the proof is complete.

3.3 Solutions of the Equation $x_{n+1} = x_{n-1} + \frac{x_{n-1}x_{n-2}}{x_{n-2} - x_{n-4}}$

Theorem 7. Suppose $\{x_n\}_{n=-4}^{\infty}$ be a solution of difference equation $x_{n+1} = x_{n-1} + \frac{x_{n-1}x_{n-2}}{x_{n-2} - x_{n-4}}$ with $x_{-4} \neq x_{-2},\ x_{-3} \neq x_{-1},\ x_0 \neq x_{-2}$. Then for $n = 0, 1, ...$

$$x_{6n-4} = \frac{(F_{2n+1}k - F_{2n-1}p)(F_{2n}r - F_{2n-2}s)(F_{2n}h - F_{2n-2}r)}{r(k-p)},$$
$$x_{6n-3} = \frac{(F_{2n}k - F_{2n-2}p)(F_{2n+1}r - F_{2n-1}s)(F_{2n+1}h - F_{2n-1}r)}{(r-s)(h-r)},$$
$$x_{6n-2} = \frac{(F_{2n+1}k - F_{2n-1}p)(F_{2n+2}r - F_{2n}s)(F_{2n}h - F_{2n-2}r)}{r(k-p)},$$
$$x_{6n-1} = \frac{(F_{2n+2}k - F_{2n}p)(F_{2n+1}r - F_{2n-1}s)(F_{2n+1}h - F_{2n-1}r)}{(r-s)(h-r)},$$
$$x_{6n} = \frac{(F_{2n+1}k - F_{2n-1}p)(F_{2n+2}r - F_{2n}s)(F_{2n+2}h - F_{2n}r)}{r(k-p)},$$
$$x_{6n+1} = \frac{(F_{2n+2}k - F_{2n}p)(F_{2n+3}r - F_{2n+1}s)(F_{2n+1}h - F_{2n-1}r)}{(r-s)(h-r)},$$

where $x_{-4} = s,\ x_{-3} = p,\ x_{-2} = r,\ x_{-1} = k$ and $x_0 = h$.

Proof. For $n = 0$ the result holds. Now assume that $n > 0$ and that assumption holds for $n - 1$. That is,

$$x_{6n-9}=\frac{\left(F_{2n-2}k-F_{2n-4}p\right)\left(F_{2n-1}r-F_{2n-3}s\right)\left(F_{2n-1}h-F_{2n-3}r\right)}{\left(r-s\right)\left(h-r\right)},$$
$$x_{6n-8}=\frac{\left(F_{2n-1}k-F_{2n-3}p\right)\left(F_{2n}r-F_{2n-2}s\right)\left(F_{2n-2}h-F_{2n-4}r\right)}{r\left(k-p\right)},$$
$$x_{6n-7}=\frac{\left(F_{2n}k-F_{2n-2}p\right)\left(F_{2n-1}r-F_{2n-3}s\right)\left(F_{2n-1}h-F_{2n-3}r\right)}{\left(r-s\right)\left(h-r\right)},$$
$$x_{6n-6}=\frac{\left(F_{2n-1}k-F_{2n-3}p\right)\left(F_{2n}r-F_{2n-2}s\right)\left(F_{2n}h-F_{2n-2}r\right)}{r\left(k-p\right)},$$
$$x_{6n-5}=\frac{\left(F_{2n}k-F_{2n-2}p\right)\left(F_{2n+1}r-F_{2n-1}s\right)\left(F_{2n-1}h-F_{2n-3}r\right)}{\left(r-s\right)\left(h-r\right)}$$

Now, it follows from Eq. (8) that

$$\begin{aligned}
x_{6n-4}&=x_{6n-6}+\frac{x_{6n-6}x_{6n-7}}{x_{6n-7}-x_{6n-9}}\\
&=x_{6n-6}\left(1+\frac{x_{6n-7}}{x_{6n-7}-x_{6n-9}}\right)\\
&=x_{6n-6}\left(1+\left(\frac{\frac{(F_{2n}k-F_{2n-2}p)(F_{2n-1}r-F_{2n-3}s)(F_{2n-1}h-F_{2n-3}r)}{(r-s)(h-r)}}{\frac{(F_{2n}k-F_{2n-2}p)(F_{2n-1}r-F_{2n-3}s)(F_{2n-1}h-F_{2n-3}r)}{(r-s)(h-r)}-\frac{(F_{2n-2}k-F_{2n-4}p)(F_{2n-1}r-F_{2n-3}s)(F_{2n-1}h-F_{2n-3}r)}{(r-s)(h-r)}}\right)\right)\\
&=x_{6n-6}\left(1+\frac{F_{2n}k-F_{2n-2}p}{F_{2n}k-F_{2n-2}p-\left(F_{2n-2}k-F_{2n-4}p\right)}\right)\\
&=x_{6n-6}\left(1+\frac{F_{2n}k-F_{2n-2}p}{F_{2n-1}k-F_{2n-3}p}\right)\\
&=\frac{(F_{2n-1}k-F_{2n-3}p)(F_{2n}r-F_{2n-2}s)(F_{2n}h-F_{2n-2}r)}{r(k-p)}\left(\frac{F_{2n+1}k-F_{2n-1}p}{F_{2n-1}k-F_{2n-3}p}\right),
\end{aligned}$$

Consequently, we have

$$x_{6n-4}=\frac{\left(F_{2n+1}k-F_{2n-1}p\right)\left(F_{2n}r-F_{2n-2}s\right)\left(F_{2n}h-F_{2n-2}r\right)}{r\left(k-p\right)}.$$

Also, from Eq. (8) we see that

$$\begin{aligned}
x_{6n-3}&=x_{6n-5}+\frac{x_{6n-5}x_{6n-6}}{x_{6n-6}-x_{6n-8}}=x_{6n-5}\left(1+\frac{x_{6n-6}}{x_{6n-6}-x_{6n-8}}\right)\\
&=x_{6n-5}\left(1+\frac{\frac{(F_{2n-1}k-F_{2n-3}p)(F_{2n}r-F_{2n-2}s)(F_{2n}h-F_{2n-2}r)}{r(k-p)}}{\frac{(F_{2n-1}k-F_{2n-3}p)(F_{2n}r-F_{2n-2}s)(F_{2n}h-F_{2n-2}r)}{r(k-p)}-\frac{(F_{2n-1}k-F_{2n-3}p)(F_{2n}r-F_{2n-2}s)(F_{2n-2}h-F_{2n-4}r)}{r(k-p)}}\right)\\
&=x_{6n-5}\left(1+\frac{F_{2n}h-F_{2n-2}r}{F_{2n}h-F_{2n-2}r-\left(F_{2n-2}h-F_{2n-4}r\right)}\right)\\
&=x_{6n-5}\left(1+\frac{F_{2n}h-F_{2n-2}r}{F_{2n-1}h-F_{2n-3}r}\right)\\
&=\frac{(F_{2n}k-F_{2n-2}p)(F_{2n+1}r-F_{2n-1}s)(F_{2n-1}h-F_{2n-3}r)}{(r-s)(h-r)}\left(\frac{F_{2n+1}h-F_{2n-1}r}{F_{2n-1}h-F_{2n-3}r}\right),
\end{aligned}$$

therefore

$$x_{6n-3}=\frac{(F_{2n}k-F_{2n-2}p)(F_{2n+1}r-F_{2n-1}s)(F_{2n+1}h-F_{2n-1}r)}{(r-s)(h-r)}$$

Similarly, other relations can be obtain and thus, the proof has been proved.

The following cases can be proved by similar way so it will be left to the readers.

4 Dynamics of Solution of $x_{n+1}=ax_{n-1}-\frac{bx_{n-1}x_{n-2}}{cx_{n-2}+dx_{n-4}}$

In this section, we get the expressions of the solution of the difference equation in the form:

$$x_{n+1}=ax_{n-1}-\frac{bx_{n-1}x_{n-2}}{cx_{n-2}+dx_{n-4}},\quad n=0,1,..., \tag{10}$$

where the initial conditions $x_{-4},x_{-3},\ x_{-2},\ x_{-1}$ *and* x_0 are arbitrary real numbers.

4.1 Local Stability of the Equilibrium Point

In this part, we study the local stability character of the solution of Eq.(10) .

Eq.(10) has a unique equilibrium point and is given by $\overline{x}=0$.

Let $F:(0,\infty)^3\rightarrow(0,\infty)$ be a function defined by

$$F(u,v,w)=au-\frac{buv}{cv+dw}.$$

Then it follows that,

$$F_u(u,v,w)=a-\frac{bv}{cv+dw},\ F_v(u,v,w)=\frac{-bduw}{(cv+dw)^2},\ F_w(u,v,w)=\frac{bduv}{(cv+dw)^2},$$

we see that

$$F_u(\overline{x},\overline{x},\overline{x})=a-\frac{b}{c+d},\quad F_v(\overline{x},\overline{x},\overline{x})=\frac{-bd}{(c+d)^2},\quad F_w(\overline{x},\overline{x},\overline{x})=\frac{bd}{(c+d)^2}.$$

The linearized equation about $\overline{x}$ is

$$y_{n+1}-\left(a-\frac{b}{c+d}\right)y_{n-1}-\left(\frac{-bd}{(c+d)^2}\right)y_{n-2}-\left(\frac{bd}{(c+d)^2}\right)y_{n-4}=0.$$

Theorem 8. Suppose that

$$\left|a-\frac{b}{c+d}\right|+\frac{2bd}{(c+d)^2}<1.$$

Then the equilibrium point of Eq.(10) is locally asymptotically stable.

4.2 Global Attractor of the Equilibrium Point of Eq.(10)

Theorem 9. The equilibrium point $\overline{x}$ of Eq. (10) is global attractor if $(1-a)(c+d)\neq -b$.

4.3 Solutions of the Equation $x_{n+1} = x_{n-1} - \frac{x_{n-1}x_{n-2}}{x_{n-2}+x_{n-4}}$

Theorem 10. Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of the difference equation $x_{n+1} = x_{n-1} - \frac{x_{n-1}x_{n-2}}{x_{n-2}+x_{n-4}}$. Then for $n = 0, 1, ...$

$$x_{6n-4} = s\prod_{i=0}^{n-1} \frac{(F_{2i}k + F_{2i+1}p)(F_{2i-1}r + F_{2i}s)(F_{2i-1}h + F_{2i}r)}{(F_{2i+1}k + F_{2i+2}p)(F_{2i}r + F_{2i+1}s)(F_{2i}h + F_{2i+1}r)},$$

$$x_{6n-3} = p\prod_{i=0}^{n-1} \frac{(F_{2i-1}k + F_{2i}p)(F_{2i}r + F_{2i+1}s)(F_{2i}h + F_{2i+1}r)}{(F_{2i}k + F_{2i+1}p)(F_{2i+1}r + F_{2i+2}s)(F_{2i+1}h + F_{2i+2}r)},$$

$$x_{6n-2} = r\prod_{i=0}^{n-1} \frac{(F_{2i}k + F_{2i+1}p)(F_{2i+1}r + F_{2i+2}s)(F_{2i-1}h + F_{2i}r)}{(F_{2i+1}k + F_{2i+2}p)(F_{2i+2}r + F_{2i+3}s)(F_{2i}h + F_{2i+1}r)},$$

$$x_{6n-1} = k\prod_{i=0}^{n-1} \frac{(F_{2i+1}k + F_{2i+2}p)(F_{2i}r + F_{2i+1}s)(F_{2i}h + F_{2i+1}r)}{(F_{2i+2}k + F_{2i+3}p)(F_{2i+1}r + F_{2i+2}s)(F_{2i+1}h + F_{2i+2}r)},$$

$$x_{6n} = h\prod_{i=0}^{n-1} \frac{(F_{2i}k + F_{2i+1}p)(F_{2i+1}r + F_{2i+2}s)(F_{2i+1}h + F_{2i+2}r)}{(F_{2i+1}k + F_{2i+2}p)(F_{2i+2}r + F_{2i+3}s)(F_{2i+2}h + F_{2i+3}r)},$$

$$x_{6n+1} = \frac{ks}{(r+s)}\prod_{i=0}^{n-1} \frac{(F_{2i+1}k + F_{2i+2}p)(F_{2i+2}r + F_{2i+3}s)(F_{2i}h + F_{2i+1}r)}{(F_{2i+2}k + F_{2i+3}p)(F_{2i+3}r + F_{2i+4}s)(F_{2i+1}h + F_{2i+2}r)}.$$

5 Dynamics of Solution of $x_{n+1} = ax_{n-1} - \frac{bx_{n-1}x_{n-2}}{cx_{n-2}-dx_{n-4}}$

In this part, we obtain the form of solution of the following difference equation

$$x_{n+1} = ax_{n-1} - \frac{bx_{n-1}x_{n-2}}{cx_{n-2} - dx_{n-4}}, \quad n = 0, 1, ..., \tag{11}$$

where the initial conditions $x_{-4},\ x_{-3},\ x_{-2},\ x_{-1}$ *and* x_0 are arbitrary non zero real numbers.

5.1 Local Stability of the Equilibrium Point

In this part, we obtain the local stability character of the solution of Eq. (11). Eq.(11) has a unique equilibrium point is $\overline{x} = 0$.

Let $F : (0, \infty)^3 \to (0, \infty)$ be a function defined by

$$F(u, v, w) = au - \frac{buv}{cv - dw},$$

Then it follows that,

$$F_u(\overline{x}, \overline{x}, \overline{x}) = a - \frac{b}{c-d}, \quad F_v(\overline{x}, \overline{x}, \overline{x}) = \frac{bd}{(c-d)^2}, \quad F_w(\overline{x}, \overline{x}, \overline{x}) = \frac{-bd}{(c-d)^2}.$$

Theorem 11. Suppose that

$$\left|a - \frac{b}{c-d}\right| + \frac{2bd}{(c-d)^2} < 1.$$

Then the equilibrium point is locally asymptotically stable.

5.2 Global Attractor of the Equilibrium Point of Eq.(11)

Theorem 12. The equilibrium point $\overline{x}$ of Eq.(11) is global attractor if $(1-a)(c-d) \neq -b$.

5.3 Solutions of the Equation $x_{n+1} = x_{n-1} - \frac{x_{n-1}x_{n-2}}{x_{n-2}-x_{n-4}}$

Theorem 13. Suppose that $\{x_n\}_{n=-4}^{\infty}$ be a solution of Eq.(11) with $(a = b = c = d = 1)$, $x_{-2} \neq x_{-4}$ $x_{-1} \neq x_{-3}$ and $x_0 \neq x_{-2}$. Then every solution of Eq.(11) is periodic with period 36. Moreover $\{x_n\}_{n=-4}^{\infty}$ takes the form

$$\left\{ \begin{array}{c} s, p, r, k, h, \frac{--ks}{r-s}, \frac{-hp}{k-p}, \frac{ksr}{(r-s)(h-r)}, \frac{-hp}{r(k-p)}, \frac{rs(k-p)}{(r-s)(h-s)}, \frac{-p(r-s)(h-r)}{r(k-p)}, \\ \frac{r^2(k-p)}{(r-s)(h-r)}, \frac{-k(r-s)(h-r)}{r(k-p)}, \frac{hr(k-p)}{(r-s)(h-r)}, \frac{ks(h-r)}{r(k-p)}, \frac{-hpr}{(r-s)(h-r)}, \frac{-ks}{k-p}, \frac{-hp}{h-r}, -s, -p, \\ -r, -k, -h, \frac{ks}{r-s}, \frac{hp}{k-p}, \frac{-krs}{(r-s)(h-r)}, \frac{hp(r-s)}{r(k-p)}, \frac{-rs(k-p)}{(r-s)(h-r)}, \frac{p(r-s)(h-r)}{r(k-p)}, \frac{-r^2(k-p)}{(r-s)(h-r)}, \\ \frac{k(r-s)(h-r)}{r(k-p)}, \frac{-hr(k-p)}{(r-s)(h-r)}, \frac{-ks(h-r)}{r(k-p)}, \frac{hpr}{(r-s)(h-r)}, \frac{ks}{k-p}, \frac{hp}{h-r}, s, p, r, k, h, ... \end{array} \right\}.$$

6 Numerical Examples

To verify the results of this paper, we consider some numerical examples as follows.

Example 1. In Figure 1, we take Eq.(5) since $a = .1$, $b = .3$, $c = .6$, $d = .5$, $x_{-4} = 2$, $x_{-3} = .6$, $x_{-2} = .7$, $x_{-1} = .3$ and $x_0 = 1.9$.

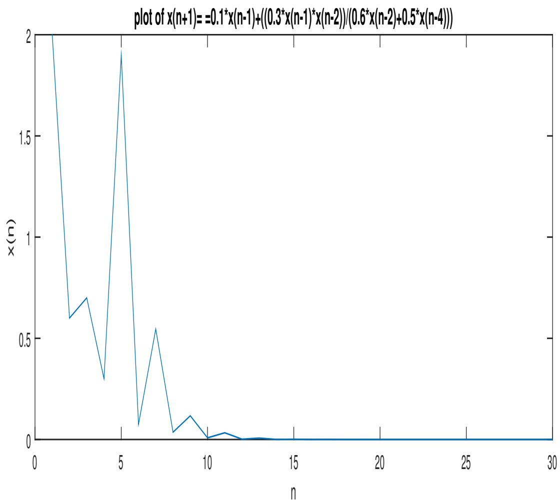

Figure 1.

Example 2. See Figure 2, we suppose for Eq.(5) that $a = 2$, $b = .3$, $c = .6$, $d = .5$, $x_{-4} = 2$, $x_{-3} = .6$, $x_{-2} = .7$, $x_{-1} = .3$ and $x_0 = 1.9$.

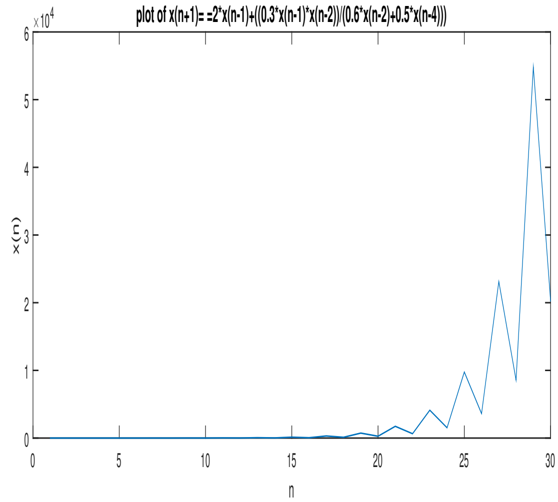

Figure 2.

Example 3. Figure 3 shows the solutions of Eq.(8) where $a = .2$, $b = .3$, $c = .6$, $d = .5$, $x_{-4} = 2$, $x_{-3} = 6$, $x_{-2} = .7$, $x_{-1} = 3$ and $x_0 = 1.9$.

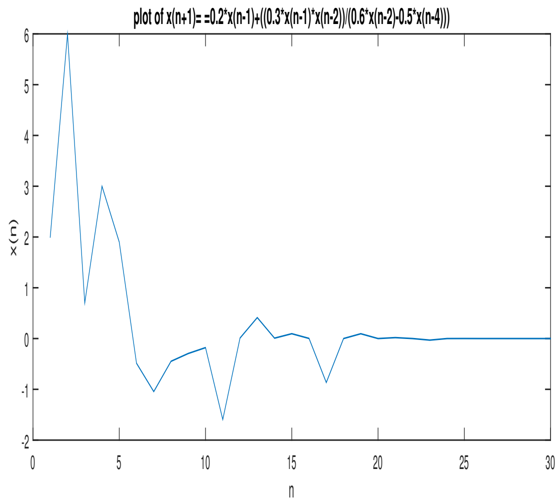

Figure 3.

Example 4. Figure 4 shows the behavior of Eq.(10) when we choose $a = .2$, $b = 3$, $c = 6$, $d = .5$, $x_{-4} = 2$, $x_{-3} = 6$, $x_{-2} = .7$, $x_{-1} = 3$ and $x_0 = 1.9$.

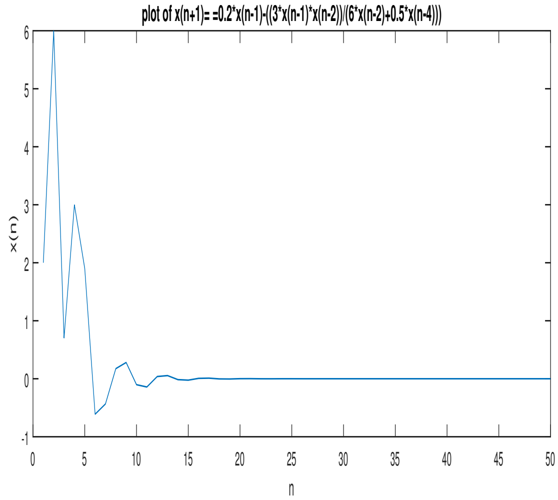

Figure 4.

Example 5. Figure 5 shows the period thirty six solutions of Eq.(11) since $a = b = c = d = 1$, $x_{-4} = 2$, $x_{-3} = .6$, $x_{-2} = .7$, $x_{-1} = 3$ and $x_0 = 0.19$.

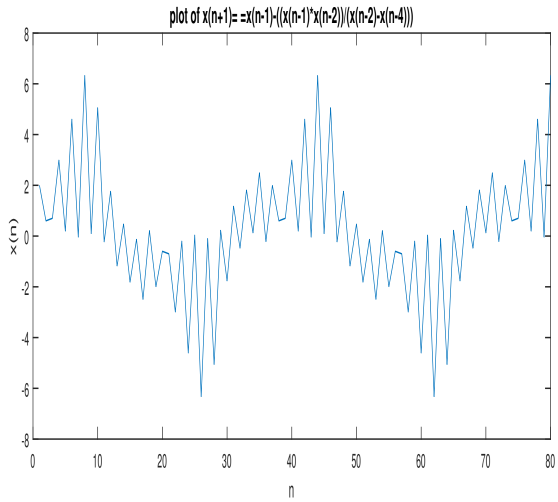

Figure 5.

Example 6. See Figure 6, we suppose for Eq.(11), that $a = c = d = 1$, $b = 2$, $x_{-4} = 2$, $x_{-3} = .6$, $x_{-2} = .7$, $x_{-1} = 3$ and $x_0 = 0.19$.

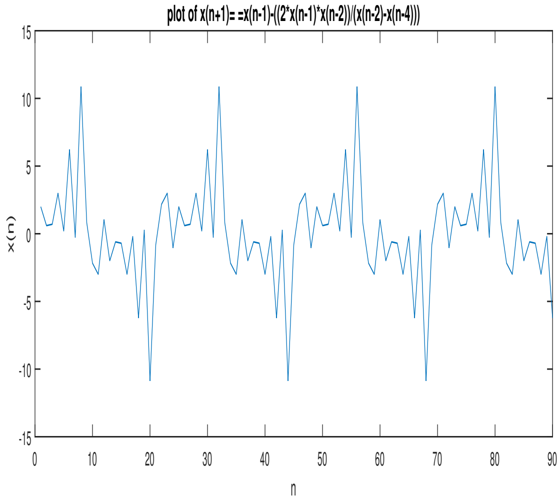

Figure 6.

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