

Any L -States Solutions of The Modified Schrödinger Equation with Generalized Hellmann–Kratzer Potential Model in The Symmetries of NRNCQM

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Abstract

In the present research paper, the approximate analytical solutions of the modified radial Schrodinger equation (MSE) have been obtained with a newly proposed potential called generalized Hellmann–Kratzer potential (GHKP) model by using the improved approximation scheme to the centrifugal term for any l -states. The potential is a superposition of the Hellmann–Kratzer potential model and new terms proportional with $(1/r^3, 1/r^4, \exp(-\alpha r)/r^2$ and $\exp(-\alpha r)/r^3)$, appears as a result of the effects of noncommutativity properties of space and phase on the Hellmann–Kratzer potential model. We applied the generalized Bopp's shift method and standard perturbation theory, in the nonrelativistic noncommutative three-dimensional real space phase (NC: 3D-RSP) instead to solving MSE directly with star product. The bound state energy eigenvalues for the some diatomic molecules such as N_2 , CO , NO and CH obtained in terms of the generalized the Gamma function, the discrete atomic quantum numbers $(j, n, l, s$ and $m)$, two infinitesimal parameters $(\Theta, \bar{\theta})$ which are induced automatically by position-position and phase-phase noncommutativity properties, in addition to, the dimensional parameters $(V_1, V, \alpha, r_e, D_e)$ of GHKP model. Furthermore, we have shown that the corresponding Hamiltonian operator in (NC: 3D-RSP) symmetries is the sum of the Hamiltonian operator of the HKP model and two operators, the first one is the modified spin-orbit interaction while the second is the modified Zeeman operator for the previous diatomic molecule.

Keywords: Schrödinger Equation, Generalized Hellmann–Kratzer Potential Model, Nonrelativistic Noncommutative Quantum Mechanics, Star Product And Generalized Bopp's Shift Method.

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1. Introduction

It is well known that the the Kratzer potential has attracted a great deal of interest over the years and has been one of the most useful models for study the molecular structure in quantum physics, recently has received a lot of attention for a wide range of interests generally. To be precise, this potential is very important in many fields of physics and chemistry such as atomic physics, molecular physics, and quantum chemistry. In particular, it is used to describe the interactions of diatomic molecules such as N_2 , CO , NO and CH in quantum mechanics. In addition, this potential has a long-range attraction behavior when dealing with relatively long distances between molecules in addition to the short-range (repulsive property) when the distance between the molecules becomes very small [1-4]. On the other hand, the Yukawa potential (the screened Coulomb potential) which is one of the exponential types and most realistic short-range potentials in nuclear Physics. It is characterized by its useful applications in many fields such as solid-state physics, condensed matter physics and charged particle in weakly non-ideal plasma [5-8]. C. O. Edet *et al.* [9-10] was studied the Hellmann–Kratzer potential model (the combined between the Kratzer potential, the Yukawa potential and the Coulomb potential) within the framework of Nikiforov and Uvarov method and obtained the nonrelativistic bound state energy eigenvalues of the diatomic molecules N_2 , CO , NO and CH . In work, motivated by several recent studies such as the non-renormalizable of the standard model, string theory, quantum gravity, the noncommutative quantum mechanics NCQM has attracted much attention [11]. The



noncommutativity of space-time was known firstly by Heisenberg and was formalized by Snyder at 1947. In the present research paper, we want to extend, the study of C. O. Edet *et al.* in ref. [10] to the nonrelativistic NCQM case to the possibility of finding other applications and more profound interpretations in the sub-atomics scales. The nonrelativistic energy levels for diatomic molecules (N_2, CO, NO and CH), which interacted with generalized Hellmann–Kratzer potential (GHKP) model in the context of the noncommutative space phase, have not been obtained yet. The main purpose of this paper is to solve the modified Schrödinger equation MSE with GHKP model:

$$V_{hk}(r) = -\frac{V_0}{r} + D_e \left(\frac{r-r_e}{r_e} \right)^2 + \frac{V_1 \exp(-ar)}{r} \rightarrow V_{hk}(\hat{r}) = V_{hk}(r) + \left(\frac{B-V_0}{2r^3} + \frac{C}{r^4} + \frac{V_1 \exp(-ar)}{2r^3} + \frac{\alpha V_1 \exp(-ar)}{2r^2} \right) \mathbb{L} \otimes \mathbb{L} \quad (1)$$

in (NC: 3D-RSP) symmetries using the generalized Bopp’s shift method which depends on the concepts that we present below in the third section. The structure of nonrelativistic noncommutative quantum mechanics (NRNCQM) based to NC canonical commutations relations in Schrödinger, Heisenberg and interactions pictures (SP, HP and IP), respectively, as follows (Throughout this paper, the natural units $c = \eta = 1$ will be used) [12-18]:

$$\begin{cases} [x_i, p_j] = [x_i(t), p_j(t)] = [x_{i_i}(t), p_{j_j}(t)] = i\delta_{ij} \\ [x_i, x_j] = [x_i(t), x_j(t)] = [x_{i_i}(t), x_{j_j}(t)] = 0 \\ [p_i, p_j] = [p_i(t), p_j(t)] = [p_{i_i}(t), p_{j_j}(t)] = 0 \end{cases} \Rightarrow \begin{cases} [\hat{x}_i^*, \hat{p}_j] = [\hat{x}_i(t)^*, \hat{p}_j(t)] = [\hat{x}_{i_i}(t)^*, \hat{p}_{j_j}(t)] = i\eta^{eff} \delta_{ij} \\ [\hat{x}_i^*, \hat{x}_j] = [\hat{x}_i(t)^*, \hat{x}_j(t)] = [\hat{x}_{i_i}(t)^*, \hat{x}_{j_j}(t)] = i\theta_{ij} \\ [\hat{p}_i^*, \hat{p}_j] = [\hat{p}_i(t)^*, \hat{p}_j(t)] = [\hat{p}_{i_i}(t)^*, \hat{p}_{j_j}(t)] = i\bar{\theta}_{ij} \end{cases} \quad (2)$$

Here $\eta^{eff} = \eta \left(1 + \frac{\theta\bar{\theta}}{4\eta^2} \right)$ is the effective Planck constant, however, the operators $\hat{\xi}_{iH}(t) = (\hat{x}_i \vee \hat{p}_i)(t)$ and $\hat{\xi}_{iI}(t) = (\hat{x}_{i_i} \vee \hat{p}_{i_i})(t)$ in (HP and IP, respectively) are depending on the corresponding operators $\hat{\xi}_{iS} = \hat{x}_i \vee \hat{p}_i$ in SP from the following projections relations:

$$\begin{cases} \hat{\xi}_{iH}(t) = \exp(i\hat{H}_{hk}(t-t_0)) \hat{\xi}_{iS} \exp(-i\hat{H}_{hk}(t-t_0)) \\ \hat{\xi}_{iI}(t) = \exp(i\hat{H}_{ohk}(t-t_0)) \hat{\xi}_{iS} \exp(-i\hat{H}_{ohk}(t-t_0)) \end{cases} \Rightarrow \begin{cases} \hat{\xi}_{iH}(t) = \exp(i\hat{H}_{nc-hk}(t-t_0)) * \hat{\xi}_{iS} * \exp(-i\hat{H}_{nc-hk}(t-t_0)) \\ \hat{\xi}_{iI}(t) = \exp(i\hat{H}_{nco-hk}(t-t_0)) * \hat{\xi}_{iS} * \exp(-i\hat{H}_{nco-hk}(t-t_0)) \end{cases} \quad (3)$$

here $\hat{\xi}_{iS} = x_i \vee p_i$, $\hat{\xi}_{iH}(t) = (x_i \vee p_i)(t)$ and $\hat{\xi}_{iI}(t) = (x_{i_i} \vee p_{i_i})(t)$ are the three representations (SP, HP and IP) in NRQM, while the dynamics of new systems $\frac{d\hat{\xi}_{iH}(t)}{dt}$ are described from the following motion equations in NRNCQM:

$$\frac{d\hat{\xi}_{iH}(t)}{dt} = [\hat{\xi}_{iH}(t), \hat{H}_{hk}] \Rightarrow \frac{d\hat{\xi}_{iH}(t)}{dt} = \left[\hat{\xi}_{iH}(t)^*, \hat{H}_{nc-hk} \right] \quad (4)$$

the two operators \hat{H}_{hk} and \hat{H}_{nc-hk} present the Hamiltonian operators for (HKP and GHKP) models in NRQM and NRNCQM, respectively. The very small two parameters $\theta^{\mu\nu}$ and $\bar{\theta}^{\mu\nu}$ (compared to the energy) are elements of two antisymmetric real matrixes and $(*)$ denote to the star product. The generalized star product between two arbitrary functions $(f, g)(x, p)$ will be $\hat{f}(\hat{x}, \hat{p})\hat{g}(\hat{x}, \hat{p}) \equiv (f * g)(x, p)$ in 3-dimensional NC space-phase [19-29]:



$$(f * g)(x, p) \approx \left(fg - \frac{i}{2} \theta^{\mu\nu} \partial_\mu^x f \partial_\nu^x g - \frac{i}{2} \bar{\theta}^{\mu\nu} \partial_\mu^p f \partial_\nu^p g \right) (x, p) \quad (5)$$

The second and the third terms in the above equation are the effects of (space-space) and (phase-phase) noncommutativity properties, respectively. The organization scheme of the recent work is given as follows: In the next section, we briefly review the ordinary SE with HKP model on based ref. [10]. Section 3 is devoted to studying the MSE by applying the generalized Bopp's shift method for GHKP model. In the next subsection, by applying standard perturbation theory to find the quantum spectrum of n^{th} excited levels in for spin-orbit interaction in the framework of the global group (NC-3D: RSP) and then, we derive the magnetic spectrum for GHKP model. In the fourth section, we resume the global spectrum and corresponding NC Hamiltonian operator for GHKP model and corresponding energy levels of the molecules N_2 , CO , NO and CH . Finally, the achieved results are briefly summarized in the last section.

2. Overview of the eigenfunctions and the energy eigenvalues for HKP model in NRQM:

In this section, we shall recall here the time-independent SE for Hellmann–Kratzer potential model $V_{hk}(r)$, which was studied by C. O. Edet *et al.* in ref. [10]:

$$V_{hk}(r) = -\frac{V_0}{r} + D_e \left(\frac{r - r_e}{r_e} \right)^2 + \frac{V_1 \exp(-\alpha r)}{r} \equiv \frac{B - V_0}{r} + \frac{C}{r^2} + D_e + \frac{V_1 \exp(-\alpha r)}{r} \quad (6)$$

where D_e is the dissociation energy, r_e is the equilibrium internuclear separation, V_0 and V_1 are the potential strength of Coulomb and Yukawa potentials, α is the screening parameter, r is the distance between the two particles, $B = -2r_e D_e$ and $C = D_e r_e^2$. If we insert this potential into the SE, we obtain:

$$\frac{d^2 U_{nl}(r)}{dr^2} + \frac{2}{r} \frac{dU_{nl}(r)}{dr} + 2\mu [E_{nl} - V_{eff-hk}] U_{nl}(r) = 0 \Rightarrow \frac{d^2 R_{nl}(r)}{dr^2} + 2\mu [E_{nl} - V_{eff-hk}] R_{nl}(r) = 0 \quad (7)$$

here $U_{nl}(r) = \frac{R_{nl}(r)}{r}$, $V_{eff-hk}(r) \equiv V_{hk}(r) + \frac{l(l+1)}{2\mu r^2}$ is effective potential, μ is the reduced mass of diatomic molecules (N_2 , CO , NO and CH), E_{nl} is the rotational–vibrational energy spectra of the diatomic molecules while n and l are the radial and orbital angular momentum quantum numbers. The complete wave function $\Psi(r, \theta, \varphi) = U_{nl}(r) Y_l^m(\theta, \varphi)$, as a function of the Jacobi polynomial $P_n^{(2\eta_n, 2\rho)}$ and the spherical harmonic functions $Y_l^m(\theta, \varphi)$ is given by [10]:

$$\Psi(r, \theta, \varphi) = N_{nl} s^{\eta_n} (1-s)^{1/2+\rho} P_n^{(2\eta_n, 2\rho)}(1-2s) Y_l^m(\theta, \varphi) \quad (8)$$

here $N_{nl} = \sqrt{\frac{2n! \eta_n \alpha \Gamma(n+2+2\eta_n+2\rho)}{\Gamma(n+1+2\eta_n) \Gamma(n+2+2\rho)}}$ is the normalization

constant, $s = \exp(-\alpha r)$, $\eta_n = \sqrt{\varepsilon_n - \beta - \chi + \delta}$, $\rho = \sqrt{1/4 + \delta}$, $\varepsilon_n = \frac{2\mu}{\alpha^2} (E_{nl} - D_e)$, $\beta = \frac{4\mu D_e r_e}{\alpha}$

$\delta = 2\mu D_e r_e^2 + l(l+1)$ and $\chi = -\frac{2\mu V_0}{\alpha}$. The rotational–vibrational energy spectra E_{nl} of the studied potential, are given by [10]:



$$E_{nl} = \frac{\alpha^2}{2\mu} \left[\delta - \frac{4\mu D_e r_e}{\alpha} \right] - V_0 \alpha + D_e - \frac{\alpha^2}{2\mu} \left[\frac{4\mu D_e r_e + 2\mu V_0 - 2\mu V_1 - \delta \alpha}{2(n + \xi) \alpha} - \frac{n + \xi}{2} \right]^2 \quad (9)$$

with $\xi = \frac{1}{2} (1 + \sqrt{1 + 4\delta})$.

3. Method and Theoretical Approach

3.1 Solution of MSE for GHKP model

In this sub-section, we shall give an overview or a brief preliminary for a GHKP model $V_{nc-ks}(r)$, in (NC: 3D-RSP) symmetries. To perform this task the physical form of MSE, it is necessary to replace ordinary three-dimensional Hamiltonian operators $\hat{H}_{hk}(p_\mu, x_\mu)$, ordinary complex wave function $\Psi(\vec{r})$ and E_{nl} (in NRQM) by three Hamiltonian operators $\hat{H}_{nc-hk}(\hat{p}_\mu, \hat{x}_\mu)$, complex wave function $\Psi(\vec{r})$ and new values E_{nc-hk} , respectively in NRNCQM. In addition to replacing the ordinary product by star product $(*)$, which allows us to construct the MSE in (NC-3D: RSP) symmetries as [30-33]:

$$\hat{H}_{hk}(p_\mu, x_\mu) \Psi(\vec{r}) = E_{nl} \Psi(\vec{r}) \Rightarrow \hat{H}_{nc-hk}(\hat{p}_\mu, \hat{x}_\mu) * \Psi(\vec{r}) = E_{nc-hk} \Psi(\vec{r}) \quad (10)$$

The Bopp's shift method employed in the solutions enables us to explore an effective way of obtaining the modified potential in NRNCQM, is based on the following commutators [28-34]:

$$[\hat{x}_\mu, \hat{x}_\nu] = [\hat{x}_\mu(t), \hat{x}_\nu(t)] = i\theta_{\mu\nu} \text{ and } [\hat{p}_\mu, \hat{p}_\nu] = [\hat{p}_\mu(t), \hat{p}_\nu(t)] = i\bar{\theta}_{\mu\nu} \quad (11)$$

The generalized positions and momentum coordinates $(\hat{x}_\mu, \hat{p}_\mu)$ in (NC: 3D-RSP) are depended with corresponding usual generalized positions and momentum coordinates (x_μ, p_μ) in NRQM by the following, respectively [24-31]:

$$(x_\mu, p_\mu) \Rightarrow (\hat{x}_\mu, \hat{p}_\mu) = \left(x_\mu - \frac{\theta^{\mu\nu}}{2} p_\nu, p_\mu + \frac{\bar{\theta}^{\mu\nu}}{2} x_\nu \right) \quad (12)$$

The above equation allows us to obtain the two operators \hat{r}^2 and \hat{p}^2 in (NC-3D: RSP) [29-34]:

$$(r^2, p^2) \Rightarrow (\hat{r}^2, \hat{p}^2) = \left(r^2 - \mathbf{L}\overset{\rho}{\Theta}, p^2 + \mathbf{L}\overset{\rho}{\bar{\Theta}} \right) \quad (13)$$

The two couplings $\mathbf{L}\overset{\rho}{\Theta}$ and $\mathbf{L}\overset{\rho}{\bar{\Theta}}$ are $(L_x \Theta_{12} + L_y \Theta_{23} + L_z \Theta_{13})$ and $(L_x \bar{\Theta}_{12} + L_y \bar{\Theta}_{23} + L_z \bar{\Theta}_{13})$, respectively and $(L_x, L_y$ and $L_z)$ are the three components of angular momentum operator $\overset{\rho}{L}$ while $\Theta_{\mu\nu} = \theta_{\mu\nu}/2$. Thus, the reduced Schrödinger equation (without star product) can be written as:

$$\hat{H}_{nc-hk}(\hat{p}_\mu, \hat{x}_\nu) * \Psi(\vec{r}) = E_{nc-hk} \Psi(\vec{r}) \Rightarrow H_{hk}(\hat{p}_\mu, \hat{x}_\nu) \Psi(\vec{r}) = E_{nc-hk} \Psi(\vec{r}) \quad (14)$$

the Hamiltonian operator $H_{nc-hk}(\hat{p}_i, \hat{x}_i)$ can be expressed as :

$$H_{hk}(p_\mu, x_\mu) \Rightarrow H_{nc-hk}(\hat{p}_\mu, \hat{x}_\mu) \equiv H \left(\hat{x}_\mu = x_\mu - \frac{\theta^{\mu\nu}}{2} p_\nu, \hat{p}_\mu = p_\mu + \frac{\bar{\theta}^{\mu\nu}}{2} x_\nu \right) \quad (15)$$

Now, we want to find the GHKP model $V_{hk}(\hat{r})$:

$$V_{hk}(r) \Rightarrow V_{hk}(\hat{r}) = \frac{B-V_0}{\hat{r}} + \frac{C}{\hat{r}^2} + D_e + \frac{V_1 \exp(-\alpha\hat{r})}{\hat{r}} \quad (16)$$

After straightforward calculations, we can obtain the important terms $(\frac{B-V_0}{\hat{r}}, \frac{C}{\hat{r}^2}$ and $\frac{V_1 \exp(-\alpha\hat{r})}{\hat{r}})$ which will be used to determine the GHKP model in (NC: 3D- RSP) symmetries as:

$$\begin{cases} \frac{B-V_0}{r} \Rightarrow \frac{B-V_0}{\hat{r}} = \frac{B-V_0}{r} + (B-V_0) \frac{\mathbb{L}\Theta}{2r^3} + O(\Theta^2) \\ \frac{C}{r} \Rightarrow \frac{C}{\hat{r}^2} = \frac{C}{r^2} + \frac{C\mathbb{L}\Theta}{r^4} + O(\Theta^2) \\ V_1 \exp(-\alpha r) \rightarrow V_1 \exp(-\alpha\hat{r}) = V_1 \exp(-\alpha r) + V_1 \frac{\alpha\mathbb{L}\Theta}{2r} \exp(-\alpha r) + O(\Theta^2) \end{cases} \quad (17)$$

this gives the following immediately result:

$$\frac{V_1 \exp(-\alpha\hat{r})}{\hat{r}} = \frac{V_1 \exp(-\alpha r)}{r} + V_1 \frac{\mathbb{L}\Theta}{2r^3} \exp(-\alpha r) + V_1 \frac{\alpha\mathbb{L}\Theta}{2r^2} \exp(-\alpha r) O(\Theta^2) \quad (18)$$

By making the substitution above equations (18) and (17) into Eq. (15), we find the global our working Hamiltonian operator $H_{nc-hk}(\hat{r})$ satisfies the equation in (NC: 3D-RSP) symmetries:

$$H_{nc-hk}(\hat{p}_\mu, \hat{x}_\mu) \equiv H_{hk}(p_\mu, x_\mu) + \left(\frac{B-V_0}{2r^3} + \frac{C}{r^4} + \frac{V_1 \exp(-\alpha r)}{2r^3} + \frac{\alpha V_1 \exp(-\alpha r)}{2r^2} \right) \mathbb{L}\Theta + \frac{\mathbb{L}\bar{\theta}}{2\mu} \quad (19)$$

where the operator $H_{hk}(p_\mu, x_\mu)$ is just the ordinary Hamiltonian operator with HKP model in NRQM :

$$H_{hk}(p_\mu, x_\nu) = \frac{p^2}{2\mu} + \frac{B-V_0}{r} + \frac{C}{r^2} + D_e + \frac{V_1 \exp(-\alpha r)}{r} \quad (20)$$

while the rest terms are proportional with two infinitesimals parameters $(\Theta$ and $\bar{\theta})$ and then we can be considered as a perturbations terms $H_{per-hk}(r)$ in (NC: 3D-RSP) symmetries as :

$$H_{per-hk}(r) = \left\{ \frac{B-V_0}{2r^3} + \frac{C}{r^4} + \frac{V_1}{2r^3} \exp(-\alpha r) + \frac{\alpha V_1}{2r^2} \exp(-\alpha r) \right\} \mathbb{L}\Theta + \frac{\mathbb{L}\bar{\theta}}{2\mu} \quad (21)$$

It is clear that the operator $H_{hk}(p_\mu, x_\nu)$ is just the Hamiltonian operator for diatomic molecules N_2 , CO , NO and CH in ordinary quantum mechanics while the generated part $H_{\text{pert-hk}}(r, \Theta, \bar{\theta})$ appears as results of deformation of noncommutativity space phase. In recent work, we can disregard the second term in $H_{\text{pert-hk}}(r, \Theta, \bar{\theta})$ because we are interested in the corrections of first order Θ and $\bar{\theta}$.

3.2 Spin-orbit Hamiltonian operator for studied diatomic molecules under GHKP model

In this subsection, we want to derive the physical form of the induced Hamiltonian $H_{\text{pert-hk}}(r, \Theta, \bar{\theta})$ due to space-phase noncommutativity. To achieve this goal, we replace both $(\overset{P}{L}\overset{P}{\Theta}$ and $\overset{P}{L}\overset{P}{\bar{\theta}}$) by useful physical forms $(\mathcal{E}\vec{S}\vec{L}$ and $\mathcal{E}\bar{\theta}\vec{S}\vec{L}$), respectively [32-37]:

$$H_{\text{so-hk}}(r, \Theta, \bar{\theta}) \equiv \mathcal{E} \left\{ \left(\frac{B-V_0}{2r^3} + \frac{C}{r^4} + \frac{V_1}{2r^3} \exp(-\alpha r) + \frac{\alpha V_1}{2r^2} \exp(-\alpha r) \right) \Theta + \frac{\bar{\theta}}{2\mu} \right\} \overset{P}{L}\overset{P}{S} \quad (22)$$

Here $\Theta = (\Theta_{12}^2 + \Theta_{23}^2 + \Theta_{13}^2)^{1/2}$, $\bar{\theta} = (\bar{\theta}_{12}^2 + \bar{\theta}_{23}^2 + \bar{\theta}_{13}^2)^{1/2}$ and $\mathcal{E} \approx 1/137$ is the atomic fine structure constant and \vec{S} denotes the spin of diatomic molecules N_2 , CO , NO and CH . Thus, the spin-orbit interactions $H_{\text{so-hk}}(r, \Theta, \bar{\theta})$ appear automatically as a result of the deformation of the space phase. Now, physically, we can rewrite the quantum spin-orbit coupling $\overset{P}{L}\overset{P}{S}$ as follows :

$$\vec{J} = \vec{L} + \vec{S} \Rightarrow 2\vec{L}\vec{S} = \vec{J}^2 - \vec{L}^2 - \vec{S}^2 \quad (23)$$

Here \vec{J} is the total momentum of the studied diatomic molecule (N_2 , CO , NO and CH). Substitution this equation into Eq. (22) yields:

$$H_{\text{pert-hk}}(r, \Theta, \bar{\theta}) = \frac{\mathcal{E}}{2} \left\{ \left(\frac{B-V_0}{2r^3} + \frac{C}{r^4} + \frac{V_1}{2r^3} \exp(-\alpha r) + \frac{\alpha V_1}{2r^2} \exp(-\alpha r) \right) \Theta + \frac{\bar{\theta}}{2\mu} \right\} (\vec{J}^2 - \vec{L}^2 - \vec{S}^2) \quad (24)$$

It is well known that, in quantum mechanics, the eigenvalues $k(j, l, s)$ of the spin-orbit coupling $\overset{P}{L}\overset{P}{S}$ are equal to $\frac{1}{2} [j(j+1) - l(l+1) - s(s+1)]$ and the eigenvalues of the total operator \vec{J} are:

$$j = \left| \frac{l-s}{4}, \frac{l+s}{4}, \frac{l-s}{4} + \frac{1}{2}, \frac{l+s}{4} \right| \quad \text{N-values}$$

Which are obtained in the interval $|l-s| \leq j \leq |l+s|$. After straightforward calculation, the radial functions $R_{nl}(r)$ satisfy the following differential equation in NRNCQ for diatomic molecules N_2 , CO , NO and CH under GHKP model :

$$\frac{d^2 R_{nl}(r)}{dr^2} + 2\mu(E_{nc-hk} - V_{nc\text{-eff}}(r))R_{nl}(r) = 0 \quad (25.1)$$



with

$$V_{nc-eff}(r) = V_{hk}(r) + \varepsilon \left\{ \left(\frac{B-V_0}{2r^3} + \frac{C}{r^4} + \frac{V_1 \exp(-\alpha r)}{2r^3} + \frac{\alpha V \exp(-\alpha r)}{2r^2} \right) \Theta + \frac{\bar{\theta}}{2\mu} \right\} \mathbf{L} \cdot \mathbf{S} + \frac{l(l+1)}{2\mu r^2} \quad (25.2)$$

We have introduced the generalized effective potential $V_{nc-eff}(r)$ in (NC: 3D-RSP) symmetries. We have seen previously that the induced spin-orbit $H_{so-hk}(r, \Theta, \bar{\theta})$ is infinitesimal compared to the principal Hamiltonian operator $H_{hk}(p, x)$ in NRNCQM for diatomic molecules N_2 , CO , NO and CH under GHKP model. This equation cannot be solved analytically for any state l because of the centrifugal term and the studied potential itself. Therefore, we must use a suitable approximations type suggested by Greene and Aldrich and Dong et al. [5, 9, 10, 38, 39]:

$$\frac{1}{r^2} \approx \frac{\alpha^2}{(1 - \exp(-\alpha r))^2} \quad (26.1)$$

Thus, we have the following direct consequences:

$$\begin{aligned} \frac{\exp(-\alpha r)}{r^2} &\approx \frac{\alpha^2 s}{(1-s)^2}, & \frac{\exp(-\alpha r)}{r^3} &\approx \frac{\alpha^2 s}{(1-s)^3} \\ \frac{1}{r^3} &\approx \frac{\alpha^3}{(1 - \exp(-\alpha r))^3} = \frac{\alpha^3}{(1-s)^3} & \text{and } \frac{1}{r^4} &\approx \frac{\alpha^4}{(1 - \exp(-\alpha r))^4} = \frac{\alpha^4}{(1-s)^4} \end{aligned} \quad (26.2)$$

This allows applying standard perturbation theory to determine the nonrelativistic energy corrections E_{so-ks} of diatomic molecules at first order of two infinitesimal parameters Θ and $\bar{\theta}$ due to noncommutativity space-phase properties.

3.3 Bound State Solution for spin-orbit operator for diatomic molecules under GHKP model

The principal goal of this subsection is to determine the energy spectrum $E_{so-hk}(n, V_1, V_0, \alpha, r_e, D_e, j, l, s) \equiv E_{so-ks}$ which produced with the effect of the operator $H_{so-hk}(r, \Theta, \bar{\theta})$ by applying standard perturbation theory at first order of $(\Theta$ and $\bar{\theta})$ and through the structure constants which specified the dimensionality of GHKP model of diatomic molecules N_2 , CO , NO and CH . Thus, we obtain the following results:

$$E_{so-hk} = N_{nl}^2 k(j, l, s) \int_0^{+\infty} s^{2\eta_n} (1-s)^{1+2\rho} [p_n^{(2\eta_n, 2\rho)}(1-2s)]^2 \left\{ \Theta \left(\frac{B-V_0}{2r^3} + \frac{C}{r^4} + \frac{V_1}{2r^3} \exp(-\alpha r) + \frac{\alpha V_1}{2r^2} \exp(-\alpha r) \right) + \frac{\bar{\theta}}{2\mu} \right\} dr \quad (27)$$

Where $s = \exp(-\alpha r)$, this allows us to obtain $dr = -\frac{1}{\alpha} \frac{ds}{s}$. After introducing a new variable $z = 1 - 2s$, we

obtain $dr = \frac{1}{\alpha} \frac{dz}{1-z}$ and $1-s = \frac{z+1}{2}$, the approximations (26.2) in that case have the following form:

$$\left\{ \begin{aligned} \frac{\exp(-\alpha r)}{r^2} &\approx \frac{\alpha^2 s}{(1-s)^2} = -2\alpha^2 \frac{z-1}{(z+1)^2}, & \frac{\exp(-\alpha r)}{r^3} &\approx \frac{\alpha^2 s}{(1-s)^3} = 4\alpha^2 \frac{1-z}{(z+1)^3} \\ \frac{1}{r^3} &\approx \frac{\alpha^3}{(1-\exp(-\alpha r))^3} = \frac{\alpha^3}{(1-s)^3} = \frac{8\alpha^3}{(z+1)^3} & \text{and } \frac{1}{r^4} &\approx \frac{\alpha^4}{(1-\exp(-\alpha r))^4} = \frac{\alpha^4}{(1-s)^4} = \frac{16\alpha^4}{(z+1)^4} \end{aligned} \right. \quad (28)$$

If we introduce the following five factors $T_i(n, r_e, D_e)$ with $(i = \overline{1,5})$ as:

$$\begin{aligned} T_1(n, r_e, D_e) &\equiv \frac{B-V_0}{2} \int_0^{+\infty} s^{2\eta_n} (1-s)^{1+2\rho} [P_n^{(2\eta_n, 2\rho)}(1-2s)]^2 \frac{1}{r^3} dr \\ T_2(n, r_e, D_e) &\equiv C \int_0^{+\infty} s^{2\eta_n} (1-s)^{1+2\rho} [P_n^{(2\eta_n, 2\rho)}(1-2s)]^2 \frac{1}{r^4} dr \\ T_3(n, r_e, D_e) &\equiv \frac{V_1}{2} \int_0^{+\infty} s^{2\eta_n} (1-s)^{1+2\rho} [P_n^{(2\eta_n, 2\rho)}(1-2s)]^2 \frac{1}{2r^3} \exp(-\alpha r) dr \\ T_4(n, r_e, D_e) &\equiv \frac{V_1\alpha}{2} \int_0^{+\infty} s^{2\eta_n} (1-s)^{1+2\rho} [P_n^{(2\eta_n, 2\rho)}(1-2s)]^2 \frac{1}{2r^2} \exp(-\alpha r) dr \\ T_5(n, r_e, D_e) &\equiv \int_0^{+\infty} s^{2\eta_n} (1-s)^{1+2\rho} [P_n^{(2\eta_n, 2\rho)}(z)]^2 dr \end{aligned} \quad (29)$$

Substituting Eq. (28) Into Eq. (29), the five factors $T_i(n, r_e, D_e)$ become like the following forms:

$$\begin{aligned} T_1(n, r_e, D_e) &\equiv 2 \frac{B-V_0}{2^{2\rho+2\eta_n}} \alpha^2 \int_0^{+\infty} (1-z)^{2\eta_n-1} (z+1)^{1+2\rho-3} [P_n^{(2\eta_n, 2\rho)}(z)]^2 dz \\ T_2(n, r_e, D_e) &\equiv \frac{8C\alpha^3}{2^{2\rho+2\eta_n}} \int_0^{+\infty} (1-z)^{2\eta_n-1} (z+1)^{1+2\rho-4} [P_n^{(2\eta_n, 2\rho)}(z)]^2 dz \\ T_3(n, r_e, D_e) &\equiv \frac{\alpha V_1}{2^{2\rho+2\eta_n}} \int_0^{+\infty} (1-z)^{2\eta_n} (z+1)^{1+2\rho-3} [P_n^{(2\eta_n, 2\rho)}(z)]^2 dz \\ T_4(n, r_e, D_e) &\equiv \frac{V_1\alpha}{2^{2+2\rho+2\eta_n}} \int_0^{+\infty} (1-z)^{2\eta_n} (z+1)^{1+2\rho} [P_n^{(2\eta_n, 2\rho)}(z)]^2 dz \\ T_5(n, r_e, D_e) &\equiv \frac{1}{2^{1+2\rho+2\eta_n} \alpha} \int_0^{+\infty} (1-z)^{2\eta_n-1} (z+1)^{1+2\rho} [P_n^{(2\eta_n, 2\rho)}(z)]^2 dz \end{aligned} \quad (30)$$

Instead of solving the modified-radial Schrödinger equation for the effective GHKP model $V_{nc\text{-eff}}(r)$ given by Eq. (25.2), we now solve the MSE for the generalized effective potential $V_{nc\text{-eff}}(r)$ given by the previous approximations:

$$V_{nc\text{-eff}}(r) = \alpha \frac{B-V_0}{1-s} + \frac{\alpha^2 C}{(1-s)^2} + D_e - \frac{A\alpha s}{1-s} + \gamma \left\{ \alpha^3 \frac{B-V_0}{2(1-s)^3} + \frac{\alpha^4 C}{(1-s)^2 4} + \frac{V_1\alpha^2 s}{2(1-s)^3} + \frac{\alpha^3 V_1}{2(1-s)^2} + \frac{\bar{\theta}}{2\mu} \right\} \frac{\rho\rho}{LS} + \frac{\alpha^2 l(l+1)}{2\mu(1-s)^2} \quad (31)$$

On arranging Eq. (27), we obtain our nonrelativistic energy corrections $E_{so\text{-hk}}(n, V_1, V_0, \alpha, r_e, D_e, j, l, s)$ at first order of two infinitesimal parameters Θ and $\bar{\theta}$ for the diatomic molecules N_2 , CO , NO and CH as:

$$E_{so\text{-hk}}(n, V_1, V_0, \alpha, r_e, D_e, j, l, s) = \frac{2n! \eta_n \alpha \Gamma(n+2+2\eta_n+2\rho)}{\Gamma(n+1+2\eta_n) \Gamma(n+2+2\rho)} k(j, l, s) \left\{ \Theta \sum_{i=1}^4 T_i(n, r_e, D_e) + \frac{\bar{\theta}}{2\mu} T_5(n, r_e, D_e) \right\} \quad (32)$$



For the ground state ($n = 0$), we have $P_0^{(\beta,\gamma)}(z) = 1$, thus the above five factors in Eq. (30) reduced to the following simple form:

$$\begin{aligned}
 T_1(n=0, r_e, D_e) &\equiv \frac{B-V_0}{2^{2\rho+2\eta_n}} 2\alpha^2 \int_0^{+\infty} (1-z)^{2\eta_n-1} (z+1)^{2\rho-2} dz, & T_2(n=0, r_e, D_e) &\equiv \frac{8C\alpha^3}{2^{2\rho+2\eta_n}} \int_0^{+\infty} (1-z)^{2\eta_n-1} (z+1)^{2\rho-3} dz \\
 T_3(n=0, r_e, D_e) &\equiv \frac{\alpha V_1}{2^{2\rho+2\eta_n}} \int_0^{+\infty} (1-z)^{2\eta_n} (z+1)^{2\rho-2} dz, & T_4(n=0, r_e, D_e) &\equiv \frac{V_1\alpha}{2^{2+2\rho+2\eta_n}} \int_0^{+\infty} (1-z)^{2\eta_n} (z+1)^{1+2\rho} dz \\
 T_5(n=0, r_e, D_e) &\equiv \frac{1}{2^{1+2\rho+2\eta_n} \alpha} \int_0^{+\infty} (1-z)^{2\eta_n-1} (z+1)^{1+2\rho} z dz
 \end{aligned} \tag{33}$$

Comparing Eq. (33) with the integral of the form [40]:

$$\begin{aligned}
 \int_{-1}^{+1} (1-p)^\alpha (1+p)^\beta P_m^{(\alpha,\beta)} P_n^{(\alpha,\beta)} dp &= 2^{\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)n!} \delta_{mn} \Rightarrow \\
 \int_{-1}^{+1} (1-p)^n + \alpha(1+p)^{n+\beta} dp &= 2^{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(2n+\alpha+\beta+1)} \text{ for } (n=0,1,\dots) \tag{34}
 \end{aligned}$$

We have five factors as:

$$\begin{aligned}
 T_1(n, r_e, D_e) &\equiv \frac{B-V_0}{2} \alpha^2 \frac{\Gamma(2\eta_n)\Gamma(2\rho-1)}{(2\eta_n+2\rho-2)\Gamma(\eta_n+2\rho-2)}, & T_2(n, r_e, D_e) &\equiv C\alpha^3 \frac{\Gamma(2\eta_n)\Gamma(2\rho-2)}{(2\eta_n+2\rho-3)\Gamma(2\eta_n+2\rho-3)} \\
 T_3(n, r_e, D_e) &\equiv \frac{\alpha V_1}{2} \frac{\Gamma(2\eta_n+1)\Gamma(2\rho-1)}{(2\eta_n+2\rho-1)\Gamma(2\eta_n+2\rho-1)}, & T_4(n, r_e, D_e) &\equiv V_1\alpha \frac{\Gamma(2\eta_n+1)\Gamma(2\rho+2)}{(2\eta_n+2\rho+2)\Gamma(2\eta_n+2\rho+2)} \\
 \text{and } T_5(n, r_e, D_e) &\equiv \frac{1}{\alpha} \frac{\Gamma(2\eta_n)\Gamma(2\rho+2)}{(2\eta_n+2\rho+1)\Gamma(2\eta_n+2\rho+1)}
 \end{aligned} \tag{35}$$

with $\varepsilon_0 = -\frac{2\mu}{\alpha^2}(E_{0l} - D_e)$. Substituting Eqs. (35) into Eq. (32), we obtain nonrelativistic energy corrections for the ground state $E_{\text{so-hk}}(n=0, V_1, V_0, \alpha, r_e, D_e, j, l, s)$ at first order of two infinitesimal parameters Θ and $\bar{\theta}$ for the diatomic molecules N_2, CO, NO and CH as:

$$E_{\text{so-hk}}(n=0, V_1, V_0, \alpha, r_e, D_e, j, l, s) = \frac{2\eta_0\alpha\Gamma(2+2\eta_0+2\rho)}{\Gamma(1+2\eta_0)\Gamma(2+2\rho)} k(j, l, s) \left\{ \Theta T(n=0, V_1, V_0, B, C, r_e, D_e) + \frac{\bar{\theta}}{2\mu} T_5(n=0, r_e, D_e) \right\} \tag{36}$$

With $T(n=0, V_1, V_0, B, C, r_e, D_e) = \sum_{i=1}^4 T_i(n=0, r_e, D_e)$. For the first excited state ($n = 1$), we have $P_1^{(\alpha,\beta)}(z) = \alpha + 1 - (\alpha + \beta + 2) \frac{1-z}{2}$, thus the five factors in Eq. (30) reduced to the following simple form:

$$\begin{aligned}
 T_1(n=1, r_e, D_e) &\equiv \frac{B-V_0}{2^{2\rho+2\eta_n}} 2\alpha^2 \int_0^{+\infty} (1-z)^{2\eta_n-1} (z+1)^{2\rho-2} [a-b(1-z)]^2 dz \\
 T_2(n=1, r_e, D_e) &\equiv \frac{8C\alpha^3}{2^{2\rho+2\eta_n}} \int_0^{+\infty} (1-z)^{2\eta_n-1} (z+1)^{2\rho-3} [a-b(1-z)]^2 dz \\
 T_3(n=1, r_e, D_e) &\equiv \frac{\alpha V_1}{2^{2\rho+2\eta_n}} \int_0^{+\infty} (1-z)^{2\eta_n} (z+1)^{2\rho-2} [a-b(1-z)]^2 dz \\
 T_4(n=1, r_e, D_e) &\equiv \frac{V_1\alpha}{2^{2+2\rho+2\eta_n}} \int_0^{+\infty} (1-z)^{2\eta_n} (z+1)^{1+2\rho} [a-b(1-z)]^2 dz \\
 T_5(n=1, r_e, D_e) &\equiv \frac{1}{\alpha 2^{1+2\rho+2\eta_n}} \int_0^{+\infty} (1-z)^{2\eta_n-1} (z+1)^{1+2\rho} [a-b(1-z)]^2 dz
 \end{aligned} \tag{37}$$

With $a = 2\eta_1 + 1$, $b = \eta_1 + \rho + 1$, $\eta_1 = \sqrt{\varepsilon_1 - \beta - \chi + \delta}$ and $\varepsilon_1 = \frac{2\mu}{\alpha^2} (E_{1l} - D_e)$. A direct simplification to Eq. (37) gives :

$$\begin{aligned}
 T_1(n=1, r_e, D_e) &\equiv \frac{B-V_0}{2^{2\rho+2\eta_n}} 2\alpha^2 (a^2 \int_0^{+\infty} (1-z)^{2\eta_n-1} (z+1)^{2\rho-2} dz - 2ab \int_0^{+\infty} (1-z)^{2\eta_n-1} (z+1)^{2\rho-1} dz + b^2 \int_0^{+\infty} (1-z)^{2\eta_n+1} (z+1)^{2\rho-2} dz) \\
 T_2(n=1, r_e, D_e) &\equiv \frac{8C\alpha^3}{2^{2\rho+2\eta_n}} (a^2 \int_0^{+\infty} (1-z)^{2\eta_n-1} (z+1)^{2\rho-3} dz - 2ab \int_0^{+\infty} (1-z)^{2\eta_n} (z+1)^{2\rho-3} dz + b^2 \int_0^{+\infty} (1-z)^{2\eta_n+1} (z+1)^{2\rho-3} dz)
 \end{aligned} \tag{38.1}$$

$$\begin{aligned}
 T_3(n=1, r_e, D_e) &\equiv \frac{\alpha V_1}{2^{2\rho+2\eta_n}} (a^2 \int_0^{+\infty} (1-z)^{2\eta_n} (z+1)^{2\rho-2} dz - 2ab \int_0^{+\infty} (1-z)^{2\eta_n+1} (z+1)^{2\rho-2} dz + b^2 \int_0^{+\infty} (1-z)^{2\eta_n+2} (z+1)^{2\rho-2} dz) \\
 T_4(n=1, r_e, D_e) &\equiv \frac{V_1\alpha}{2^{2+2\rho+2\eta_n}} (a^2 \int_0^{+\infty} (1-z)^{2\eta_n} (z+1)^{1+2\rho} dz - 2ab \int_0^{+\infty} (1-z)^{2\eta_n+1} (z+1)^{1+2\rho} dz + b^2 \int_0^{+\infty} (1-z)^{2\eta_n+2} (z+1)^{1+2\rho} dz)
 \end{aligned} \tag{38.2}$$

$$T_5(n=1, r_e, D_e) \equiv \frac{1}{2^{1+2\rho+2\eta_n} \alpha} (a^2 \int_0^{+\infty} (1-z)^{2\eta_n-1} (z+1)^{1+2\rho} dz - 2ab \int_0^{+\infty} (1-z)^{2\eta_n} (z+1)^{1+2\rho} dz + b^2 \int_0^{+\infty} (1-z)^{2\eta_n+1} (z+1)^{1+2\rho} dz)$$

Comparing Eqs. (38.1) and (38.2) with the integral (34), we have the five factors as:

$$\begin{aligned}
 T(n=1, r_e, D_e) &\equiv 2(B-V_0)\alpha^2 \left(\frac{a^2}{4} \frac{\Gamma(2\eta_1)\Gamma(2\rho-1)}{(\gamma-2)\Gamma(\gamma-2)} - ab \frac{\Gamma(2\eta_1+1)\Gamma(2\rho-1)}{(\gamma-1)\Gamma(\gamma-1)} + b^2 \frac{\Gamma(2\eta_1+2)\Gamma(2\rho-1)}{\gamma\Gamma(\gamma)} \right) \\
 T_2(n=1, r_e, D_e) &\equiv 4C\alpha^3 \left(\frac{a^2}{16} \frac{\Gamma(2\eta_1)\Gamma(2\rho-2)}{(\gamma-3)\Gamma(\gamma-3)} - ab \frac{\Gamma(2\eta_1+1)\Gamma(2\rho-2)}{(\gamma-2)\Gamma(\gamma-2)} + b^2 \frac{\Gamma(2\eta_1+2)\Gamma(2\rho-2)}{(\gamma-1)\Gamma(\gamma-1)} \right) \\
 T_3(n=1, r_e, D_e) &\equiv \alpha V_1 \left(\frac{a^2}{2} \frac{\Gamma(2\eta_1+1)\Gamma(2\rho-1)}{(\gamma-1)\Gamma(\gamma-1)} - 2ab \frac{\Gamma(2\eta_1+2)\Gamma(2\rho-1)}{\gamma\Gamma(\gamma)} + 2b^2 \frac{\Gamma(2\eta_1+3)\Gamma(2\rho-1)}{(\gamma+1)\Gamma(\gamma+1)} \right)
 \end{aligned} \tag{39.1}$$

$$\begin{aligned}
 T_4(n=1, r_e, D_e) &\equiv 4V_1\alpha \left(a^2 \frac{\Gamma(2\eta_1+1)\Gamma(2\rho+2)}{(\gamma+2)\Gamma(\gamma+2)} - 4ab \frac{\Gamma(2\eta_1+2)\Gamma(2\rho+2)}{(\gamma+3)\Gamma(\gamma+3)} + 4b^2 \frac{\Gamma(2\eta_1+3)\Gamma(2\rho+2)}{(\gamma+4)\Gamma(\gamma+4)} \right) \\
 T_5(n=1, r_e, D_e) &\equiv \frac{1}{\alpha} \left(a^2 \frac{\Gamma(2\eta_1)\Gamma(2\rho+2)}{(\gamma+1)\Gamma(\gamma+1)} - 4ab \frac{\Gamma(2\eta_1+1)\Gamma(2\rho+2)}{(\gamma+2)\Gamma(\gamma+2)} + 4b^2 \frac{\Gamma(2\eta_1+2)\Gamma(2\rho+2)}{(\gamma+3)\Gamma(\gamma+3)} \right)
 \end{aligned} \tag{39.2}$$

With $\gamma = 2\eta_1 + 2\rho$. This allows us to obtain nonrelativistic energy corrections for the first excited state $E_{so-hk}(n=1, V_1, V_0, \alpha, r_e, D_e, j, l, s)$ produced by spin-orbit operator $H_{so-ks}(r, \Theta, \bar{\theta})$ at first order of two infinitesimal parameters Θ and $\bar{\theta}$ for the diatomic molecules N_2, CO, NO and CH :



$$E_{so-ks}(n=1, V_1, V_0, B, C, j, l, s) = \frac{2\eta_1 \alpha \Gamma(3+2\eta_n+2\rho)}{\Gamma(2+2\eta_n)\Gamma(3+2\rho)} k(j, l, s) \epsilon \left\{ \Theta T(n=1, V_1, V_0, B, C, r_e, D_e) + \frac{\bar{\theta}}{2\mu} T_5(n=1, r_e, D_e) \right\} \quad (40)$$

With $T(n=1, V_1, V_0, B, C, r_e, D_e) = \sum_{i=1}^4 T_i(n=1, r_e, D_e)$. Thus, for any excited state, the nonrelativistic energy corrections $E_{so-hk}(n, V_1, V_0, \alpha, r_e, D_e, j, l, s)$ which produced by spin-orbit operator $H_{so-hk}(r, \Theta, \bar{\theta})$ at first order of two infinitesimal parameters Θ and $\bar{\theta}$ for the diatomic molecules N_2, CO, NO and CH can be written as:

$$E_{so-hk}(n, V_1, V_0, \alpha, r_e, D_e, j, l, s) = B_n^2 k(j, l, s) \left\{ \Theta T(n, V_1, V_0, B, C, r_e, D_e) + \frac{\bar{\theta}}{2\mu} T_5(n, r_e, D_e) \right\} \quad (41)$$

With $T(n, V_1, V_0, B, C, r_e, D_e) = \sum_{i=1}^4 T_i(n, r_e, D_e)$. This allow us to obtain the following important physical results for diatomic molecules N_2, CO, NO and CH under GHKP model:

$$H_{so-hk}(r, \Theta, \bar{\theta}) \Psi_{n,lm}(r, \theta, \varphi) = E_{so-hk}(n, V_1, V_0, \alpha, r_e, D_e, j, l, s) \Psi_{n,lm}(r, \theta, \varphi) \quad (42)$$

3.4 Bound state solution for modified Zeeman effect for GHKP model

In this subsection, it is possible to obtain the second automatically symmetry for diatomic molecules N_2, CO, NO and CH under GHKP model. This physical phenomenon is induced of the influence of an external uniform magnetic field $\vec{\mathfrak{N}}$, if we make the following two simultaneous transformations to ensure that previous calculations are not reputed :

$$(\Theta, \bar{\theta}) \rightarrow (\lambda, \bar{\sigma}) \mathfrak{N} \quad (43)$$

here λ and $\bar{\sigma}$ are just two infinitesimal real proportional constants, and to simplify calculations without compromising physical content we choose the magnetic field $\vec{\mathfrak{N}} = \mathfrak{N} \vec{k}$. Then we transform the spin-orbit coupling to new physical phenomena as follows:

$$\left\{ \left(\frac{B-V_0}{2r^3} + \frac{C}{r^4} + \frac{V_1 \exp(-\alpha r)}{2r^3} + \frac{\alpha V_1 \exp(-\alpha r)}{2r^2} \right) \frac{\bar{\theta}}{2\mu} \right\} \bar{L} \rightarrow \left\{ \lambda \left(\frac{B-V_0}{2r^3} + \frac{C}{r^4} + \frac{V_1 \exp(-\alpha r)}{2r^3} + \frac{\alpha V_1 \exp(-\alpha r)}{2r^2} \right) + \frac{\bar{\sigma}}{2\mu} \right\} \mathfrak{N} L_z \quad (44)$$

This allowed deriving the modified magnetic Hamiltonian operator $H_{z-hk}(r, \lambda, \bar{\sigma})$ for previous diatomic molecules under GHKP model in global (NC: 3D-RSP) symmetries as:

$$H_{z-hk}(r, \lambda, \bar{\sigma}) = \left\{ \lambda \left(\frac{B-V_0}{2r^3} + \frac{C}{r^4} + \frac{V_1 \exp(-\alpha r)}{2r^3} + \frac{\alpha V_1 \exp(-\alpha r)}{2r^2} \right) + \frac{\bar{\sigma}}{2\mu} \right\} \mathfrak{N}_{mod-z} \quad (45)$$

Here $\mathfrak{N}_z \equiv -\vec{S} \vec{\mathfrak{N}}$ denote to Zeeman effect in commutative quantum mechanics, while $\mathfrak{N}_{mod-z} \equiv \vec{\mathfrak{N}} \vec{J} - \mathfrak{N}_z$ is the modified Zeeman effect. To obtain the exact NC magnetic modifications of energy for ground state, first excited state and n^{th} excited states of diatomic molecules N_2, CO, NO and CH under GHKP model

$E_{mag-hk}(n=0, m, l, V_1, V_0, \alpha, r_e, D_e)$, $E_{mag-hk}(n=1, l, m, V_1, V_0, \alpha, r_e, D_e)$ and $E_{mag-hk}(n, l, m, V_1, V_0, \alpha, r_e, D_e)$ and just replace $k(j, l, s)$ and Θ in the Eqs. (36), (40) and (41) by the following parameters m and χ , respectively:

$$\begin{aligned}
 E_{mag-hk}(n=0, l, m, V_1, V_0, \alpha, r_e, D_e) &= \frac{2\eta_0\alpha\Gamma(2+2\eta_0+2\rho)}{\Gamma(1+2\eta_n)\Gamma(2+2\rho)} \varepsilon \mathfrak{N} \left\{ \lambda T(n=0, V_1, V_0, B, C, r_e, D_e) + \frac{\bar{\sigma}}{2\mu} T_5(n=0, r_e, D_e) \right\} m \\
 E_{mag-hk}(n=1, l, m, V_1, V_0, \alpha, r_e, D_e) &= \frac{2\eta_1\alpha\Gamma(3+2\eta_n+2\rho)}{\Gamma(2+2\eta_n)\Gamma(3+2\rho)} \varepsilon \mathfrak{N} \left\{ \lambda T(n=1, V_1, V_0, B, C, r_e, D_e) + \frac{\bar{\sigma}}{2\mu} T_5(n=1, r_e, D_e) \right\} m \\
 E_{mag-hk}(n, l, m, V_1, V_0, \alpha, r_e, D_e) &= \frac{2n!\eta_n\alpha\Gamma(n+2+2\eta_n+2\rho)}{\Gamma(n+1+2\eta_n)\Gamma(n+2+2\rho)} \varepsilon \mathfrak{N} \left\{ \lambda T(n, V_1, V_0, B, C, r_e, D_e) + \frac{\bar{\sigma}}{2\mu} T_5(n, r_e, D_e) \right\} m
 \end{aligned} \tag{46}$$

We have $-l \leq m \leq +l$, which allow us to fix $(2l+1)$ values for discrete numbers m . It should be noted that the results obtained in Eq. (43) could find it by direct calculation $E_{mag-hk} = \langle \Psi(r, \theta, \varphi) | H_{m-hk}(r, \chi, \bar{\sigma}) | \Psi(r, \theta, \varphi) \rangle$ that takes the following explicit relation:

$$E_{mag-hk}(n, j, m, s) = N_m^2 \mathfrak{N} m \int_0^{+\infty} s^{2\eta_n} (1-s)^{1+2\rho} \left[P_n^{(2\eta_n, 2\rho)}(1-2s) \right]^2 \left\{ \lambda \left(\frac{B-V_0}{2r^3} + \frac{C}{r^4} + \frac{V_1 \exp(-\alpha r)}{2r^3} + \frac{\alpha V_1 \exp(-\alpha r)}{2r^2} \right) + \frac{\bar{\sigma}}{2\mu} \right\} dr \tag{47}$$

It is clear that, the Eq. (47) can be rewritten as follows:

$$E_{mag-hk}(n, j, m, s, V_1, V_0, \alpha, r_e, D_e) = \frac{2n!\eta_n\alpha\Gamma(n+2+2\eta_n+2\rho)}{\Gamma(n+1+2\eta_n)\Gamma(n+2+2\rho)} \varepsilon \mathfrak{N} m \left\{ \lambda \sum_{i=1}^4 T_i(n, r_e, D_e) + \frac{\bar{\sigma}}{2\mu} T_5(n, r_e, D_e) \right\} \tag{48}$$

The 5-factors $T_i(n, r_e, D_e)$ are given by Eq. (30). Then we find the magnetic specters of energy produced by the operator $H_{m-ks}(r, \lambda, \bar{\sigma})$ for the ground state and first excited states repeating the same calculations in the previous subsection. We end this subsection by addressing the important results :

$$H_{z-hk}(r, \Theta, \bar{\theta}) \Psi_{n,lm}(r, \theta, \varphi) = E_{mag-hk}(n, j, m, s, V_1, V_0, \alpha, r_e, D_e) \Psi_{n,lm}(r, \theta, \varphi) \tag{49}$$

4. Results and discussion

In the previous subsections, we obtained the solution of the MSE for GHKP model, which is given in Eq. (25.2) by using the generalized Bopp's shift method and standard perturbation theory. The energy eigenvalues are calculated in the 3D space-phase. The modified eigenenergies for the ground state, first excited state and n^{th} excited states of diatomic molecules N_2, CO, NO and CH under GHKP model $E_{nc-hk}(n=0, V_1, V_0, \alpha, r_e, D_e, j, l, m, s)$, $E_{nc-hk}(n=1, V_1, V_0, \alpha, r_e, D_e, j, l, m, s)$ and $E_{nc-hk}(n, V_1, V_0, \alpha, r_e, D_e, j, l, m, s)$ with spin \vec{S} are obtained in this paper on based to our original results presented on the Eqs. (36), (40), (41) and (46), in addition to the ordinary energy E_{nl} for Hellmann–Kratzer potential model which presented in the Eq. (9) take the form:

$$\begin{aligned}
 E_{nc-hk}(n=0, V_1, V_0, \alpha, r_e, D_e, j, l, m, s) &= E_{0l} + \alpha x \frac{2\eta_0 \Gamma(2+2\eta_0+2\rho)}{\Gamma(1+2\eta_0)\Gamma(2+2\rho)} \left\{ \Theta k(j, l, s) + B\lambda m \Gamma(n=0, V_1, V_0, B, C, r_e, D_e) + \frac{T_5(n=0, r_e, D_e)}{2\mu} (k(j, l, s)\bar{\theta} + N\bar{\sigma}m) \right\} \\
 E_{nc-hk}(n=1, V_1, V_0, \alpha, r_e, D_e, j, l, m, s) &= E_{1l} + \alpha x \frac{2\eta_1 \Gamma(3+2\eta_1+2\rho)}{\Gamma(2+2\eta_1)\Gamma(3+2\rho)} \left\{ \Theta k(j, l, s) + B\lambda m \Gamma(n=1, V_1, V_0, B, C, r_e, D_e) + \frac{T_5(n=1, r_e, D_e)}{2\mu} (k(j, l, s)\bar{\theta} + B\bar{\sigma}m) \right\} \\
 E_{nc-hk}(n, V_1, V_0, \alpha, r_e, D_e, j, l, m, s) &= E_{nl} + \alpha x \frac{2n! \eta_n \Gamma(n+2+2\eta_n+2\rho)}{\Gamma(n+1+2\eta_n)\Gamma(n+2+2\rho)} \left\{ \Theta k(j, l, s) + B\lambda m \Gamma(n, V_1, V_0, B, C, r_e, D_e) + \frac{T_5(n, r_e, D_e)}{2\mu} (k(j, l, s)\bar{\theta} + B\bar{\sigma}m) \right\}
 \end{aligned} \tag{50}$$

Where E_{0l} and E_{1l} are the energy of ground state and first excited state of diatomic molecules N_2 , CO , NO and CH in the symmetries of quantum mechanics under Hellmann–Kratzer potential model:

$$\begin{aligned}
 E_{0l} &= \frac{\alpha^2}{2\mu} \left[\delta - \frac{4\mu D_e r_e}{\alpha} \right] - V_0 \alpha + D_e - \frac{\alpha^2}{2\mu} \left[\frac{4\mu D_e r_e + 2\mu V_0 - 2\mu V_1 - \delta \alpha}{2\xi \alpha} - \frac{\xi}{2} \right]^2 \\
 E_{1l} &= \frac{\alpha^2}{2\mu} \left[\delta - \frac{4\mu D_e r_e}{\alpha} \right] - V_0 \alpha + D_e - \frac{\alpha^2}{2\mu} \left[\frac{4\mu D_e r_e + 2\mu V_0 - 2\mu V_1 - \delta \alpha}{2(1+\xi)\alpha} - \frac{1+\xi}{2} \right]^2
 \end{aligned} \tag{51}$$

Thus, the total energy $E_{nc-ks}(n, V_1, V_0, \alpha, r_e, D_e, j, l, m, s)$ for GHKP model in (NC: 3D-RSP) symmetries, is the sum of the ordinary part of energy E_{nl} and the two corrections of energy $E_{so-hk}(n, j, l, s, V_1, V_0, B, C)$ and $E_{mag-hk}(n, j, m, s, V_1, V_0, B, C)$. This is one of the main objectives of our research. Finally, we end this section by introducing the important result of this work as :

$$\begin{aligned}
 \{H_{hk}(r) + H_{so-hk}(r, \Theta, \bar{\theta}) + H_{z-hk}(r, \lambda, \bar{\sigma})\} \Psi_{nlm}(r, \theta, \varphi) &= \\
 = \{E_{nl} + E_{so-hk}(n, V_1, V_0, \alpha, r_e, D_e, j, l, s) + E_{z-hk}(n, V_1, V_0, \alpha, r_e, D_e, m)\} \Psi_{nlm}(r, \theta, \varphi)
 \end{aligned} \tag{52}$$

Now it is very necessary to return to some special cases, we adjust some parameters of eq. (1) to have the following.

1-Generalized Hellmann potential model: For $D_e \rightarrow 0$, Eq. (1) becomes the generalized Hellmann potential model of the form (under investigation):

$$V_{hk}(\hat{r}) \rightarrow V_{hp}(\hat{r}) = -\frac{V_0}{r} + \frac{V_1 \exp(-\alpha r)}{r} + \left(-\frac{V_0}{2r^3} + \frac{V_1 \exp(-\alpha r)}{2r^3} + \frac{\alpha V_1 \exp(-\alpha r)}{2r^2} \right) \mathbb{L} \mathbb{O} \mathbb{O} \tag{53}$$

Its corresponding energy eigenvalues, in (NC: 3D-RSP) symmetries, is obtained as (under investigation):

$$E_{nc-hp}(n, k(j, l, s), a, b, \delta, j, l, m, s) = E_{hp-nl} + N_{nl}^2 \epsilon \mathbb{N} \left(\lambda T(n, l, \delta) + \frac{\bar{\sigma}}{2\mu} T_4(n, l, \delta) \right) m + N_{nl}^2 \epsilon k(j, l, s) \left(\Theta T(n, l, \delta) + \frac{\bar{\theta}}{2\mu} T_4(n, l, \delta) \right) \tag{54}$$

where E_{hp-nl} , in ordinary quantum mechanics, is given by [41]

$$E_{hp-nl} = -\frac{\delta^2}{2\mu} \left[\left(\frac{(2\mu/\delta)(a-b) - (n+l+1)^2 - l(l+1)}{2(n+l+1)} \right)^2 - l(l+1) + \frac{2\mu a}{\delta} \right] \tag{55}$$

2-Generalized Kratzer potential model: For $(V_0, V_1) \rightarrow (0, 0)$, Eq. (1) becomes the Generalized Kratzer potential model of the form [42]:



$$V_{hk}(\hat{r}) \rightarrow V_{hk}(\hat{r}) = D_e \left(\frac{r - r_e}{r_e} \right)^2 + D_e \left(\frac{r_e^2}{r^4} - \frac{r_e}{r^3} \right) \Theta \bar{\Theta} \quad (56)$$

Its corresponding energy eigenvalues, in (NC: 3D-RSP) symmetries, is obtained as:

$$E_{nc-hp} = E_{kp-nl} + \varepsilon \frac{2n! \eta_n \alpha \Gamma(n+2+2\eta_n+2\rho)}{\Gamma(n+1+2\eta_n) \Gamma(n+2+2\rho)} \left\{ (\Theta k(j, l, s) + B \lambda m) \Gamma(n, V_1 \rightarrow 0, V_0 \rightarrow 0, r_e, D_e) + \frac{1}{2\mu} (k(j, l, s) \bar{\Theta} + B \bar{\sigma} m) \Gamma_5(n, r_e) \right\} \quad (57)$$

Where E_{kp-nl} , in ordinary quantum mechanics, is given by [43] :

$$E_{kp-nl} = D_e - \frac{1}{2\mu} \left[(4\mu D_e r_e)^2 \left(1 + 2n + \sqrt{1 + 4(2\mu D_e r_e + l(l+1))} \right)^{-2} \right] \quad (58)$$

Paying attention to the behavior of the spectrums that appears in Eq. (50) ($E_{nc-hk}(n=0, V_1, V_0, \alpha, r_e, D_e, j, l, m, s)$, $E_{nc-hk}(n=1, V_1, V_0, \alpha, r_e, D_e, j, l, m, s)$ $E_{nc-hk}(n, V_1, V_0, \alpha, r_e, D_e, j, l, m, s)$), it is possible to recover the results of commutative space (9), which was obtained in [10], when we consider the two simultaneously limits $(\Theta, \bar{\Theta}) \rightarrow (0, 0)$.

5. Conclusions

In this paper, three-dimensional modified radial Schrödinger equation has been performed for the generalized Hellmann–Kratzer potential model by using the improved approximation scheme to the centrifugal term for any l-states and generalized Bopp’s shift method, in addition to the standard perturbation theory in (NC: 3D-RSP) symmetries, we resume the main obtained results:

- The energy eigenvalues of the bound states $E_{nc-hk}(n=0, V_1, V_0, \alpha, r_e, D_e, j, l, m, s)$, $E_{nc-hk}(n=1, V_1, V_0, \alpha, r_e, D_e, j, l, m, s)$ and $E_{nc-hk}(n, V_1, V_0, \alpha, r_e, D_e, j, l, m, s)$ for the ground state, first excited state and n^{th} excited states of diatomic molecules N_2 , CO , NO and CH under GHKP model with spin \vec{S} have been analytically found via both generalized Bopp’s shift method and standard perturbation theory within the previous approximations type suggested by Greene and Aldrich and Dong et al.
- The Hamiltonian operator $H_{nc-ks}(r, \Theta, \bar{\Theta})$ in (NC: 3D-RSP) is the sum of Hamiltonian operator of GHKP model $H_{ks}(r)$ and two operators, the first one is the modified spin-orbit interaction $H_{so-hk}(r, \Theta, \bar{\Theta})$ while the second is the modified Zeeman operator $H_{z-hk}(r, \lambda, \bar{\sigma})$ for the diatomic molecules N_2 , CO , NO and CH .

It has been shown that, the MSE presents useful rich spectrums for improved understanding the diatomic molecules N_2 , CO , NO and CH influenced by the GHKP model and we have seen also that the modified of spin-orbit and modified Zeeman effect appeared du the presence of the two infinitesimal parameters $(\Theta, \bar{\Theta})$ and $(\lambda, \bar{\sigma})$ which are induced by position-position noncommutativity property of space. It should be noted that the results obtained in this research would be identical with corresponding results in ordinary quantum mechanics when the two parameters $(\Theta, \bar{\Theta})$ are reduced to the values $(0, 0)$. They are also identical to some of the special cases in the same field of our research. The results of our present work give us some special cases for example generalized Hellmann potential model and generalized Kratzer potential model.



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