Certain Integral Representations for Hypergeometric Functions of Four Variables

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Abstract

In the present work, we first introduce five new quadruple hypergeometric series and then we give integral representations of Euler type and Laplace type for these new hypergeometric series, which we denote by $X_{21}^{(4)}, X_{22}^{(4)}, X_{23}^{(4)}, X_{24}^{(4)}, X_{25}^{(4)}$.

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1. Introduction

Many applications of Gauss hypergeometric functions to partitions, number theory, combinatorics and mathematical physics, etc. have inspired a large number of authors to investigate hypergeometric functions of two or more variables (for example [1-9]). The developments in many areas of mathematics such as representation theory, geometry, algebraic geometry, combinatorics, number theory, mirror symmetry, etc. have led to increasing interest in the study of hypergeometric functions of several variables. Moreover, hypergeometric functions are seen in several applications of physical and chemical problems ([10-12]) for instance, in the solutions of degenerate second-order partial differential equations in many problems in gas dynamics, in the problem of adiabatic flat-parallel gas flow without whirlwind and in many other problems [13].

Appell defined four hypergeometric functions of two variables denoted by F_1, F_2, F_3, F_4 (see [14]). Horn [14] introduced ten hypergeometric functions of two variables namely $G_1, G_2, G_3, H_1, ..., H_7$. Then, Lauricella gave fourteen complete triple hypergeometric functions denoted by $F_1, F_2, ..., F_{14}$ (see [14, section 1.4 and 1.5]). F_1, F_2, F_5 and F_9 of these hypergeometric functions correspond to the Lauricella hypergeometric functions of three variables $F_A^{(3)}, F_B^{(3)}, F_C^{(3)}$ and $F_D^{(3)}$, respectively. Saran [8] presented systematic study of the triple hypergeometric functions of Lauricella given by the symbols $F_E, F_F, ..., F_T$. Srivastava's triple hypergeometric functions are given by H_A, H_B and H_C which aren't included among Lauricella's hypergeometric functions [14, section 1.5]. Exton [3] introduced twenty distinct triple hypergeometric functions namely $X_i\ (i = 1, ..., 20)$. By the motivation of double and triple hypergeometric functions, Exton [4] defined twenty one complete hypergeometric functions of four variables by symbols $K_1, K_2, ..., K_{21}$. In [9], Sharma and Parihar introduced eighty three complete hypergeometric $F_1^{(4)}, F_2^{(4)}, ..., F_{83}^{(4)}$ of four variables.

Motivated essentially by the works by Exton ([3], [4, Chapter 3]) and Sharma and Parihar [9], we introduced in [15] thirty new quadruple hypergeometric series $X_i^{(4)}(i = 1, ..., 30)$, five of them defined below

$$X_{21}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_2,c_3;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+q}(a_2)_{q+n+p}(a_3)_p}{(c_1)_{m+n}(c_2)_p(c_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{1.1}$$

$$X_{22}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_2,c_1,c_1,c_3;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+q}(a_2)_{q+n+p}(a_3)_p}{(c_1)_{n+p}(c_2)_m(c_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{1.2}$$

$$X_{23}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_1,c_2;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+q}(a_2)_{q+n+p}(a_3)_p}{(c_1)_{m+n+p}(c_2)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{1.3}$$

$$X_{24}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_3;c_1,c_2,c_1,c_3;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+q}(a_2)_{n+p}(a_3)_{p+q}}{(c_1)_{m+p}(c_2)_n(a_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{1.4}$$

$$X_{25}^{(4)}(a_1,a_1,a_3,a_1,a_1,a_2,a_4,a_2;c_1,c_2,c_1,c_3;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+q}(a_2)_{n+q}(a_3)_p(a_4)_p}{(c_1)_{m+p}(c_2)_n(c_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}. \tag{1.5}$$

We have organized the rest of this paper in the following way: Section 2 introduces numerous integral representations of Euler type which include Appell's functions of two variables F_2, F_3 and F_4 [14 and 16], the Horn's function H_4 of two variables (see [14]), the Gaussian hypergeometric function ${}_2F_1$ [14], the Exton's triple series X_2, X_3, X_4, X_{13}, X_{14}, X_{15}, X_{16} and X_{20} (see [4]), the Lauricella's triple series $F_C^{(3)}$ and F_F ([5], [8]), and the quadruple series $X_{21}^{(4)}$, Sharma and Parihar hypergeometric function of four variables $F_{14}^{(4)}$ [9] and Lauricella hypergeometric function of four variables $F_C^{(4)}$ [14] for the new quadruple functions denoted by $X_i^{(4)}$ (i = 21, 22, 23, 24, 25). Section 3 presents Laplace type integrals for each series $X_i^{(4)}$ (i = 21, 22, 23, 24, 25).

2. Integral representations of Euler-Type

Now, by means of the Gauss hypergeometric function ${}_2F_1$, Appell hypergeometric functions F_2, F_3 and F_4, Horn's function H_4 of two variables, the Exton's triple series X_2, X_3, X_4, X_{13}, X_{14}, X_{15}, X_{16} and X_{20}, the Lauricella's triple series $F_C^{(3)}$ and F_F and the quadruple series $F_{14}^{(4)}$, $X_{21}^{(4)}$ and $F_C^{(4)}$, we investigate some further integral representations of Euler-type for $X_i^{(4)}$ (i = 21, 22, 23, 24, 25) as follows:

$$X_{21}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_2,c_3;x,y,z,u)=\frac{\Gamma(a_1+a_2+a_3)\Gamma(c_1)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a)\Gamma(c_1-a)}$$
$$\times\int_0^1\int_0^1\int_0^1 \alpha^{a_1-1}(1-\alpha)^{a_2-1}\beta^{a_1+a_2-1}(1-\beta)^{a_3-1}\gamma^{a-1}(1-\gamma)^{c_1-a-1}$$
$$\times F_C^{(4)}\left(\frac{a_1+a_2+a_3}{2},\frac{a_1+a_2+a_3+1}{2};a,c_1-a,c_2,c_3;\lambda_1 x,\lambda_2 y,\lambda_3 z,\lambda_4 u\right)d\alpha\, d\beta\, d\gamma \tag{2.1}$$
$$\left(\lambda_1=4\alpha^2\beta^2\gamma,\ \lambda_2=4\alpha(1-\alpha)\beta^2(1-\gamma),\ \lambda_3=4(1-\alpha)\beta(1-\beta),\ \lambda_4=4\alpha(1-\alpha)\beta^2\right),$$
$$\left(\mathrm{Re}(a_i)>0,\ i=(1,2,3),\ \mathrm{Re}(a)>0,\ \mathrm{Re}(c_1-a)>0\right);$$

$$X_{21}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_2,c_3;x,y,z,u)=\frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}$$
$$\times\int_0^1 \alpha^{a_1-1}(1-\alpha)^{a_2-1}$$
$$\times X_2\left(a_1+a_2,a_3;c_1,c_3,c_2;\alpha^2 x+\alpha(1-\alpha)y,\alpha(1-\alpha)u,(1-\alpha)z\right)d\alpha \tag{2.2}$$
$$\left(\mathrm{Re}(a_1)>0,\ \mathrm{Re}(a_2)>0\right);$$

$$X_{21}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_2,c_3;x,y,z,u)=\frac{2\,\Gamma(c_3)}{\Gamma(a_1)\Gamma(c_3-a_1)}$$
$$\times\int_0^{\frac{\pi}{2}}\left(\sin^2\alpha\right)^{a_1-\frac{1}{2}}\left(\cos^2\alpha\right)^{c_3-a_1-\frac{1}{2}}\left(1-u\sin^2\alpha\right)^{-a_2}$$
$$\times X_{14}\left(1+a_1-c_3,a_2,a_3;c_1,c_2;x\tan^4\alpha,-\frac{y\tan^2\alpha}{\left(1-u\sin^2\alpha\right)},\frac{z}{\left(1-u\sin^2\alpha\right)}\right)d\alpha \tag{2.3}$$
$$\left(\mathrm{Re}(a_1)>0,\ \mathrm{Re}(c_3-a_1)>0\right);$$

$$X_{21}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_2,c_3;x,y,z,u)=\frac{\Gamma(c_1)(S-T)^{a_1}(R-T)}{\Gamma(a_1)\Gamma(c_1-a_1)(S-R)^{2c_1-a_1-a_2-2}}$$
$$\times\int_R^S(\alpha-R)^{a_1-1}(\alpha-T)^{1+a_1+a_2-2c_1}$$
$$\times\left[(R-T)(S-R)(S-\alpha)(\alpha-T)+(S-T)^2(\alpha-R)^2x\right]^{c_1-a_1-1}\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)y\right]^{-a_2}$$
$$\times F_2(a_2,a_3,1+a_1-c_1;c_2,c_3;\lambda_1 z,\lambda_2 u)\,d\alpha \tag{2.4}$$
$$\left(\lambda_1=\frac{(S-R)(\alpha-T)}{\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)y\right]},\right.$$
$$\left.\lambda_2=-\frac{(S-R)^2(S-T)(\alpha-R)(\alpha-T)^2}{\left[(R-T)(S-R)(S-\alpha)(\alpha-T)+(S-T)^2(\alpha-R)^2x\right]\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)y\right]}\right),$$
$$(\mathrm{Re}(a_1)>0,\ \mathrm{Re}(c_1-a_1)>0,T<R<S);$$

$$X_{21}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_2,c_3;x,y,z,u)=\frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(a_1)\Gamma(a_3)\Gamma(c_1-a_1)\Gamma(c_2-a_3)}$$
$$\times\int_0^\infty\int_0^\infty\left(e^{-\alpha}\right)^{a_1}\left(e^{-\beta}\right)^{a_3}\left(1-e^{-\beta}\right)^{c_2-a_3-1}\left[\left(1-e^{-\alpha}\right)+x\,e^{-2\alpha}\right]^{c_1-a_1-1}\left(1-y\,e^{-\alpha}-z\,e^{-\beta}\right)^{-a_2}$$
$$\times{}_2F_1\left(1+a_1-c_1,a_2;c_3;-\frac{u\,e^{-\alpha}}{\left[\left(1-e^{-\alpha}\right)+x\,e^{-2\alpha}\right]\left(1-y\,e^{-\alpha}-z\,e^{-\beta}\right)}\right)d\alpha\,d\beta \tag{2.5}$$
$$(\mathrm{Re}(a_1)>0,\mathrm{Re}(a_3)>0,\mathrm{Re}(c_1-a_1)>0,\mathrm{Re}(c_2-a_3)>0);$$

$$X_{22}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_2,c_1,c_1,c_3;x,y,z,u)=\frac{\Gamma(a_1+a_3)}{\Gamma(a_1)\Gamma(a_3)}$$
$$\times\int_0^\infty\left(e^{-\alpha}\right)^{a_1}\left(1-e^{-\alpha}\right)^{a_3-1}$$
$$\times X_4\left(a_1+a_3,a_2;c_2,c_1,c_3;x\,e^{-2\alpha},y\,e^{-\alpha}+z\left(1-e^{-\alpha}\right),u\,e^{-\alpha}\right)d\alpha \tag{2.6}$$
$$(\mathrm{Re}(a_1)>0,\mathrm{Re}(a_3)>0);$$

$$X_{22}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_2,c_1,c_1,c_3;x,y,z,u)=\frac{\Gamma(c_3)}{\Gamma(a_1)\Gamma(c_3-a_1)}$$
$$\times\int_0^1\alpha^{a_1-1}(1-\alpha)^{c_3-a_1-1}(1-\alpha\,u)^{-a_2}$$
$$\times X_{15}\left(1+a_1-c_3,a_2,a_3;c_2,c_1;\frac{\alpha^2x}{(1-\alpha)^2},-\frac{\alpha\,y}{(1-\alpha)(1-\alpha\,u)},\frac{z}{(1-\alpha\,u)}\right)d\alpha \tag{2.7}$$
$$(\mathrm{Re}(a_1)>0,\mathrm{Re}(c_3-a_1)>0);$$

$$X_{22}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_2,c_1,c_1,c_3;x,y,z,u)=\frac{\Gamma(c_2)}{\Gamma(a_1)\Gamma(c_2-a_1)}$$
$$\times\int_0^\infty \alpha^{a_1-1}(1+\alpha)^{1+a_1-2c_2}\left[(1+\alpha)+\alpha^2 x\right]^{c_2-a_1-1}$$
$$\times F_F\left(a_2,a_2,a_2,1+a_1-c_2,a_3,1+a_1-c_2;c_3,c_1,c_1;\lambda u,z,\lambda y\right)d\alpha$$
$$\left(\lambda=-\frac{\alpha(1+\alpha)}{\left[(1+\alpha)+\alpha^2 x\right]}\right),\tag{2.8}$$
$$(\mathrm{Re}(a_1)>0,\ \mathrm{Re}(c_2-a_1)>0);$$

$$X_{22}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_2,c_1,c_1,c_3;x,y,z,u)=\frac{2\,\Gamma(c_1)}{\Gamma(a_2)\Gamma(c_1-a_2)}$$
$$\times\int_0^{\frac{\pi}{2}}(\sin^2\alpha)^{a_2-\frac{1}{2}}(\cos^2\alpha)^{c_1-a_2-\frac{1}{2}}(1-y\sin^2\alpha)^{-a_1}(1-z\sin^2\alpha)^{-a_3}$$
$$\times H_4\left(a_1,1+a_2-c_1;c_2,c_3;\frac{x}{(1-y\sin^2\alpha)^2},-\frac{u\tan^2\alpha}{(1-y\sin^2\alpha)}\right)d\alpha\tag{2.9}$$
$$(\mathrm{Re}(a_2)>0,\ \mathrm{Re}(c_1-a_2)>0);$$

$$X_{22}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_2,c_1,c_1,c_3;x,y,z,u)=\frac{4\,\Gamma(c_1)\Gamma(c_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(c_1-a_2)\Gamma(c_2-a_1)}$$
$$\times\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}(\sin^2\alpha)^{a_1-\frac{1}{2}}(\cos^2\alpha)^{c_2-a_1-\frac{1}{2}}(\sin^2\beta)^{a_2-\frac{1}{2}}(\cos^2\beta)^{c_1-a_2-\frac{1}{2}}$$
$$\times(1+x\sin^2\alpha\tan^2\alpha+y\tan^2\alpha\sin^2\beta)^{c_2-a_1-1}(1-z\sin^2\beta)^{-a_3}\tag{2.10}$$
$$\times{}_2F_1\left(1+a_1-c_2,1+a_2-c_1;c_3;\frac{u\tan^2\alpha\tan^2\beta}{(1+x\sin^2\alpha\tan^2\alpha+y\tan^2\alpha\sin^2\beta)}\right)d\alpha\,d\beta$$
$$(\mathrm{Re}(a_1)>0,\ \mathrm{Re}(a_2)>0,\ \mathrm{Re}(c_1-a_2)>0,\ \mathrm{Re}(c_2-a_1)>0);$$

$$X_{23}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_1,c_2;x,y,z,u)=\frac{2\,\Gamma(c_1)}{\Gamma(a)\Gamma(c_1-a)}$$
$$\times\int_0^{\frac{\pi}{2}}(\sin^2\alpha)^{a-\frac{1}{2}}(\cos^2\alpha)^{c_1-a-\frac{1}{2}}\tag{2.11}$$
$$\times X_{21}^{(4)}\left(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;a,a,c_1-a,c_2;x\sin^2\alpha,y\sin^2\alpha,z\cos^2\alpha,u\right)d\alpha$$
$$(\mathrm{Re}(a)>0,\ \mathrm{Re}(c_1-a)>0);$$

$$X_{23}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_1,c_2;x,y,z,u)=\frac{\Gamma(c_1)(1+M)^{a_3}}{\Gamma(a_3)\Gamma(c_1-a_3)}$$
$$\times\int_0^1 \alpha^{a_3-1}(1-\alpha)^{c_1-a_3-1}(1+M\alpha)^{a_2-c_1}\left[(1+M\alpha)-(1+M)\alpha z\right]^{-a_2}$$
$$\times X_3\left(a_1,a_2;c_1-a_3,c_2;\frac{(1-\alpha)x}{(1+M\alpha)},\frac{(1-\alpha)y}{\left[(1+M\alpha)-(1+M)\alpha z\right]},\frac{(1+M\alpha)u}{\left[(1+M\alpha)-(1+M)\alpha z\right]}\right)d\alpha \tag{2.12}$$
$$\left(\operatorname{Re}(a_3)>0,\ \operatorname{Re}(c_1-a_3)>0,\ M>-1\right);$$

$$X_{23}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_1,c_2;x,y,z,u)=\frac{\Gamma(c_2)}{\Gamma(a_2)\Gamma(c_2-a_2)}$$
$$\times\int_0^\infty \left(e^{-\alpha}\right)^{a_2}\left(1-e^{-\alpha}\right)^{c_2-a_2-1}\left(1-u\,e^{-\alpha}\right)^{-a_1}$$
$$\times X_{13}\left(a_1,1+a_2-c_2,a_3;c_1;\frac{x}{\left(1-u\,e^{-\alpha}\right)^2},-\frac{y\,e^{-\alpha}}{\left(1-e^{-\alpha}\right)\left(1-u\,e^{-\alpha}\right)},-\frac{z\,e^{-\alpha}}{\left(1-e^{-\alpha}\right)}\right)d\alpha \tag{2.13}$$
$$\left(\operatorname{Re}(a_2)>0,\ \operatorname{Re}(c_2-a_2)>0\right);$$

$$X_{23}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_1,c_2;x,y,z,u)=\frac{2\,\Gamma(c_2)M^{a_1}}{\Gamma(a_1)\Gamma(c_2-a_1)}$$
$$\times\int_0^{\frac{\pi}{2}}\left(\sin^2\alpha\right)^{a_1-\frac12}\left(\cos^2\alpha\right)^{c_2-a_1-\frac12}\left(\cos^2\alpha+M\sin^2\alpha\right)^{a_2-c_2}$$
$$\times\left[\left(\cos^2\alpha+M\sin^2\alpha\right)-M\,u\sin^2\alpha\right]^{-a_2}$$
$$\times X_{13}\left(1+a_1-c_2,a_2,a_3;c_1;M^2\,x\tan^4\alpha,\lambda_1 y,\lambda_2 z\right)d\alpha \tag{2.14}$$
$$\left(\lambda_1=-\frac{M\left(\cos^2\alpha+M\sin^2\alpha\right)\tan^2\alpha}{\left[\left(\cos^2\alpha+M\sin^2\alpha\right)-M\,u\sin^2\alpha\right]},\lambda_2=-\frac{\left(\cos^2\alpha+M\sin^2\alpha\right)}{\left[\left(\cos^2\alpha+M\sin^2\alpha\right)-M\,u\sin^2\alpha\right]}\right),$$
$$\left(\operatorname{Re}(a_1)>0,\ \operatorname{Re}(c_2-a_1)>0,\ M>0\right);$$

$$X_{23}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_1,c_2;x,y,z,u)=\frac{\Gamma(a_1+a_2+a_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}$$
$$\times\int_0^1\int_0^1 \alpha^{a_1-1}(1-\alpha)^{a_2-1}\beta^{a_1+a_2-1}(1-\beta)^{a_3-1}$$
$$\times F_4\left(\frac{a_1+a_2+a_3}{2},\frac{a_1+a_2+a_3+1}{2};c_1,c_2;4\alpha^2\beta^2x+4\alpha(1-\alpha)\beta^2y+4(1-\alpha)\beta(1-\beta)z,4\alpha(1-\alpha)\beta^2u\right)d\alpha\,d\beta \tag{2.15}$$
$$\left(\operatorname{Re}(a_i)>0,\ i=(1,2,3)\right);$$

$$X_{24}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_3;c_1,c_2,c_1,c_3;x,y,z,u)=\frac{2\,\Gamma(c_3)}{\Gamma(a_1)\Gamma(c_3-a_1)}$$
$$\times\int_0^{\frac{\pi}{2}}\left(\sin^2\alpha\right)^{a_1-\frac12}\left(\cos^2\alpha\right)^{c_3-a_1-\frac12}\left(1-u\sin^2\alpha\right)^{-a_3}$$
$$\times X_{16}\left(1+a_1-c_3,a_2,a_3;c_1,c_2;x\tan^4\alpha,-y\tan^2\alpha,\frac{z}{\left(1-u\sin^2\alpha\right)}\right)d\alpha \tag{2.16}$$
$$\left(\operatorname{Re}(a_1)>0,\ \operatorname{Re}(c_3-a_1)>0\right);$$

$$X_{24}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_3;c_1,c_2,c_1,c_3;x,y,z,u)=\frac{\Gamma(c_3)(S-T)^{a_3}(R-T)^{c_3-a_3}}{\Gamma(a_3)\Gamma(c_3-a_3)(S-R)^{c_3-a_1-1}}$$

$$\times\int_R^S(\alpha-R)^{a_3-1}(S-\alpha)^{c_3-a_3-1}(\alpha-T)^{a_1-c_3}\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)u\right]^{-a_1}$$

$$\times X_{16}(a_1,a_2,1+a_3-c_3;c_1,c_2;\lambda_1 x,\lambda_2 y,\lambda_3 z)\,d\alpha \tag{2.17}$$

$$\left(\lambda_1=\frac{(S-R)^2(\alpha-T)^2}{\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)u\right]^2},\lambda_2=\frac{(S-R)(\alpha-T)}{\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)u\right]},\right.$$

$$\left.\lambda_3=-\frac{(R-T)(S-\alpha)}{(S-R)(\alpha-T)}\right),$$

$$\left(\mathrm{Re}(a_3)>0,\ \mathrm{Re}(c_3-a_3)>0, T<R<S\right);$$

$$X_{24}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_3;c_1,c_2,c_1,c_3;x,y,z,u)=\frac{\Gamma(a_1+a_2+a_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}$$

$$\times\int_0^\infty\int_0^\infty\left(e^{-\alpha}\right)^{a_1}\left(1-e^{-\alpha}\right)^{a_2-1}\left(e^{-\beta}\right)^{a_1+a_2}\left(1-e^{-\beta}\right)^{a_3-1}$$

$$\times F_C^{(3)}\left(\frac{a_1+a_2+a_3}{2},\frac{a_1+a_2+a_3+1}{2};c_1,c_2,c_3;\lambda_1 x+\lambda_2 z,\lambda_3 y,\lambda_4 u\right)d\alpha\,d\beta \tag{2.18}$$

$$\left(\lambda_1=4\,e^{-2(\alpha+\beta)},\lambda_2=4\left(1-e^{-\alpha}\right)e^{-\beta}\left(1-e^{-\beta}\right),\lambda_3=4\,e^{-(\alpha+2\beta)}\left(1-e^{-\alpha}\right),\lambda_4=4\,e^{-(\alpha+\beta)}\left(1-e^{-\beta}\right)\right),$$

$$\left(\mathrm{Re}(a_i)>0,\ i=(1,2,3)\right);$$

$$X_{24}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_3;c_1,c_2,c_1,c_3;x,y,z,u)=\frac{\Gamma(c_2)\Gamma(c_3)}{\Gamma(a_2)\Gamma(a_3)\Gamma(c_2-a_2)\Gamma(c_3-a_3)}$$

$$\times\int_0^1\int_0^1\alpha^{a_2-1}(1-\alpha)^{c_2-a_2-1}\beta^{a_3-1}(1-\beta)^{c_3-a_3-1}(1-\alpha\,y-\beta\,u)^{-a_1} \tag{2.19}$$

$$\times F_3\left(\frac{a_1}{2},1+a_2-c_2,\frac{a_1+1}{2},1+a_3-c_3;c_1;\frac{4\,x}{(1-\alpha\,y-\beta\,u)^2},\frac{\alpha\,\beta\,z}{(1-\alpha)(1-\beta)}\right)d\alpha\,d\beta$$

$$\left(\mathrm{Re}(a_2)>0,\mathrm{Re}(a_3)>0,\mathrm{Re}(c_2-a_2)>0,\mathrm{Re}(c_3-a_3)>0\right);$$

$$X_{24}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_3;c_1,c_2,c_1,c_3;x,y,z,u)=\frac{\Gamma(c_2)\Gamma(c_3)}{\Gamma(a_2)\Gamma(a_3)\Gamma(c_2-a_2)\Gamma(c_3-a_3)}$$

$$\times\int_0^\infty\int_0^\infty\alpha^{a_2-1}(1+\alpha)^{a_1-c_2}\beta^{a_3-1}(1+\beta)^{a_1-c_3}\left[(1+\alpha)(1+\beta)-\alpha(1+\beta)y-(1+\alpha)\beta\,u\right]^{-a_1} \tag{2.20}$$

$$\times F_3\left(\frac{a_1}{2},1+a_2-c_2,\frac{a_1+1}{2},1+a_3-c_3;c_1;\frac{4(1+\alpha)^2(1+\beta)^2x}{\left[(1+\alpha)(1+\beta)-\alpha(1+\beta)y-(1+\alpha)\beta\,u\right]^2},\alpha\,\beta\,z\right)d\alpha\,d\beta$$

$$\left(\mathrm{Re}(a_2)>0,\mathrm{Re}(a_3)>0,\mathrm{Re}(c_2-a_2)>0,\mathrm{Re}(c_3-a_3)>0\right);$$

$$X_{25}^{(4)}(a_1,a_1,a_3,a_1,a_1,a_2,a_4,a_2;c_1,c_2,c_1,c_3;x,y,z,u)=\frac{2\,\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}$$

$$\times\int_0^{\frac{\pi}{2}}\left(\sin^2\alpha\right)^{a_1-\frac{1}{2}}\left(\cos^2\alpha\right)^{a_2-\frac{1}{2}}$$

$$\times F_{14}^{(4)}\left(\frac{a_1+a_2}{2},\frac{a_1+a_2}{2},\frac{a_1+a_2}{2},a_3,\frac{a_1+a_2+1}{2},\frac{a_1+a_2+1}{2},\frac{a_1+a_2+1}{2},a_4;c_1,c_2,c_3,c_1;\lambda_1 x,\lambda_2 y,\lambda_2 u,z\right)d\alpha \tag{2.21}$$

$$\left(\lambda_1=4\sin^4\alpha,\lambda_2=\sin^2 2\alpha\right),$$

$$\left(\mathrm{Re}(a_1)>0,\mathrm{Re}(a_2)>0\right);$$

$$X_{25}^{(4)}(a_1,a_1,a_3,a_1,a_1,a_2,a_4,a_2;c_1,c_2,c_1,c_3;x,y,z,u)=\frac{\Gamma(c_1)(S-T)^{a_3}(R-T)^{c_1-a_3}}{\Gamma(a_3)\Gamma(c_1-a_3)(S-R)^{c_1-a_4-1}}$$
$$\times\int_R^S(\alpha-R)^{a_3-1}(S-\alpha)^{c_1-a_3-1}(\alpha-T)^{a_4-c_1}\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)z\right]^{-a_4}$$
$$\times X_4\left(a_1,a_2;c_1-a_3,c_2,c_3;\frac{(R-T)(S-\alpha)x}{(S-R)(\alpha-T)},y,u\right)d\alpha \tag{2.22}$$
$$\left(\mathrm{Re}(a_3)>0,\ \mathrm{Re}(c_1-a_3)>0,T<R<S\right);$$

$$X_{25}^{(4)}(a_1,a_1,a_3,a_1,a_1,a_2,a_4,a_2;c_1,c_2,c_1,c_3;x,y,z,u)=\frac{\Gamma(c_3)}{\Gamma(a_2)\Gamma(c_3-a_2)}$$
$$\times\int_0^\infty\alpha^{a_2-1}(1+\alpha)^{a_1-c_3}\left[(1+\alpha)-\alpha\,u\right]^{-a_1}$$
$$\times X_{20}\left(a_1,1+a_2-c_3,a_3,a_4;c_1,c_2;\frac{(1+\alpha)^2x}{\left[(1+\alpha)-\alpha\,u\right]^2},-\frac{\alpha(1+\alpha)y}{\left[(1+\alpha)-\alpha\,u\right]},z\right)d\alpha \tag{2.23}$$
$$\left(\mathrm{Re}(a_2)>0,\mathrm{Re}(c_3-a_2)>0\right);$$

$$X_{25}^{(4)}(a_1,a_1,a_3,a_1,a_1,a_2,a_4,a_2;c_1,c_2,c_1,c_3;x,y,z,u)=\frac{\Gamma(a_1+a_2+a_3+a_4)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}$$
$$\times\int_0^1\int_0^1\int_0^1\alpha^{a_1-1}(1-\alpha)^{a_2-1}\beta^{a_1+a_2-1}(1-\beta)^{a_3-1}\gamma^{a_1+a_2+a_3-1}(1-\gamma)^{a_4-1}$$
$$\times F_C^{(3)}\left(\frac{a_1+a_2+a_3+a_4}{2},\frac{a_1+a_2+a_3+a_4+1}{2};c_1,c_2,c_3;\lambda_1x+\lambda_2z,\lambda_3y,\lambda_3u\right)d\alpha\,d\beta\,d\gamma \tag{2.24}$$
$$\left(\lambda_1=4\,\alpha^2\beta^2\gamma^2,\lambda_2=4(1-\beta)\gamma(1-\gamma),\lambda_3=4\,\alpha(1-\alpha)\beta^2\gamma^2\right),$$
$$\left(\mathrm{Re}(a_i)>0,\ i=(1,2,3,4)\right);$$

$$X_{25}^{(4)}(a_1,a_1,a_3,a_1,a_1,a_2,a_4,a_2;c_1,c_2,c_1,c_3;x,y,z,u)=\frac{\Gamma(c_1)\Gamma(c_3)}{\Gamma(a_2)\Gamma(a_4)\Gamma(c_1-a_4)\Gamma(c_3-a_2)}$$
$$\times\int_0^\infty\int_0^\infty e^{-(a_4\alpha+a_2\beta)}\left(1-e^{-\alpha}\right)^{c_1-a_4-1}\left(1-e^{-\beta}\right)^{c_3-a_2-1}\left(1-z\,e^{-\alpha}\right)^{-a_3}\left(1-u\,e^{-\beta}\right)^{-a_1}$$
$$\times H_4\left(a_1,1+a_2-c_3;c_1-a_4,c_2;\frac{x\left(1-e^{-\alpha}\right)}{\left(1-u\,e^{-\beta}\right)^2},-\frac{y\,e^{-\beta}}{\left(1-e^{-\beta}\right)\left(1-u\,e^{-\beta}\right)}\right)d\alpha\,d\beta \tag{2.25}$$
$$\left(\mathrm{Re}(a_2)>0,\mathrm{Re}(a_4)>0,\mathrm{Re}(c_1-a_4)>0,\mathrm{Re}(c_3-a_2)>0\right).$$

Proof of the integral representations of Euler-type

Once substituting the series definition of the special function in each integrand and then, changing the order of the integral and the summation, and finally taking into account the following integral representations of the Beta function and their various associated Eulerian integrals (see, for example, [17 p. 9–11], [18, 19, Section 1.1] and [16, p. 26 and p. 86, Problem 1]), we derive each of the integral representations from (2.1) to (2.25).

$$B(a,b)=\begin{cases}\int_0^1 t^{a-1}(1-t)^{b-1}\,dt\,, & (\mathrm{Re}(a)>0,\,\mathrm{Re}(b)>0),\\ \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} & (a,b\in C\setminus Z_0^-),\end{cases} \tag{2.26}$$

$$B(a,b)=\int_0^1 \alpha^{a-1}(1-\alpha)^{b-1}\,d\alpha=\int_0^\infty \left(e^{-\alpha}\right)^a\left(1-e^{-\alpha}\right)^{b-1}d\alpha$$
$$(\mathrm{Re}(a)>0,\,\mathrm{Re}(b)>0), \tag{2.27}$$

$$B(a,b)=2\int_0^{\frac{\pi}{2}}(\sin\alpha)^{2a-1}(\cos\alpha)^{2b-1}\,d\alpha=\int_0^\infty \frac{\alpha^{a-1}}{(1+\alpha)^{a+b}}\,d\alpha \tag{2.28}$$
$$(\mathrm{Re}(a)>0,\,\mathrm{Re}(b)>0),$$

$$B(a,b)=\frac{(S-T)^a(R-T)^b}{(S-R)^{a+b-1}}\int_R^S \frac{(\alpha-R)^{a-1}(S-\alpha)^{b-1}}{(\alpha-T)^{a+b}}\,d\alpha \quad (T<R<S)$$
$$=(1+M)^a\int_0^1 \frac{\alpha^{a-1}(1-\alpha)^{b-1}}{(1+M\alpha)^{a+b}}\,d\alpha \qquad (M>-1) \tag{2.29}$$
$$(\mathrm{Re}(a)>0,\,\mathrm{Re}(b)>0).$$

3. Integrals of Laplace-Type

Here, we represent the quadruple series $X_i^{(4)}(i=21, 22, 23, 24, 25)$ in terms of single, double and triple integrals by means of Laplace transform. Following are these integral representations of the quadruple series:

$$X_{21}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_2,c_3;x,y,z,u)=\frac{1}{\Gamma(a_1)\Gamma(a_2)}\int_0^\infty\int_0^\infty e^{-(s+t)}\,s^{a_1-1}\,t^{a_2-1}$$
$$\times{}_0F_1\left(-;c_1;s^2x+sty\right)\,{}_1F_1(a_3;c_2;tz)\,{}_0F_1(-;c_3;stu)\,ds\,dt\,,\quad (\mathrm{Re}(a_1)>0,\,\mathrm{Re}(a_2)>0); \tag{3.1}$$

$$X_{22}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_2,c_1,c_1,c_3;x,y,z,u)=\frac{1}{\Gamma(a_1)\Gamma(a_3)}\int_0^\infty\int_0^\infty e^{-(s+t)}\,s^{a_1-1}\,t^{a_3-1}$$
$$\times\Psi_2(a_2;c_1,c_3;sy+tz,su)\,{}_0F_1\left(-;c_2;s^2x\right)ds\,dt\,,\qquad (\mathrm{Re}(a_1)>0,\,\mathrm{Re}(a_3)>0); \tag{3.2}$$

$$X_{23}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_1,c_2;x,y,z,u)=\frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s+t+v)}\,s^{a_1-1}\,t^{a_2-1}\,v^{a_3-1}$$
$$\times{}_0F_1\left(-;c_1;s^2x+sty+tvz\right)\,{}_0F_1(-;c_2;stu)\,ds\,dt\,dv\,,\qquad (\mathrm{Re}(a_1)>0,\,\mathrm{Re}(a_2)>0,\,\mathrm{Re}(a_3)>0); \tag{3.3}$$

$$X_{24}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_3;c_1,c_2,c_1,c_3;x,y,z,u)=\frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(s+t+v)}\, s^{a_1-1}\, t^{a_2-1}\, v^{a_3-1}$$
$$\times {}_0F_1\left(-;c_1;s^2x+tvz\right)\, {}_0F_1\left(-;c_2;sty\right)\, {}_0F_1\left(-;c_3;svu\right)\, ds\, dt\, dv\ ,\quad (\mathrm{Re}(a_1)>0,\mathrm{Re}(a_2)>0,\mathrm{Re}(a_3)>0); \tag{3.4}$$

$$X_{25}^{(4)}(a_1,a_1,a_3,a_1,a_1,a_2,a_4,a_2;c_1,c_2,c_1,c_3;x,y,z,u)=\frac{1}{\Gamma(a_1)}\int_0^\infty e^{-s}\, s^{a_1-1}$$
$$\times \Xi_2\left(a_3,a_4;c_1;z,s^2x\right)\Psi_2\left(a_2;c_2,c_3;sy,su\right)\, ds\ ,\qquad (\mathrm{Re}(a_1)>0); \tag{3.5}$$

Where $\left({}_0F_1\ ,{}_1F_1\right),\Psi_2$ and Ξ_2 denote the confluent hypergeometric functions and the Humbert functions defined, respectively, by

$${}_0F_1\left(-;c;x\right)=\sum_{m=0}^{\infty}\frac{1}{(c)_m}\frac{x^m}{m!},$$

$${}_1F_1\left(a;c;x\right)=\sum_{m=0}^{\infty}\frac{(a)_m}{(c)_m}\frac{x^m}{m!},$$

$$\Psi_2\left(a;b,c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}}{(b)_m(c)_n}\frac{x^m}{m!}\frac{y^n}{n!}$$

and

$$\Xi_2\left(a,b;c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_m(b)_m}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}.$$

Proof of the integral representations of Laplace-type

To prove the integral representations (3.1) to (3.5), it is enough to consider the expressions of confluent hypergeometric functions and the Humbert functions given above and then to use Gamma integral formula

$$\Gamma(z)=\int_0^\infty e^{-t}\, t^{z-1}\, dt\, .$$

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