Approximation of solutions of monotone variational inequality problems with applications in real Hilbert spaces

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Abstract

In this paper, variational inequality problem related to monotone operators is studied, and an iterative algorithm which is a modification of extragradient method is proposed for approximation of solution (assuming existence) of the variational inequality problem. Weak and strong convergence theorems are obtained and as applications, the iterative scheme proposed is shown to also approximate fixed points of pseudocontractive mappings, zeros of monotone mappings and solutions of equilibrium problems. A numerical example is given to show the functionality of the studied scheme. The obtained results improve and unify the corresponding results of several authors.

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Introduction

Let H be a real Hilbert space. A mapping $A : D(A) \subseteq H \to H$ is called *monotone* if for any $x, y \in D(A)$, we have that

$$\langle Ax - Ay, x - y \rangle \geq 0.$$

A mapping $A : D(A) \subseteq H \to H$ is called *m-strongly monotone* (or simply *strongly monotone*) if there exists a constant $m > 0$ such that for any $x, y \in D(A)$,

$$\langle Ax - Ay, x - y \rangle \geq m\|x - y\|^2.$$

A mapping $A : D(A) \subseteq H \to H$ is called *γ-inverse strongly monotone* (or simply *inverse strongly monotone*) if there exists a constant $\gamma > 0$ such that for any $x, y \in D(A)$,

$$\langle Ax - Ay, x - y \rangle \geq \gamma\|Ax - Ay\|^2,$$

while a mapping $A : D(A) \subseteq H \to H$ is called *L-Lipschitz continuous* if there exists a constant $L > 0$ such that for all $x, y \in D(A)$,

$$\|Ax - Ay\| \leq L\|x - y\|.$$

We shall say that a mapping $T : D(T) \subseteq H \to H$ is *pseudocontractive (strongly pseudocontractive)* if and only if the mapping $A = I - T$ is monotone (strongly monotone). Theory of zeros of monotone mappings and fixed point theory of pseudocontractive mappings have been studied extensively by many authors (see, for example, [15] and references therein).

Let C be a nonempty closed and convex subset of a real Hilbert space H, let $A : H \to H$ be a mapping. We consider the classical variational inequality problem (VIP), which is to find a point $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0 \quad \forall x \in C, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in H.

The VIP is one of the central problems in nonlinear analysis (see [2, 7, 21, 14, 16, 19]). Monotone operators have turned out to be an important tool in the study of various problems arising in the domain of optimization, nonlinear analysis, differential equations, and related fields (see [3, 22]). Therefore, numerical methods for VIP with monotone operator have been extensively studied in the literature; see [7, 10] and references therein. In this section we briefly consider the development of projection methods for monotone variational inequality problems that provided weak convergence of the sequence generated to a solution of (1). The simplest iterative procedure is the well-known projected gradient method given by

$$x_0 \in C, x_{n+1} = P_C(x_n - \lambda Ax_n), n \geq 0$$

where $P_C : H \to C$ denotes the metric projection onto the set C and λ is a positive number. In order to converge, however, this method requires the restrictive assumption that A is strongly (or inverse strongly) monotone. The extragradient method proposed by Korpelevich [11] and Antipin [1] is given by $x_0 \in C$,

$$\{y_n = P_C(x_n - \lambda Ax_n), x_{n+1} = P_C(x_n - \lambda Ay_n), n \geq 0 \tag{2}$$

(where $\lambda \in (0, \frac{1}{L})$, and L is the Lipschitz constant of the mapping A) took care of the difficulty. The extragradient method has received a great deal of attention by many authors, who improved on it in various ways; see, e.g., [5, 8, 9, 18] and references therein. The following extension of extragradient method was proposed in [5] and it is given by $x_0 \in C$,

$$\{y_n = P_C(x_n - \lambda Ax_n), T_n = \{w \in H \ : \langle x_n - \lambda Ax_n - y_n, w - y_n \rangle \leq 0, x_{n+1} = P_{T_n}(x_n - \lambda Ay_n), n \geq 0 \tag{3}$$

where $\lambda \in (0, \frac{1}{L})$, and L is the Lipschitz constant of the mapping A. According to Censor *et al.* [5], since the second projection in (3) can be found in a closed form, this method is more applicable when computation of projection onto the closed convex set C is a nontrivial problem.

An alternative to the extragradient method or its modification is the following remarkable scheme proposed by Tseng in [20]:

$$\{x_0 \in Cy_n = P_C(x_n - \lambda Ax_n), x_{n+1} = y_n + \lambda(Ax_n - Ay_n), \tag{4}$$

where $\lambda \in (0, \frac{1}{L})$, L the Lipschitz constant of the mapping A. Algorithms (3) and (4) have the same complexity per iteration because computation of one projection onto the set C and two values of A are needed.

Popov [18] proposed an ingenious method which is similar to the extragradient method but uses on every iteration only one value of the mapping A. Using the idea from [13], Malitsky and Semenov [12] improved the Popov's algorithm and obtained the following scheme:

$$\{x_1, y_0, y_1 \in C, T_n = \{w \in H : \langle x_n - \lambda Ay_{n-1} - y_n, w - y_n \rangle \leq 0\}, x_{n+1} = P_{T_n}(x_n - \lambda Ay_n), y_{n+1} = P_C(x_{n+1} - \lambda Ay_n), n \geq 1 \tag{5}$$

where $\lambda \in (0, \frac{1}{3L})$, L the Lipschitz constant of the mapping A. It is easy to see that this method needs only one projection onto the set C (as in (3) or (4)) and only one value of A per iteration. The latter makes the algorithm (5) very attractive for cases where a computation of operator A is expensive. This often happens, for example, in a huge-scale VIP or a VIP that arises from optimal control.

Very recently, Malitsky proposed the following scheme (see [13]):

$$x_0, x_1 \in C, x_{n+1} = P_C(x_n - \lambda A(2x_n - x_{n-1})), \ n \geq 1 \tag{6}$$

where $\lambda \in (0, \frac{\sqrt{2}-1}{L})$, and L the Lipschitz constant of the mapping A. Algorithm e6 is far better than algorithm e5 in the sense that it does not require the computation of the set $T_n, \ n \in N$ at each point of iteration. Moreover, each stage of computation involves one value of P_C and one value of the operator A. The sequence generated by e6, however, converged weakly to solution of VIP eq1. This is a draw back and thus the need to improve performance of algorithm e6.

The aim of this paper is to introduce and study a generalization of algorithm e6 and examine means of obtaining strong convergence of our scheme to a solution of VIP eq1. This research is motivated by the work of Malisky [13]. Our theorems augment and improve the result of Malisky and that of other authors mentioned above.

1 Preliminaries

We now turn to presentation of important tools and results that will enhance establishment of our main theorems. Let us proceed as follows: Let C be a nonempty closed convex subset of a real Hilbert space H; then the following are equivalent

$$P_C : H \to C \text{ is the projection operator of } H \text{ onto } C. \tag{7}$$

$$\forall\, x \in H, \quad \langle y - P_C x, x - P_C x\rangle \le 0 \;\forall\, y \in C. \tag{8}$$

$$\forall\, x \in H, \quad \|P_C x - y\|^2 \le \|x - y\|^2 - \|x - P_C x\|^2 \;\forall\, y \in C. \tag{9}$$

Proof: We show that eq implies eqq. Suppose that $P_C : H \to C$ is the projection operator of H onto C; then $\forall\, x \in H$,

$$\|x - P_C x\| \le \|x - y\| \;\forall\, y \in C.$$

In particular, using the convexity of C, we have that for $\lambda \in [0, 1]$,

$$\|x - P_C x\| \le \|x - y_\lambda\|,$$

where $y_\lambda = \lambda y + (1 - \lambda)P_C x \in C$. Thus,

$$\forall\, x \in H,\; \|x - P_C x\|^2 \le \|x - y_\lambda\|^2.$$

This implies that,

$$\begin{aligned}
\langle x - P_C x, x - P_C x\rangle &\le \langle x - y_\lambda, x - y_\lambda\rangle \\
&= \langle x - \lambda y - (1 - \lambda)P_C x, x - \lambda y - (1 - \lambda)P_C x\rangle \\
&= \langle x - P_C x - \lambda(y - P_C x), x - P_C x - \lambda(y - P_C x)\rangle \\
&= \langle x - P_C x, x - P_C x\rangle - 2\lambda \langle y - P_C x, x - P_C x\rangle + \lambda^2\|P_C x - y\|^2
\end{aligned}$$

so that

$$\langle y - P_C x, x - P_C x\rangle \le \frac{\lambda}{2}\|P_C x - y\|^2.$$

So, as $\lambda \to 0$, we obtain that

$$\forall\, x \in H,\; \langle y - P_C x, x - P_C x\rangle \le 0 \;\forall\, y \in C.$$

Next, we show that eqq implies equ. Suppose that

$$\forall\, x \in H,\; \langle y - P_C x, x - P_C x\rangle \le 0 \;\forall\, y \in C;$$

then

$$\forall\, x \in H,\; \langle y - x + x - P_C x, x - P_C x\rangle \le 0 \;\forall\, y \in C;$$

This implies that

$$\begin{aligned}
0 &\ge \langle y - x, x - P_C x\rangle + \langle x - P_C x, x - P_C x\rangle \\
&= \langle y - x, x - y + y - P_C x\rangle + \langle x - P_C x, x - P_C x\rangle \\
&= \langle y - x, x - y\rangle + \langle y - x, y - P_C x\rangle + \langle x - P_C x, x - P_C x\rangle \\
&= -\|x - y\|^2 + \|x - P_C x\|^2 + \langle y - x, y - P_C x\rangle \\
&= -\|x - y\|^2 + \|x - P_C x\|^2 + \langle y - P_C x, y - P_C x\rangle + \langle P_C x - x, y - P_C x\rangle
\end{aligned}$$

Thus,

$$\langle y - P_C x, y - P_C x\rangle + \langle P_C x - x, y - P_C x\rangle \le \|x-y\|^2 - \|x - P_C x\|^2.$$

This implies that

$$\|y - P_C x\|^2 - \langle y - P_C x, x - P_C x\rangle \le \|x-y\|^2 - \|x - P_C x\|^2.$$

Therefore,

$$\|y - P_C x\|^2 \le \|y - P_C x\|^2 - \langle y - P_C x, x - P_C x\rangle \le \|x-y\|^2 - \|x - P_C x\|^2,$$

which implies that

$$\|y - P_C x\|^2 \le \|x-y\|^2 - \|x - P_C x\|^2.$$

Finally, we show that equ implies eq. Suppose that

$$\forall\, x \in H,\ \|P_C x - y\|^2 \le \|x-y\|^2 - \|x - P_C x\|^2 \;\forall\, y \in C;$$

then

$$\forall\, x \in H,\ \|x - P_C x\|^2 \le \|x-y\|^2 - \|P_C x - y\|^2 \le \|x-y\|^2 \;\forall\, y \in C.$$

This implies that

$$\forall\, x \in H,\ \|x - P_C x\| \le \|x-y\| \;\forall\, y \in C.$$

The result therefore easily follows. This completes the proof.

Let H be a real Hilbert space and let $u, v \in H$; then

$$\|2u - v\|^2 = 2\|u\|^2 - \|v\|^2 + 2\|u-v\|^2. \tag{10}$$

(Opial [17]) Let $\{x_n\}$ be a sequence in H such that $x_n \rightharpoonup x$; then for all $y \ne x$

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

where $x_n \rightharpoonup x$ as $n \to \infty$ if and only if $\{x_n\}_{n\ge1}$ converges weakly to x.

Let $\{a_n\}$ and $\{b_n\}$ be two real sequences; then

$$\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \le \liminf_{n\to\infty} (a_n + b_n).$$

Let $\{a_n\}_{n\ge1}$ and $\{b_n\}_{n\ge1}$ be two real sequences such that $a_n \ne 0$, for all $n \in N$, and $a_n \to a^*$ as $n \to \infty$, for some $a^* \ne 0$. Suppose $\lim_{n\to\infty} a_n b_n = 0$; then $\lim_{n\to\infty} b_n = 0$.

Let C be a nonempty convex subset of a real Hilbert space H. Assume that $A : C \to H$ is a continuous and monotone mapping; then x^* is a solution of VIP (1) if and only if

$$\langle Ax, x - x^*\rangle \ge 0 \;\forall\, x \in C. \tag{11}$$

Proof: Suppose $x^* \in C$ is a solution of VIP (1), then

$$\left\langle Ax^*, x - x^*\right\rangle \ge 0 \;\forall\, x \in C.$$

Since A is a monotone mapping, we have that

$$\left\langle Ax - Ax^*, x - x^*\right\rangle \ge 0 \;\forall\, x \in C.$$

This implies that

$$\left\langle Ax, x - x^*\right\rangle \ge \left\langle Ax^*, x - x^*\right\rangle \ge 0 \;\forall\, x \in C.$$

Thus,

$$\left\langle Ax, x - x^*\right\rangle \ge 0 \;\forall\, x \in C,$$

so that eqr holds.

Conversely, suppose that $x^* \in C$ satisfies eqr, that is, suppose that

$$\langle Ax, x - x^* \rangle \geq 0 \, \forall \, x \in C,$$

then for any $t \in]0,1[$, we obtain by convexity of C that $x^* + t(x - x^*) \in C$ for all $x \in C$. It thus follows that

$$0 \leq \langle A(x^* + t(x - x^*)), t(x - x^*) \rangle \ \forall \, x \in C.$$

This is equivalent to

$$0 \leq \big\langle Ax^*, t(x - x^*) \big\rangle + \langle A(x^* + t(x - x^*)) - Ax^*, t(x - x^*) \rangle \ \forall \, x \in C. \tag{12}$$

Thus, for all $x \in C$, using the fact that $t > 0$, we obtain from eric that

$$\begin{aligned} 0 &\leq \big\langle Ax^*, x - x^* \big\rangle + \langle A(x^* + t(x - x^*)) - Ax^*, x - x^* \rangle \\ &\leq \big\langle Ax^*, x - x^* \big\rangle + \|A(x^* + t(x - x^*)) - Ax^*\| \|x - x^*\|. \end{aligned}$$

Thus,

$$0 \leq \big\langle Ax^*, x - x^* \big\rangle + \|A(x^* + t(x - x^*)) - Ax^*\| \|x - x^*\|.$$

Since A is continuous; and thus continuous at $x^* \in C$, then as $t \to 0$, we obtain from the last inequality that

$$\big\langle Ax^*, x - x^* \big\rangle \geq 0,$$

that is, x^* is a solution of VIP (1). This completes the proof.
The following gives the relationship between the VIP eq1 and a fixed point problem.

Let H be a real Hilbert space, let C be a closed convex subset of H, let $S \subset H$ be the solution set of the variational inequality problem (1); then $x^* \in S$ if and only if for all $\lambda > 0$, x^* is a fixed point of the operator $P_C(I - \lambda A) : H \to C$, that is, $x^* = P_C(x^* - \lambda Ax^*)$

Proof: Let $x^* \in S$ and $\lambda > 0$, then $\big\langle Ax^*, x - x^* \big\rangle \geq 0$ for all $x \in C$, so that

$$\big\langle -\lambda Ax^*, x - x^* \big\rangle \leq 0 \, \forall \, x \in C$$

Thus,

$$\big\langle x^* - \lambda Ax^*, x - x^* \big\rangle \leq \big\langle x^*, x - x^* \big\rangle \, \forall \, x \in C.$$

This implies that

$$\big\langle x^* - \lambda Ax^* - x^*, x - x^* \big\rangle \leq 0 \, \forall \, x \in C. \tag{13}$$

So, by Lemma 1 eqq, we obtain from 2w that $x^* = P_C(x^* - \lambda Ax^*)$.

Conversely, if $x^* = P_C(x^* - \lambda Ax^*)$, then

$$\big\langle x^*, x - x^* \big\rangle \geq \big\langle x^* - \lambda Ax^*, x - x^* \big\rangle, \ \forall \, x \in C,$$

so that

$$\big\langle Ax^*, x - x^* \big\rangle \geq 0, \ \forall \, x \in C.$$

This completes the proof.

2 Main Result

We now turn to the heart of this work. Let us start with the following Lemma:

Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be sequences of nonnegative terms, let $\{\eta_n\}_{n=1}^{\infty}$ be a sequence in $]0,1[$ and $\beta \in]0,1[$ and let

$$\gamma_n = \frac{1 - 2\eta_n + \sqrt{1 + 4\eta_n^2}}{2} \ \text{ and } \ \delta_\mathrm{n} = \frac{-1 + 2\eta_\mathrm{n} + \sqrt{1 + 4\eta_\mathrm{n}^2}}{2} \ \ \forall \, \mathrm{n} \in \mathrm{N};$$

then the following are equivalent:
1. $a_{n+1} + b_{n+1} \leq (1 - 2\eta_n)a_n + \eta_n a_{n-1} + \beta b_n$
2. $a_{n+1} + \delta_n a_n + b_{n+1} \leq \gamma_n(a_n + \delta_n a_{n-1}) + \beta b_n$.

Proof. Observe that for any $n \in N$,

$$\begin{aligned}
a_{n+1} + \delta_n a_n + b_{n+1} &\leq \gamma_n(a_n + \delta_n a_{n-1}) + \beta b_n \\
\Leftrightarrow a_{n+1} + b_{n+1} &\leq (\gamma_n - \delta_n)a_n + \gamma_n\delta_n a_{n-1} + \beta b_n \\
\Leftrightarrow a_{n+1} + b_{n+1} &\leq \left(\frac{1 - 2\eta_n + \sqrt{1+4\eta_n^2}}{2} - \frac{-1 + 2\eta_n + \sqrt{1+4\eta_n^2}}{2}\right) a_n \\
&\quad + \left(\frac{1 - 2\eta_n + \sqrt{1+4\eta_n^2}}{2}\right)\left(\frac{-1 + 2\eta_n + \sqrt{1+4\eta_n^2}}{2}\right) a_{n-1} \\
&\quad + \beta b_n \\
\Leftrightarrow a_{n+1} + b_{n+1} &\leq (1 - 2\eta_n)a_n + \eta_n a_{n-1} + \beta b_n
\end{aligned}$$

This completes the proof.

2.1 Strong convergence theorem for $m-$strongly monotone mapping

Let C be a nonempty closed convex subset of a real Hilbert space, H. Let $A : H \to H$ be m-strongly monotone and L-Lipschitz continuous mapping. Fix $m_1 \in]0, m]$. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence generated iteratively by

$$\begin{aligned}
x_0, y_0 &\in H, \\
x_{n+1} &= P_C(x_n - \lambda_n A y_n), \\
y_{n+1} &= 2x_{n+1} - x_n, \quad \forall\, n \geq 0,
\end{aligned}$$

where $\{\lambda_n\}_{n=0}^{\infty}$ is a decreasing sequence in $]a, b[$, for some $a, b \in \left]0, \min\left\{\frac{1}{4m_1}, \frac{\sqrt{2}}{4L}\right\}\right[$; then $\{x_n\}_{n=0}^{\infty}$ converges strongly to some $z \in S$, where S is the solution set of (1).

Proof. Observe that since A is m-strongly monotone, the variational inequality problem (1) has a unique solution, say z. So, by Lemma 1 equ, we have that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \|x_n - \lambda_n A y_n - z\|^2 - \|x_n - \lambda_n A y_n - x_{n+1}\|^2 \\
&= \|x_n - z\|^2 + \langle x_n - z, -\lambda_n A y_n \rangle - \langle \lambda_n A y_n, x_n - z \rangle \\
&\quad + \langle \lambda_n A y_n, \lambda_n A y_n \rangle - \|x_n - x_{n+1}\|^2 - \langle x_n - x_{n+1}, -\lambda_n A y_n \rangle \\
&\quad + \langle \lambda_n A y_n, x_n - x_{n+1} \rangle - \langle \lambda_n A y_n, \lambda_n A y_n \rangle \\
&= \|x_n - z\|^2 - \|x_n - x_{n+1}\|^2 - 2\lambda_n \langle A y_n, x_n - z \rangle \\
&\quad + 2\lambda_n \langle A y_n, x_n - x_{n+1} \rangle \\
&= \|x_n - z\|^2 - \|x_n - x_{n+1}\|^2 - 2\lambda_n \langle A y_n, x_{n+1} - z \rangle. \qquad (14)
\end{aligned}$$

Also by Lemma 1, we have that

$$\begin{aligned}
\|y_n - z\|^2 &= 2\|x_n - z\|^2 - \|x_{n-1} - z\|^2 + 2\|x_n - x_{n-1}\|^2 \\
&\geq 2\|x_n - z\|^2 - \|x_{n-1} - z\|^2.
\end{aligned}$$

From this and from strong monotonicity of A, we conclude that with $m_1 \in (0, m]$,

$$\begin{aligned}
& 2\lambda_n \left(\langle A y_n - Az, y_n - z \rangle - m_1(2\|x_n - z\|^2 - \|x_{n-1} - z\|^2)\right) \\
\geq\ & 2\lambda_n(\langle A y_n - Az, y_n - z \rangle - m\|y_n - z\|^2) \geq 0 \qquad (15)
\end{aligned}$$

Then using (15) in (14), we get

$$\|x_{n+1} - z\|^2 \leq \|x_n - z\|^2 - \|x_{n+1} - x_n\|^2 - 2\lambda_n \langle A y_n, x_{n+1} - z \rangle$$

$$\begin{aligned}
&+2\lambda_n \langle Ay_n - Az, y_n - z\rangle - 2\lambda_n m_1(2\|x_n - z\|^2 - \|x_{n-1} - z\|^2) \\
=\ & \|x_n - z\|^2 - \|x_{n+1} - x_n\|^2 + 2\lambda_n \langle Ay_n, y_n - x_{n+1}\rangle \\
&-2\lambda_n \langle Az, y_n - z\rangle - 2\lambda_n m_1(2\|x_n - z\|^2 - \|x_{n-1} - z\|^2) \\
=\ & (1 - 4\lambda_n m_1)\|x_n - z\|^2 - \|x_{n+1} - x_n\|^2 \\
&+2\lambda_n m_1\|x_{n-1} - z\|^2 + 2\lambda_n \langle Ay_n - Ay_{n-1}, y_n - x_{n+1}\rangle \\
&+2\lambda_n \langle Ay_{n-1}, y_n - x_{n+1}\rangle - 2\lambda_n \langle Az, y_n - z\rangle . \qquad (16)
\end{aligned}$$

Now, let us estimate $2\lambda_n\langle Ay_{n-1}, y_n - x_{n+1}\rangle$. Observe that since $x_{n+1}, x_{n-1} \in C$, we obtain by Lemma 1 eqq that

$$\langle x_n - x_{n-1} + \lambda_{n-1}Ay_{n-1}, x_n - x_{n+1}\rangle \leq 0, \qquad (17)$$

$$\langle x_n - x_{n-1} + \lambda_{n-1}Ay_{n-1}, x_n - x_{n-1}\rangle \leq 0. \qquad (18)$$

Adding inequalities chizy and kamdili, we obtain that

$$\langle x_n - x_{n-1} + \lambda_{n-1}Ay_{n-1}, y_n - x_{n+1}\rangle \leq 0. \qquad (19)$$

It immediately follows from muy that

$$\begin{aligned}
2\lambda_{n-1} \langle Ay_{n-1}, y_n - x_{n+1}\rangle &\leq 2\langle x_n - x_{n-1}, x_{n+1} - y_n\rangle \\
&= 2\langle y_n - x_n, x_{n+1} - y_n\rangle \\
&= \|x_{n+1} - x_n\|^2 - \|x_n - y_n\|^2 - \|x_{n+1} - y_n\|^2.
\end{aligned}$$

Observe that

$$2\lambda_n \langle Ay_{n-1}, y_n - x_{n+1}\rangle = \frac{\lambda_n}{\lambda_{n-1}} \cdot 2\lambda_{n-1} \langle Ay_{n-1}, y_n - x_{n+1}\rangle \qquad (20)$$

So, from inequality muy1, we have that

$$\begin{aligned}
2\lambda_n \langle Ay_{n-1}, y_n - x_{n+1}\rangle &\leq \frac{\lambda_n}{\lambda_{n-1}}[\|x_{n+1} - x_n\|^2 \\
&\quad -\|x_n - y_n\|^2 - \|x_{n+1} - y_n\|^2]. \qquad (21)
\end{aligned}$$

Next, we estimate $2\lambda_n \langle Ay_n - Ay_{n-1}, x_{n+1} - y_n\rangle$. Observe that since A is L-Lipschitz continuous mapping,

$$\begin{aligned}
2\lambda_n \langle Ay_n - Ay_{n-1}, y_n - x_{n+1}\rangle &\leq 2\lambda_n\|Ay_n - Ay_{n-1}\|\|x_{n+1} - y_n\| \\
&\leq 2\lambda_n L\|y_n - y_{n-1}\|\|x_{n+1} - y_n\| \\
&\leq \lambda_n L\left(\frac{1}{\sqrt{2}}\|y_n - y_{n-1}\|^2 + \sqrt{2}\|x_{n+1} - y_n\|^2\right) \\
&\leq \lambda_n L\Big[\frac{1}{\sqrt{2}}(\|y_n - x_n\|^2 + 2\|y_n - x_n\|\|x_n - y_{n-1}\| \\
&\quad +\|x_n - y_{n-1}\|^2) + \sqrt{2}\|x_{n+1} - y_n\|^2\Big] \\
&\leq \lambda_n L\Big[\frac{1}{\sqrt{2}}\Big(\|y_n - x_n\|^2 + (1+\sqrt{2})\|y_n - x_n\|^2 \\
&\quad +(\sqrt{2} - 1)\|x_n - y_{n-1}\|^2 + \|x_n - y_{n-1}\|^2\Big) \\
&\quad +\sqrt{2}\|x_{n+1} - y_n\|^2\Big] \\
&= \lambda_n L(1+\sqrt{2})\|y_n - x_n\|^2 + \lambda_n L\|x_n - y_{n-1}\|^2 \\
&\quad +\sqrt{2}\lambda_n L\|x_{n+1} - y_n\|^2 \qquad (22)
\end{aligned}$$

Using (21) and (22), we deduce from (16) that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq (1 - 4\lambda_n m_1)\|x_n - z\|^2 - \|x_{n+1} - x_n\|^2 + 2\lambda_n m_1\|x_{n-1} - z\|^2 \\
&\quad +\lambda_n L(1+\sqrt{2})\|y_n - x_n\|^2 + \lambda_n L\|x_n - y_{n-1}\|^2 + \sqrt{2}\lambda_n L\|x_{n+1} - y_n\|^2 \\
&\quad +\frac{\lambda_n}{\lambda_{n-1}}\Big[\|x_{n+1} - x_n\|^2 - \|x_n - y_n\|^2 - \|x_{n+1} - y_n\|^2\Big] - 2\lambda_n \langle Az, y_n - z\rangle .
\end{aligned}$$

Thus,

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\le (1-4\lambda_n m_1)\|x_n-z\|^2+\left(\frac{\lambda_n}{\lambda_{n-1}}-1\right)\|x_{n+1}-x_n\|^2+2\lambda_n m_1\|x_{n-1}-z\|^2\\
&+\left(\lambda_n L(1+\sqrt{2})-\frac{\lambda_n}{\lambda_{n-1}}\right)\|y_n-x_n\|^2+\left(\sqrt{2}\lambda_n L-\frac{\lambda_n}{\lambda_{n-1}}\right)\|x_{n+1}-y_n\|^2\\
&+\lambda_n L\|x_n-y_{n-1}\|^2-2\lambda_n\langle Az,y_n-z\rangle\\
&\le (1-4\lambda_n m_1)\|x_n-z\|^2+2\lambda_n m_1\|x_{n-1}-z\|^2\\
&+\left(\lambda_n L(1+\sqrt{2})-\frac{\lambda_n}{\lambda_{n-1}}\right)\|y_n-x_n\|^2\\
&+\left(\sqrt{2}\lambda_n L-\frac{\lambda_n}{\lambda_{n-1}}\right)\|x_{n+1}-y_n\|^2+\lambda_n L\|x_n-y_{n-1}\|^2\\
&-2\lambda_n\langle Az,y_n-z\rangle .
\end{aligned}$$

But $\lambda_n L(1+\sqrt{2})-\frac{\lambda_n}{\lambda_{n-1}}\le 0$. Therefore,

$$\begin{aligned}
\|x_{n+1}-z\|^2 &+\left(\frac{\lambda_n-\sqrt{2}\lambda_n\lambda_{n-1}L}{\lambda_{n-1}}\right)\|x_{n+1}-y_n\|^2+4\lambda_n\langle Az,x_n-z\rangle\\
&\le (1-4\lambda_n m_1)\|x_n-z\|^2+2\lambda_n m_1\|x_{n-1}-z\|^2\\
&+\lambda_n L\|x_n-y_{n-1}\|^2-2\lambda_n\langle Az,y_n-z\rangle+4\lambda_n\langle Az,x_n-z\rangle\\
&= (1-4\lambda_n m_1)\|x_n-z\|^2+2\lambda_n m_1\|x_{n-1}-z\|^2\\
&+\lambda_n L\|x_n-y_{n-1}\|^2+2\lambda_n\langle Az,x_{n-1}-z\rangle\\
&\le (1-4\lambda_n m_1)\|x_n-z\|^2+2\lambda_n m_1\|x_{n-1}-z\|^2\\
&+\lambda_{n-1} L\|x_n-y_{n-1}\|^2+2\lambda_{n-1}\langle Az,x_{n-1}-z\rangle\\
&\le (1-4\lambda_n m_1)\|x_n-z\|^2+2\lambda_n m_1\|x_{n-1}-z\|^2\\
&+max\left\{\frac{\lambda_{n-1}\lambda_{n-2}L}{\lambda_{n-1}-\sqrt{2}\lambda_{n-1}\lambda_{n-2}L},\frac{1}{2}\right\}\\
&\times\Bigg[\left(\frac{\lambda_{n-1}-\sqrt{2}\lambda_{n-1}\lambda_{n-2}L}{\lambda_{n-2}}\right)\|x_n-y_{n-1}\|^2\\
&+4\lambda_{n-1}\langle Az,x_{n-1}-z\rangle\Bigg]
\end{aligned}$$

Since $\lambda_n<\frac{\sqrt{2}}{4L}$, it follows from the last inequality that

$$\begin{aligned}
\|x_{n+1}-z\|^2 &+\left(\frac{\lambda_n-\sqrt{2}\lambda_n\lambda_{n-1}L}{\lambda_{n-1}}\right)\|x_{n+1}-y_n\|^2+4\lambda_n\langle Az,x_n-z\rangle\\
&\le (1-4\lambda_n m_1)\|x_n-z\|^2+2\lambda_n m_1\|x_{n-1}-z\|^2\\
&+\frac{\sqrt{2}}{2}\Bigg[\left(\frac{\lambda_{n-1}-\sqrt{2}\lambda_{n-1}\lambda_{n-2}L}{\lambda_{n-2}}\right)\|x_n-y_{n-1}\|^2\\
&+4\lambda_{n-1}\langle Az,x_{n-1}-z\rangle\Bigg]
\end{aligned}\tag{23}$$

Now, set

$$\begin{aligned}
a_n &= \|x_n-z\|^2,\ \ \beta=\frac{\sqrt{2}}{2},\ \ \eta_n=2\lambda_n m_1,\ \ \text{and}\\
b_n &= \left(\frac{\lambda_{n-1}-\sqrt{2}\lambda_{n-1}\lambda_{n-2}L}{\lambda_{n-2}}\right)\|x_n-y_{n-1}\|^2+4\lambda_{n-1}\langle Az,x_{n-1}-z\rangle
\end{aligned}$$

then (23) becomes

$$a_{n+1} + b_{n+1} \leq (1 - 2\eta_n)a_n + \eta_n a_{n-1} + \beta b_n, \quad \forall\, n \in N. \tag{24}$$

Thus, if we set

$$\gamma_n = \frac{1 - 4\lambda_n m_1 + \sqrt{1 + 16\lambda_n^2 m_1^2}}{2} = \frac{1 - 2\eta_n + \sqrt{1 + 4\eta_n^2}}{2}.$$

and

$$\delta_n = \frac{-1 + 4\lambda_n m_1 + \sqrt{1 + 16\lambda_n^2 m_1^2}}{2} = \frac{-1 + 2\eta_n + \sqrt{1 + 4\eta_n^2}}{2},$$

then by Lemma 2, (24) is equivalent to

$$a_{n+1} + \delta_n a_n + b_{n+1} \leq \gamma_n(a_n + \delta_n a_{n-1}) + \beta b_n. \tag{25}$$

Moreover, observe that $\lambda_n \in \left]0, \min\left\{\frac{1}{4m_1}, \frac{\sqrt{2}}{4L}, \right\}\right[$ implies that

$$\lambda_n < \frac{1}{4m_1} \iff \frac{\sqrt{2}}{2} < \frac{1 - 4\lambda_n m_1 + \sqrt{1 + 16\lambda_n^2 m_1^2}}{2},$$

that is, $\gamma_n > \beta$, so that (25) gives

$$\begin{aligned} a_{n+1} + \delta_n a_n + b_{n+1} &\leq \gamma_n(a_n + \delta_n a_{n-1} + b_n) \\ &= \gamma_n(a_n + \delta_{n-1} a_{n-1} + b_n) + \gamma_n(\delta_n - \delta_{n-1})a_{n-1}. \end{aligned} \tag{26}$$

Since $\{\lambda_n\}$ is decreasing, we have that $\gamma_n(\delta_n - \delta_{n-1})a_{n-1} \leq 0$, and since the function f given by $f(x) = \frac{1-2x+\sqrt{1+4x^2}}{2}$ is strictly decreasing on $[0, 1]$ with $f(0) = 1$, we obtain that since $0 < 2am_1 < 2\lambda_n m_1 < 1$,

$$\gamma_n < \frac{1 - 4am_1 + \sqrt{1 + 16a^2 m_1^2}}{2} := \gamma < 1.$$

So we obtain from (26) that

$$\begin{aligned} a_{n+1} + \delta_n a_n + b_{n+1} &\leq \gamma_n(a_n + \delta_{n-1} a_{n-1} + b_n) \\ &\leq \gamma(a_n + \delta_{n-1} a_{n-1} + b_n) \\ &\ \vdots \\ &\leq \gamma^n(a_1 + \delta_0 a_0 + b_1). \end{aligned}$$

This implies that $a_{n+1} \leq \gamma^n M$, where $M = a_1 + \delta_0 a_0 + b_1 > 0$. Hence, $\{x_n\}_{n=1}^{\infty}$ converges strongly to z. This completes the proof.

We note that Theorem 2.1 holds for the class of m-strongly monotone mappings which is a proper subclass of class of monotone mappings. A natural question is: *under what condition will the conclusion of Theorem 2.1 be obtained for the entire class of monotone mappings?* The next Theorem gives an affirmative answer to this question.

Let C be a nonempty closed convex subset of a real Hilbert space, H. Let $A : H \to H$ be a monotone and Lipschitz continuous operator. Let $\{x_n\}$ be any sequence generated iteratively by

$$\begin{aligned} x_0, y_0 &\in H, \\ x_{n+1} &= P_C(x_n - \lambda_n A y_n), \\ y_{n+1} &= 2x_{n+1} - x_n, \quad \forall n \geq 0, \end{aligned} \tag{27}$$

where $\{\lambda_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence such that $0 < a < \lambda_n < b \leq \frac{\sqrt{2}-1}{L}$. Suppose that the solution set, S, of problem (1) is nonempty, then
$\lim\limits_{n\to\infty} \|x_{n+1} - x_n\| = 0$. If in addition, there exists $n_0 \in N$ such that $\forall\ n \geq n_0$, $\|x_{n+1} - x_n\| \leq ck^n$ for some $c > 0$, $k \in (0, 1)$; then $\{x_n\}_{n\geq 1}$ converges strongly to some $x^* \in S$.

Proof. Observe that for fixed $z \in S$, we obtain from eq1** and eqw that

$$\|x_{n+1} - z\|^2 \leq \|x_n - z\|^2 - \|x_n - x_{n+1}\|^2 - 2\lambda_n \big\langle Ay_n, x_{n+1} - z\big\rangle. \tag{28}$$

Since A is monotone, we have that $2\lambda_n \langle Ay_n - Az, y_n - z\rangle \geq 0$. Thus, adding $2\lambda_n \langle Ay_n - Az, y_n - z\rangle$ to the right of (28) yields

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \|x_n - z\|^2 - \|x_n - x_{n+1}\|^2 + 2\lambda_n \langle Ay_n - Az, y_n - z\rangle - 2\lambda_n \langle Ay_n, x_{n+1} - z\rangle \\
&= \|x_n - z\|^2 - \|x_n - x_{n+1}\|^2 + 2\lambda_n \langle Ay_n, y_n - x_{n+1}\rangle - 2\lambda_n \langle Az, y_n - x_{n+1}\rangle \\
&\quad +2\lambda_n \langle Ay_n, x_{n+1} - z\rangle - 2\lambda_n \langle Az, x_{n+1} - z\rangle - 2\lambda_n \langle Ay_n, x_{n+1} - z\rangle \\
&= \|x_n - z\|^2 - \|x_n - x_{n+1}\|^2 + 2\lambda_n \langle Ay_n, y_n - x_{n+1}\rangle \\
&\quad -2\lambda_n \langle Az, y_n - x_{n+1}\rangle - 2\lambda_n \langle Az, x_{n+1} - z\rangle \\
&= \|x_n - z\|^2 - \|x_n - x_{n+1}\|^2 + 2\lambda_n \langle Ay_n, y_n - x_{n+1}\rangle - 2\lambda_n \langle Az, y_n - z\rangle \\
&= \|x_n - z\|^2 - \|x_n - x_{n+1}\|^2 + 2\lambda_n \langle Ay_n - Ay_{n-1}, y_n - x_{n+1}\rangle \\
&\quad +2\lambda_n \langle Ay_{n-1}, y_n - x_{n+1}\rangle - 2\lambda_n \langle Az, y_n - z\rangle.
\end{aligned} \tag{29}$$

Using the estimates of $\langle Ay_{n-1}, y_n - x_{n+1}\rangle$ and $\langle Ay_n - Ay_{n-1}, y_n - x_{n+1}\rangle$ (respectively given by (21) and (22)) in (29), we obtain that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \|x_n - z\|^2 - \|x_{n+1} - x_n\|^2 + \lambda_n L(1 + \sqrt{2})\|y_n - x_n\|^2 \\
&\quad +\lambda_n L\|x_n - y_{n-1}\|^2 + \sqrt{2}\lambda_n L\|x_{n+1} - y_n\|^2 + \frac{\lambda_n}{\lambda_{n-1}}\|x_{n+1} - x_n\|^2 \\
&\quad -\frac{\lambda_n}{\lambda_{n-1}}\|x_n - y_n\|^2 - \frac{\lambda_n}{\lambda_{n-1}}\|x_{n+1} - y_n\|^2 - 2\lambda_n \langle Az, y_n - z\rangle. \\
&\leq \|x_n - z\|^2 - \left(\frac{\lambda_n}{\lambda_{n-1}} - \lambda_n L(1 + \sqrt{2})\right)\|x_n - y_n\|^2 + \lambda_n L\|x_n - y_{n-1}\|^2 \\
&\quad -\left(\frac{\lambda_n}{\lambda_{n-1}} - \sqrt{2}\lambda_n L\right)\|x_{n+1} - y_n\|^2 - 2\lambda_n \langle Az, y_n - z\rangle.
\end{aligned} \tag{30}$$

Since

$$\begin{aligned}
\langle Az, y_n - z\rangle &= 2\langle Az, x_n - z\rangle - \langle Az, x_{n-1} - z\rangle \\
&\geq \langle Az, x_n - z\rangle - \langle Az, x_{n-1} - z\rangle,
\end{aligned} \tag{31}$$

we deduce from (30) that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &+ \lambda_n L\|x_{n+1} - y_n\|^2 + 2\lambda_n \langle Az, x_n - z\rangle \\
&\leq \|x_n - z\|^2 - \left(\frac{\lambda_n}{\lambda_{n-1}} - \lambda_n L(1 + \sqrt{2})\right)\|x_n - y_n\|^2 + \lambda_n L\|x_n - y_{n-1}\|^2 \\
&\quad -\left(\frac{\lambda_n}{\lambda_{n-1}} - \sqrt{2}\lambda_n L\right)\|x_{n+1} - y_n\|^2 - 2\lambda_n \langle Az, y_n - z\rangle + \lambda_n L\|x_{n+1} - y_n\|^2 \\
&\quad +2\lambda_n \langle Az, x_n - z\rangle \\
&= \|x_n - z\|^2 - \left(\frac{\lambda_n}{\lambda_{n-1}} - \lambda_n L(1 + \sqrt{2})\right)\|x_n - y_n\|^2 + \lambda_n L\|x_n - y_{n-1}\|^2 \\
&\quad -\left(\frac{\lambda_n}{\lambda_{n-1}} - \lambda_n L(1 + \sqrt{2})\right)\|x_{n+1} - y_n\|^2 + 2\lambda_n(\langle Az, x_n - z\rangle - \langle Az, y_n - z\rangle) \\
&\leq \|x_n - z\|^2 + \lambda_n L\|x_n - y_{n-1}\|^2 + 2\lambda_n \langle Az, x_{n-1} - z\rangle \\
&\quad -\left(\frac{\lambda_n}{\lambda_{n-1}} - \lambda_n L(1 + \sqrt{2})\right)(\|x_n - y_n\|^2 + \|x_{n+1} - y_n\|^2).
\end{aligned}$$

But $\|x_n - y_n\| = \|x_n - x_{n-1}\|$. Therefore,

$$\begin{aligned}
\frac{\|x_n - x_{n-1}\|^2 + \|x_{n+1} - y_n\|^2}{\left(\frac{\lambda_n}{\lambda_{n-1}} - \lambda_n L(1 + \sqrt{2})\right)^{-1}} &\leq \|x_n - z\|^2 + \lambda_n L\|x_n - y_{n-1}\|^2 + 2\lambda_n \langle Az, x_{n-1} - z\rangle \\
&\quad -(\|x_{n+1} - z\|^2 + \lambda_n L\|x_{n+1} - y_n\|^2 + 2\lambda_n \langle Az, x_n - z\rangle).
\end{aligned}$$

So that

$$\begin{aligned} \frac{\|x_n - x_{n-1}\|^2 + \|x_{n+1} - y_n\|^2}{\left(\frac{\lambda_n}{\lambda_{n-1}} - \lambda_n L(1+\sqrt{2})\right)^{-1}} &\leq \|x_n - z\|^2 + \lambda_n L\|x_n - y_{n-1}\|^2 + 2\lambda_n \langle Az, x_{n-1} - z\rangle \\ &-\|x_{n+1} - z\|^2 - \lambda_{n+1} L\|x_{n+1} - y_n\|^2 - 2\lambda_{n+1}\langle Az, x_n - z\rangle \\ &+\|x_{n+1} - z\|^2 + \lambda_{n+1} L\|x_{n+1} - y_n\|^2 + 2\lambda_{n+1}\langle Az, x_n - z\rangle \\ &-\|x_{n+1} - z\|^2 - \lambda_n L\|x_{n+1} - y_n\|^2 - 2\lambda_n\langle Az, x_n - z\rangle . \end{aligned}$$

Since $\{\lambda_n\}_{n\geq 1}$ is monotone decreasing, we have that

$$\begin{aligned} \frac{\|x_n - x_{n-1}\|^2 + \|x_{n+1} - y_n\|^2}{\left(\frac{\lambda_n}{\lambda_{n-1}} - \lambda_n L(1+\sqrt{2})\right)^{-1}} &\leq \|x_n - z\|^2 + \lambda_n L\|x_n - y_{n-1}\|^2 + 2\lambda_n \langle Az, x_{n-1} - z\rangle \\ &-\|x_{n+1} - z\|^2 - \lambda_{n+1} L\|x_{n+1} - y_n\|^2 \\ &-2\lambda_{n+1}\langle Az, x_n - z\rangle . \end{aligned} \tag{32}$$

It therefore follows from (32) that for any $p \in N$,

$$\begin{aligned} \sum_{n=1}^{p}\left(\frac{\|x_n - x_{n-1}\|^2 + \|x_{n+1} - y_n\|^2}{\left(\frac{\lambda_n}{\lambda_{n-1}} - \lambda_n L(1+\sqrt{2})\right)^{-1}}\right) &\leq \|x_1 - z\|^2 + \lambda_1 L\|x_1 - y_0\|^2 + 2\lambda_1\langle Az, x_0 - z\rangle \\ &-(\|x_{p+1} - z\|^2 + \lambda_{p+1} L\|x_{p+1} - y_p\|^2 \\ &+2\lambda_{p+1}\langle Az, x_p - z\rangle). \end{aligned}$$

This implies that

$$\begin{aligned} \sum_{n=1}^{p}\left(\frac{\|x_n - x_{n-1}\|^2 + \|x_{n+1} - y_n\|^2}{\left(\frac{\lambda_n}{\lambda_{n-1}} - \lambda_n L(1+\sqrt{2})\right)^{-1}}\right) &\leq \|x_1 - z\|^2 + \lambda_1 L\|x_1 - y_0\|^2 \\ &+2\lambda_1\langle Az, x_0 - z\rangle . \end{aligned}$$

so that as $p \to \infty$, we have that

$$\begin{aligned} \sum_{n=1}^{\infty}\left(\frac{\|x_n - x_{n-1}\|^2 + \|x_{n+1} - y_n\|^2}{\left(\frac{\lambda_n}{\lambda_{n-1}} - \lambda_n L(1+\sqrt{2})\right)^{-1}}\right) &\leq \Big[\|x_1 - z\|^2 + \lambda_1 L\|x_1 - y_0\|^2 \\ &+2\lambda_1\langle Az, x_0 - z\rangle\Big] < +\infty. \end{aligned}$$

This implies that $(\|x_n-x_{n-1}\|^2+\|x_{n+1}-y_n\|^2)\left(\frac{\lambda_n}{\lambda_{n-1}} - \lambda_n L(1+\sqrt{2})\right) \to 0$ as $n \to \infty$. But $\lim_{n\to\infty}\left(\frac{\lambda_n}{\lambda_{n-1}} - \lambda_n L(1+\sqrt{2})\right)$ exists and it is not equal to zero. Thus, by Lemma 1, we have that $\lim_{n\to\infty}(\|x_n - x_{n-1}\|^2 + \|x_{n+1} - y_n\|^2) = 0$. This implies that $\lim_{n\to\infty}\|x_n - x_{n-1}\| = 0 \iff \lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty}\|x_{n+1} - y_n\| = 0$.

We note here that the sequence $\{x_n\}_{n\geq 1}$ is bounded. This follows from inequality eqt*, which gives

$$\begin{aligned} b_{n+1} &:= \|x_{n+1} - z\|^2 + \lambda_{n+1} L\|x_{n+1} - y_n\|^2 + 2\lambda_{n+1}\big\langle Az, x_{n+1} - z\big\rangle \\ &\leq \|x_n - z\|^2 + \lambda_{n+1} L\|x_n - y_{n-1}\|^2 + 2\lambda_n\big\langle Az, x_n - z\big\rangle =: b_n \ \forall\, n \in N. \end{aligned} \tag{33}$$

Thus, the sequence $\{b_n\}_{n\geq 1}$ is monotone decreasing sequence of non-negative real numbers which is bounded above by b_1. It is easy to see that

$$\|x_n - z\|^2 \leq b_n \leq b_1 \ \forall\, n \in N. \tag{34}$$

Hence, the sequnce $\{\|x_n - z\|\}_{n\geq 1}$ is bounded. Boundednes of $\{x_n\}_{n\geq 1}$ thus follows.

Now, suppose there exists $n_0 \in N$ such that for any $n \geq n_0$, $\|x_{n+1} - x_n\| \leq ck^n$ for some $c > 0$ and $k \in (0, 1)$. Let $m, n \in N$. We may assume without loss of generality that $m \geq n$, then

$$\begin{aligned} \|x_n - x_m\| &= \|x_n - x_{n+1} + x_{n+1} - x_{n+2} + \cdots + x_{m-2} - x_{m-1} + x_{m-1} - x_m\| \\ &\leq \sum_{j=n}^{m-1} \|x_j - x_{j+1}\| \\ &\leq c\sum_{j=n}^{m-1} k^j \to 0 \quad \text{as m, n} \to \infty. \end{aligned}$$

So that, $\|x_n - x_m\| \to 0$ as $m, n \to \infty$. Hence, $\{x_n\}_{n\geq 1}$ is Cauchy, and since H is Hilbert, we obtain that $x_n \to x^*$ for some $x^* \in H$; and since $\|x_{n+1} - y_n\| \to 0$ as $n \to \infty$, we obtain that $y_n \to x^*$ as $n \to \infty$.
Next, we show that $x^* \in S$. Let $y \in C$, then using Lemma 1 and monotonicity of A, we have that

$$\begin{aligned} 0 &\leq \langle x_{n+1} - x_n + \lambda_n Ay_n, y - x_{n+1}\rangle \\ &= \langle x_{n+1} - x_n, y - x_{n+1}\rangle + \lambda_n \langle Ay_n, y - x_{n+1}\rangle \\ &= \langle x_{n+1} - x_n, y - x_{n+1}\rangle + \lambda_n \langle Ay_n, y - y_n\rangle + \lambda_n \langle Ay_n, y_n - x_{n+1}\rangle \\ &\leq \langle x_{n+1} - x_n, y - x_{n+1}\rangle + \lambda_n \langle Ay, y - y_n\rangle + \lambda_n \langle Ay_n, y_n - x_{n+1}\rangle \\ &= \langle x_{n+1} - x_n, y - x_{n+1}\rangle + \lambda_n \langle Ay, y - x^*\rangle \\ &\quad + \lambda_n \langle Ay, x^* - y_n\rangle + \lambda_n \langle Ay_n, y_n - x_{n+1}\rangle \\ &\leq \|x_{n+1} - x_n\|\|y - x_{n+1}\| + \lambda_n \langle Ay, y - x^*\rangle \\ &\quad + \lambda_n\|Ay\|\|x^* - y_n\| + \lambda_n\|Ay_n\|\|y_n - x_{n+1}\|. \end{aligned} \tag{35}$$

Thus, using the fact that A is Lipschitz and the fact that $\{x_n\}_{n\geq 1}$ is bounded, we obtain from obinna and for some $M > 0$that

$$\begin{aligned} 0 &\leq \|x_{n+1} - x_n\|\|y - x_{n+1}\| + \lambda_n \langle Ay, y - x^*\rangle \\ &\quad + \lambda_n\|Ay\|\|x^* - y_n\| + \lambda_n\|Ay_n\|\|y_n - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\|\|y - x_{n+1}\| + \lambda_n \langle Ay, y - x^*\rangle + \lambda_n\|Ay\|\|x^* - y_n\| \\ &\quad + \lambda_n(L\|y_n - x^*\| + \|Ax^*\|)\|y_n - x_{n+1}\| \\ &\leq M\|x_{n+1} - x_n\| + \lambda_n \langle Ay, y - x^*\rangle + \lambda_n\|Ay\|\|x^* - y_n\| \\ &\quad + \lambda_n(L\|y_n - x^*\| + \|Ax^*\|)\|y_n - x_{n+1}\| \\ &\leq M\|x_{n+1} - x_n\| + \lambda_n \langle Ay, y - x^*\rangle + b\|Ay\|\|x^* - y_n\| \\ &\quad + b(L\|y_n - x^*\| + \|Ax^*\|)\|y_n - x_{n+1}\| \end{aligned} \tag{36}$$

But $\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x^* - y_n\| = \lim_{n\to\infty} \|y_n - x_{n+1}\| = 0$ and since $\{\lambda_n\}_{n\geq 1}$ is a bounded monotone sequence of real numbers, $\lim_{n\to\infty} \lambda_n$ exists; and since $0 < a < \lambda_n < b$, we have that $\lim_{n\to\infty} \lambda_n > 0$. So, taking limit as $n \to \infty$ on both sides of (36) we obtain that

$$0 \leq \lim_{n\to\infty} \lambda_n \langle Ay, y - x^*\rangle, \ \forall y \ \in C$$

This implies that,

$$\langle Ay, y - x^*\rangle \geq 0 \ \forall y \ \in C. \tag{37}$$

Using Lemma 1 and (37), we obtain that $x^* \in S$. Hence, $\{x_n\}_{n=1}^{\infty}$ converges strongly to a solution of problem (1). This completes the proof.

3 Weak Convergence Result

Let C be a nonempty closed convex subset of a real Hilbert space, H. Let $A : H \to H$ be a monotone and Lipschitz continuous operator. Let $\{x_n\}$ be any sequence generated iteratively by

$$x_0, y_0 \ \in \ C,$$

$$\begin{aligned} x_{n+1} &= P_C(x_n - \lambda_n A y_n), \\ y_{n+1} &= 2x_{n+1} - x_n, \quad \forall n \geq 0, \end{aligned}$$

where $\{\lambda_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence such that $0 < a < \lambda_n < b \leq \frac{\sqrt{2}-1}{L}$. Suppose that the solution set, S, of problem (1) is nonempty; then the sequence $\{x_n\}$ converges weakly to some $x^* \in S$.

Proof. From Theorem (2.1) we got that $\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0$ and that the sequence $\{x_n\}_{n\geq 1}$ is bounded. We also got that $\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ of $\{x_n\}$ which converges weakly to some $x^* \in H$. Since $\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0$, it is easy to see that $\{y_{n_i}\}_{i=1}^{\infty}$ also converges weakly to x^*. We now show that $x^* \in S$. From Lemma (1) it follows that

$$\langle x_{n_i+1} - x_{n_i} + \lambda_{n_i} A y_{n_i}, y - x_{n_i+1} \rangle \geq 0 \quad \forall\, y \in C \tag{38}$$

Thus, using the fact that A is monotone, we obtain that for all $y \in C$,

$$\begin{aligned} 0 &\leq \langle x_{n_i+1} - x_{n_i}, y - x_{n_i+1} \rangle + \lambda_{n_i} \langle A y_{n_i}, y - y_{n_i} \rangle + \lambda_{n_i} \langle A y_{n_i}, y_{n_i} - x_{n_i+1} \rangle \\ &\leq \langle x_{n_i+1} - x_{n_i}, y - x_{n_i+1} \rangle + \lambda_{n_i} \langle A y, y - y_{n_i} \rangle + \lambda_{n_i} \langle A y_{n_i}, y_{n_i} - x_{n_i+1} \rangle \\ &\leq \|x_{n_i+1} - x_{n_i}\|(\|y\| + M) + \lambda_{n_i} \langle A y, y - y_{n_i} \rangle + M \|y_{n_i} - x_{n_i+1}\| \\ &= \|x_{n_i+1} - x_{n_i}\|(\|y\| + M) + \lambda_{n_i} \big\langle A y, y - x^* \big\rangle \\ &\quad + \lambda_{n_i} \big\langle A y, x^* - y_{n_i} \big\rangle + M \|y_{n_i} - x_{n_i+1}\|, \end{aligned} \tag{39}$$

for some $M > 0$. Taking limit as $i \to \infty$ in (39) and using the fact that $\lim_{i\to\infty} \|x_{n_i+1} - x_{n_i}\| = \lim_{i\to\infty} \|y_{n_i+1} - y_{n_i}\| = 0$, $\lim_{i\to\infty} \lambda_{n_i} > 0$ and $\{y_{n_i}\}_{i=1}^{\infty}$ converges weakly to x^* we obtain that

$$0 \leq \langle A y, y - x^* \rangle \quad \forall\, y \in C,$$

which implies that $x^* \in S$.

We next show that $\{x_n\}$ converges weakly to x^*. From (32), it is clear that the sequence

$$\{\|x_n - z\|^2 + 2\lambda_n L(2 + \sqrt{2})\|x_n - y_{n-1}\|^2 + 2\lambda_n \langle A z, x_{n-1} - z \rangle\}$$

is bounded and monotone, so that it is convergent. But $\{\|x_n - y_{n-1}\|^2\}$ is also convergent, therefore,

$$\{\|x_n - z\|^2 + 2\lambda_n \langle A z, x_{n-1} - z \rangle\}$$

is convergent. Assume for contradiction that $\{x_n\}$ does not converge weakly to x^*. Let $\bar{x}$ be a weak cluster point of $\{x_n\}_{n\geq 0}$ such that $\bar{x} \neq x^*$. Let $\{x_{n_k}\}_{k\geq 1}$ be a subsequence of $\{x_n\}_{n\geq 0}$ such that $x_{n_k} \rightharpoonup \bar{x}$ as $k \to \infty$. Then by Lemma 1 and Lemma 1 we have that

$$\begin{aligned} \lim_{n\to\infty} (\|x_n - \bar{x}\|^2 + 2\lambda_n \langle A\bar{x}, x_n - \bar{x} \rangle) &= \lim_{k\to\infty} (\|x_{n_k} - \bar{x}\|^2 + 2\lambda_n \langle A\bar{x}, x_{n_k} - \bar{x} \rangle) \\ &= \lim_{k\to\infty} \|x_{n_k} - \bar{x}\|^2 = \liminf_{k\to\infty} \|x_{n_k} - \bar{x}\|^2 \\ &< \liminf_{k\to\infty} \|x_{n_k} - x^*\|^2 \\ &\leq \liminf_{k\to\infty} \|x_{n_k} - x^*\|^2 + 2 \liminf_{k\to\infty} \lambda_{n_k} \langle A x^*, x_{n_k} - x^* \rangle \\ &\leq \liminf_{k\to\infty} (\|x_{n_k} - x^*\|^2 + 2\lambda_n \langle A x^*, x_{n_k} - x^* \rangle) \\ &= \lim_{n\to\infty} (\|x_n - x^*\|^2 + 2\lambda_n \langle A x^*, x_n - x^* \rangle). \end{aligned}$$

Similarly, we can deduce that

$$\lim_{n\to\infty} (\|x_n - x^*\|^2 + 2\lambda_n \langle A x^*, x_n - x^* \rangle) < \lim_{n\to\infty} (\|x_n - \bar{x}\|^2 + 2\lambda_n \langle A\bar{x}, x_n - \bar{x} \rangle).$$

But this is impossible. Hence, $\{x_n\}_{n\geq 1}$ converges weakly to some $x^* \in S$. This completes the proof.

The following consequence of Theorem 3 coincides with the weak convergence theorem of Malitsky [13]:

Let C be a nonempty closed convex subset of a real Hilbert space, H. Let $A : H \to H$ be a monotone and Lipschitz continuous operator. Let $\{x_n\}$ be any sequence generated iteratively by

$$\begin{aligned} x_0, y_0 &\in C, \\ x_{n+1} &= P_C(x_n - \lambda A y_n), \\ y_{n+1} &= 2x_{n+1} - x_n, \quad \forall n \geq 0, \end{aligned}$$

where $\lambda \in]0, \frac{\sqrt{2}-1}{L}[$. Suppose that the solution set, S, of problem (1) is nonempty; then the sequence $\{x_n\}$ converges weakly to some $x^* \in S$.

4 Applications

4.1 Approximation of zeros and fixed points of nonlinear mappings

Let $A : C \to H$ be a monotone L-Lipschitz mapping. Recall from Theorem 1 that $x^* \in S$ if and only if $x^* = P_C(I - \lambda A)x^*$, $\lambda > 0$. So that if $C = H$, then the projection mapping becomes the identity map, and in this case; $x^* \in S$ if and only if $x^* = x^* - \lambda A x^*$ if and only if $Ax^* = 0$. Going by this, we assert that an iterative scheme given by

$$\begin{aligned} x_0, y_0 &\in H \\ x_{n+1} &= x_n - \lambda_n A y_n, \\ y_{n+1} &= 2x_{n+1} - x_n, \quad \forall\, n \geq 0. \end{aligned} \tag{40}$$

approximates a zero of the monotone operator A. Thus, we have the following Theorems. Let H be a real Hilbert space, let $A : H \to H$ be m-strongly monotone and Lipschitz continuous mapping such that $Z(A) = \{x^* \in H : Ax^* = 0\} \neq \emptyset$. Let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated iteratively by eqcor, where $\{\lambda_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence in $]a, b[$ for some $a, b \in \Big]0, \min\big\{\frac{1}{4m_1}, \frac{\sqrt{2}}{4L}\big\}\Big[$; then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $x^* \in Z(A)$.

Proof. This follows as in the proof of Theorem 2.1 with $P_C \equiv P_H = I$, the identity operator on H.

Let H be a real Hilbert space, let $A : H \to H$ be a monotone and Lipschitz continuous operator. Let $\{x_n\}$ be any sequence generated iteratively by eqcor, where $\{\lambda_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence in $]a, b[$ for some $a, b \in \Big]0, \frac{\sqrt{2}-1}{L}\Big[$. Suppose that $Z(A) = \{x^* \in H : Ax^* = 0\} \neq \emptyset$, then $\lim\limits_{n\to\infty} \|x_{n+1} - x_n\| = 0$. If in addition, there exists $n_0 \in N$ such that $\forall\, n \geq n_0$, $\|x_{n+1} - x_n\| \leq ck^n$ for some $c > 0$, $k \in (0, 1)$; then $\{x_n\}_{n\geq 1}$ converges strongly to some $x^* \in Z(A)$.

Proof. This follows as in the proof of Theorem 2.1 with $P_C \equiv P_H = I$, the identity operator on H.

Let C be a nonempty closed convex subset of a real Hilbert space, H. Let $A : H \to H$ be a monotone and Lipschitz continuous operator. Let $\{x_n\}$ be any sequence generated iteratively by

$$\begin{aligned} x_0, y_0 &\in H, \\ x_{n+1} &= x_n - \lambda_n A y_n, \\ y_{n+1} &= 2x_{n+1} - x_n, \quad \forall n \geq 0, \end{aligned}$$

where $\{\lambda_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence such that $0 < a < \lambda_n < b \leq \frac{\sqrt{2}-1}{L}$. Suppose that $Z(A) = \{x^* \in H : Ax^* = 0\} \neq \emptyset$; then the sequence $\{x_n\}$ converges weakly to some $x^* \in Z(A)$.

Proof: Follows as in the proof of Theorem 3 with $P_C = P_H = I$, the identity mapping of H.

Recall that a mapping A is monotone if and only if $T = I - A$ is *pseudocontractive*. With this connection, it is easy to see that fixed point theory of pseudocontrcative mappings coincides with theory of zeros of monotone mappings. The following theorem is thus an immediate consequence of Theorem 4.1.

Let H be a real Hilbert space. Let $T : H \to H$ be a Lipschitz continuous pseudocontractive mapping such that $Fix(T) = \{x \in H : Tx = x\} \neq \emptyset$. Let the sequence $\{x_n\}_{n\geq 1}$ be generated iteratively by

$$\begin{aligned} x_0, y_0 &\in H \\ x_{n+1} &= x_n - \lambda_n(I-T)y_n, \\ y_{n+1} &= 2x_{n+1} - x_n, \quad \forall n \geq 0, \end{aligned}$$

where $\{\lambda_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence in $]a, b[$ for some $a, b \in \left]0, \frac{\sqrt{2}-1}{L}\right[$. Then $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. If in addition, there exists $n_0 \in N$ such that $\forall\, n \geq n_0$, $\|x_{n+1} - x_n\| \leq ck^n$ for some $c > 0$, $k \in (0, 1)$; then $\{x_n\}_{n\geq 1}$ converges strongly to some $x^* \in F(T)$.
Proof: Follows from Theorem 4.1 with $A = I - T$.

Let H be a real Hilbert space. Let $T : H \to H$ be a Lipschitz continuous pseudocontractive mapping such that $Fix(T) = \{x \in H : Tx = x\} \neq \emptyset$. Let the sequence $\{x_n\}_{n\geq 1}$ be generated iteratively by

$$\begin{aligned} x_0, y_0 &\in H \\ x_{n+1} &= x_n - \lambda_n(I-T)y_n, \\ y_{n+1} &= 2x_{n+1} - x_n, \quad \forall n \geq 0, \end{aligned}$$

where $\{\lambda_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence in $]a, b[$ for some $a, b \in \left]0, \frac{\sqrt{2}-1}{L}\right[$. Suppose that $Fix(T) = \{x \in H : Tx = x\} \neq \emptyset$; then the sequence $\{x_n\}$ converges weakly to some $x^* \in Fix(T)$.

Proof: This is an immediate consequence of Theorem 4.1 with $A = I - T$.

4.2 Approximation of solution of classical equilibrium problem

Let C be a closed convex nonempty subset of a real Hilbert space H and let $F : C \times C \to R$ be a function. The classical *equilibrium problem* (EP) for a bifunction F is to find $u^* \in C$ such that

$$F(u^*, y) \geq 0 \;\forall\, y \in C. \tag{41}$$

The set of solutions for EP er is denoted by

$$EP(F) = \{u \in C : F(u, y) \geq 0 \;\forall\, y \in C\}.$$

The classical equilibrium problem (EP) includes (see [4]) as special cases the monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, vector equilibrium problems, Nash equilibria in noncooperative games. Furthermore, there are several other problems, for example, the complementarity problems and fixed point problems, which can also be written in the form of the classical equilibrium problem. In other words, the classical equilibrium problem is a unifying model for several problems arising from engineering, physics, statistics, computer science, optimization theory, operations research, economics and many other fields. For the past 20 years or so, many existence results have been published for various equilibrium problems (see e.g., [4]).

For solving the equilibrium problem (EP) for a bifunction $F : E \times E \to R$, we assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in E$,

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in E$,

(A3) for each $x, y \in E$, $\lim_{t\to 0} F(tz + (1-t)x, y) \leq F(x, y)$,

(A4) for each $x \in E$, $y \mapsto F(x, y)$ is convex and lower semi-continuous.

(Blum and Oettli [4]) The resolvent of a bifunction $F : C \times C \to R$ is the set-valued operator $J_F : H \to 2^C$ defined by

$$J_F(x) := \{z \in C : F(z, y) + \langle z - x, y - z\rangle \geq 0, \ \forall\, y \in C\}.$$

(Combettes and Hirstoaga, [6]) Suppose that $F : C \times C \to R$ satisfies Conditions $A_1 - A_4$. Let J_F be the resolvent of the bifunction F as given in Definition 4.2; then

1. J_F is single valued.
2. J_F is firmly nonexpansive, that is, for any $x, y \in H$,

$$\|J_F x - J_F y\|^2 \leq \big\langle x - y, J_F x - J_F y\big\rangle.$$

3. $Fix(J_F) = EP(F)$, that is, the fixed point set of J_F is equal to the solution set of the equilibrium problem.
4. $EP(F)$ is closed and convex.

It is well known that every firmly nonexpansive mapping is pseudocontractive. Thus, we obtain in particular that J_F is pseudocontractive; and as a consequence of Theorem 4.1, we obtain the following corollary:

Let C be a nonempty closed convex subset of a real Hilbert space H, let $F : C \times C \to R$ satisfy Condition (4.2), let J_F and $EP(F)$ be as in Lemma 4.2. Let the sequence$\{x_n\}_{n\geq 1}$ be generated iteratively by

$$\begin{aligned} x_0, y_0 &\in H \\ x_{n+1} &= x_n - \lambda_n (I - J_F) y_n, \\ y_{n+1} &= 2x_{n+1} - x_n, \quad \forall n \geq 0, \end{aligned}$$

where $\{\lambda_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence in $]a, b[$ for some $a, b \in \Big]0, \sqrt{2} - 1\Big[$. Suppose that $EP(F) \neq \emptyset$, then $\lim\limits_{n\to\infty} \|x_{n+1} - x_n\| = 0$. If in addition, there exists $n_0 \in N$ such that $\forall\, n \geq n_0$, $\|x_{n+1} - x_n\| \leq ck^n$ for some $c > 0$, $k \in (0, 1)$; then $\{x_n\}_{n\geq 1}$ converges strongly to some $x^* \in Fix(J_F) = EP(F)$.
The following corollary easily follows from Theorem 4.1

Let C be a nonempty closed convex subset of a real Hilbert space H, let $F : C \times C \to R$ satisfy Condition (4.2), let J_F and $EP(F)$ be as in Lemma 4.2. Let the sequence$\{x_n\}_{n\geq 1}$ be generated iteratively by

$$\begin{aligned} x_0, y_0 &\in H \\ x_{n+1} &= x_n - \lambda_n (I - J_F) y_n, \\ y_{n+1} &= 2x_{n+1} - x_n, \quad \forall n \geq 0, \end{aligned}$$

where $\{\lambda_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence in $]a, b[$ for some $a, b \in \Big]0, \sqrt{2} - 1\Big[$. Suppose that $EP(F) \neq \emptyset$; then the sequence $\{x_n\}_{n\geq 1}$ converges weakly to some $x^* \in Fix(J_F) = EP(F)$.

5 Numerical Example

Let $A : R^2 \to R^2$ be defined for $(x, y) \in R^2$ by

$$A(x, y) = (2x + 1 - y, x + 2y).$$

It could be easily shown that he mapping A is Lipschtz and strongly monotone. To see this, let $x = (x_1, x_2)$, $y = (y_1, y_2) \in R^2$. Then

$$\begin{aligned} \|Ax - Ay\|^2 &= [2(x_1 - y_1) - (x_2 - y_2)]^2 \\ &\quad + [(x_1 - y_1) - 2(x_2 - y_2)]^2 \\ &= 5[(x_1 - y_1)^2 + (x_2 - y_2)^2] \\ \|Ax - Ay\| &= \sqrt{5}\|x - y\|, \end{aligned}$$

showing that A is Lipschitz. Moreover,

$$\begin{aligned}\langle x-y, Ax-Ay\rangle &= \langle (x_1-y_1, x_2-y_2), (2(x_1-y_1)-(x_2-y_2), (x_1-y_1)+2(x_2-y_2))\rangle \\ &= 2[(x_1-y_1)^2+(x_2-y_2)^2] \\ &= 2\|x-y\|^2,\end{aligned}$$

showing that A is $m-$strongly monotone with $m=2$. Observe that $(-0.4, 0.2)$ is a zero of the operator A. Now, fix $m_1 = 1 \in]0, m[$ and let $\lambda_n = \frac{1}{2n}+\frac{1}{4\sqrt{10}}$. Observe that $\{\lambda_n\}_{n\geq 1}$ is a decreasing sequence $0 < a < \lambda_n < \min\left\{\frac{1}{4m1}, \frac{\sqrt{2}}{4L}\right\} = \min\left\{\frac{1}{4}, \frac{1}{2\sqrt{10}}\right\} = \frac{1}{2\sqrt{10}}$ for all $n \geq 7$, where $a = \frac{1}{4\sqrt{10}}$.

From $x_0 = (1,2)$ and $y_0 = (-1,3) \in R^2$, let $\{x_n\}_{n\geq 0}$ be iteratively generated by

$$x_{n+1} = x_n - \lambda_n A y_n,\ y_{n+1} = 2x_{n+1} - x_n, \tag{42}$$

then with $x^* = (-0.4, 0.2) \in A^{-1}(0)$, the following graph shows the behaviour of $\|x_n - x^*\|$ and $\|y_n - x^*\|$

The above figure is drawn with the aid of MATLAB R2008b. Values of $n \in N$ are plotted on the horizontal axis, while the values of $\|x_n - x^*\|$ and $\|y_n - x^*\|$ are plotted on the vertical axis. The blue curve represents the graph of $\|x_n - x^*\|$ while the green curve denotes the graph of $\|y_n - x^*\|$.

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