Characterization of Maps on Positive Semidefinite Choi Matrices

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Abstract:

Several investigations have been done on positive maps on their algebraic structures with more emphasis on completely positive maps. In this study, we have described the structure of the Choi matrices for $2-$positive maps on positive semidefinite matrices and the conditions for complete positivity of positive linear maps from n to $n+1$. The motivation behind these objectives is work done by Majewski and Marciniak on the structure of positive maps ϕ from $\mathcal{M}_n$ to $\mathcal{M}_n + 1 (2 \geq 2)$ between matrix algebras.

Keywords: Positive map; completely positive; Choi matrix.

1 Introduction

The theory of completely positive maps has been developed by operator algebraists and mathematical physicists over the last four decades. The two major theorems of Stinespring [13] and Arveson [1], hold in much the study of completely positive maps. Supported by the [12], Choi-Kraus theorem originally by Man-Duon Choi in [4] and [5] known as the Choi's Theorem which is used in the description of completely positive linear maps.

Theorem 1.1 [3, Choi-Kraus Theorem 3.1.1] *Let $\phi : \mathcal{M}_n \to \mathcal{M}_m$ be a completely positive linear map. Then there exist $V_j \in \mathbb{C}^{n\times m}, 1 \leq j \leq nm$, such that $\phi(A) = \sum_{j=1}^{nm} V_j^* A V_j$.*

In the proof, it is shown that if a linear map $\phi : \mathcal{M}_n \to \mathcal{M}_m$ is n-positive, then it is completely positive and that if the block matrix $[\phi(E_{ij})]$ is positive, then ϕ is completely positive.

Theorem 1.2 [12, Theorem 1.1.23] *Let $\phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C})$ be a linear. The following are equivalent;*

(i). ϕ is $n-$positive.

(ii). the matrix operator entries $C_\phi = (I \otimes \phi)(\sum_{ij} E_{ij} \otimes A_{ij}) = \sum_{ij} E_{ij} \otimes \phi(A_{ij}) \in \mathcal{M}_n \otimes \mathcal{M}_m(\mathbb{C})$ is positive where $E_{ij} \in \mathbb{C}^{n\times n}$ is the matrix with 1 in the ij-th entry and zeros elsewhere. The matrix C_ϕ is called the Choi's matrix of ϕ.

(iii). ϕ is completely positive.

The map ϕ is positive if and only if the Choi matrix C_ϕ is block-positive ($n-$positive), and ϕ is completely positive if and only if C_ϕ is positive. Størmer's [14, Theorem 3.6] result includes Arveson's extension theorem for completely

positive maps [1], as completely positive maps are those which are n-positive and completely positive maps. According to Størmer (Theorem 1.2.4, Theorem 1.2.5 and Remark 1.2.6), if A and B are C^*-algebras and either A or B is abelian, then every positive map ϕ from A to B is completely positive (or completely copositive).
In the case of block $n-$positive matrices the mapping $[\phi(A_{ij})]_{i,j=1}^n$ is justified in [10] with the property that such ϕ maps with finite dimensions are decomposable.

Theorem 1.3 [10, Theorem 1.1] *Let $\phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C})$ be a linear map. Then*

(i). the map ϕ is positive if and only if the matrix $[\phi(A_{ij})]_{i,j=1}^n$ is block n-positive;

(ii). the map ϕ is completely positive (respectively completely co-positive) if and only if $[\phi(A_{ij})]_{i,j=1}^n$ (respectively $[\phi(A_{ji})]_{i,j=1}^n$) is a positive element of $\mathcal{M}_n(\mathcal{M}_m(\mathbb{C}))$.

In [15] Theorem 1., Størmer gives a natural generalization to positive maps that if $\mathcal{M}_n$ is a finite dimensional C^*-algebra and ϕ from $\mathcal{M}_n$ to $\mathcal{M}_n$ is a positive unital map, then the set $\mathcal{M}_n = \{A \in \mathcal{M}_n : \phi(A) = A\}$ has a natural structure as a Jordan algebra for trace preserving maps ϕ.

1.1 Preliminaries

Definition 1.4 *Let A be a $n \times n$ square matrix, A is positive semidefinite if, for any vector v with real components, $\langle v, Av\rangle \geq 0$ for all $v \in \mathbb{R}^n$ or equivalently A is Hermitian and all its eigenvalues are non negative and positive definite if, in addition, $\langle v, Av\rangle > 0$ for all $v \neq 0$.*

We denote the set of positive semidefinite matrices of order n by $\mathcal{M}_n$, that is $A \in \mathcal{M}_n$.

Definition 1.5 *A linear map ϕ is from $\mathcal{M}_n(\mathbb{C})$ to $\mathcal{M}_m(\mathbb{C})$ is called positive if $\phi(\mathcal{M}_n(\mathbb{C})) \subseteq \mathcal{M}_m(\mathbb{C})$.*

The identity map on $\mathcal{M}_n(\mathbb{C})$ and the transpose map on $\mathcal{M}_n(\mathbb{C})$ are denoted by I_n and τ_n respectively.

Definition 1.6 *A positive linear map ϕ is $n-$*positive *if and only if the map $I_n \otimes \phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ is positive for all $n \geq 1$.*

A mathematically convenient way to express n-positivity is by using a block matrix notation. Let $[A_{ij}]_j^n$ be positive semidefinite block matrix with $A_{ij} \in \mathcal{M}_n(\mathbb{C})$, then $(\phi \otimes I)([A_{ij}])$ is the induced map, represented by the block matrix $\phi([A_{ij}])$.

Definition 1.7 *A positive linear map ϕ is $n-$*copositive *if and only if the map $\tau_n \otimes \phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ is positive.*

Definition 1.8 *A map is* completely positive *if for every n it is n-positive and* completely copositive *if for every n it is n-copositive.*

Since the linear map $\phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C})$ is positive, then for every positive semidefinite matrix $A \in \mathcal{M}_n(\mathbb{C})$, we have $\phi(A) \geq 0$. We note that there are positive maps that are not completely positive. Stinespring [13] and Arveson [1] give examples of positive linear maps that fail to be completely positive.

Definition 1.9 *Let $\phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C})$ be a linear map. Let (E_{ij}) with $i, j = 1, ..., n$ be a complete set of matrix units for $\mathcal{M}_n(\mathbb{C})$. Then the Choi matrix for ϕ is the operator*

$$C_\phi = (I \otimes \phi)(\sum_{ij} E_{ij} \otimes E_{ij}) = \sum_{ij} E_{ij} \otimes \phi(E_{ij}) \in \mathbb{C}^{nm \times nm}.$$

Remark 1.10 *The map $\phi \to C_\phi$ is linear, injective and is surjective, and given an operator $\sum E_{ij} \otimes A_{ij} \in \mathcal{M}_n \otimes \mathcal{M}_m$, then we can define a linear map ϕ by $\phi(E_{ij}) = A_{ij}$. This map is often called the Jamiolkowski isomorphism [16]. We therefore observe that the Choi matrix depends on the choice of matrix units (E_{ij}).*

Theorem 1.11 *Let A be an invertible matrix. The self-adjoint block matrix $M = \begin{bmatrix} A & C \\ C^* & B \end{bmatrix}$ is positive if and only if A is positive and*

$$C^* A^{-1} C \leq B.$$

See [6] for the proof.

Theorem 1.12 *Let A be an invertible matrix . The determinant of the block matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is computed as follows.*

$$\det M = (\det A) \det(D - CA^{-1}B).$$

See [6] for the proof.

Remark 1.13 *In Theorem 1.12 and Theorem 1.14, $B - C^* A^{-1} C$ and $D - CA^{-1}B$ are Schur complements of A in matrices M respectively.*

Theorem 1.14 *. Let $A \in \mathbb{F}^{n \times n}, b \in \mathbb{F}^n$, and $a \in \mathbb{R}$, and define*

$$\mathcal{A} = \begin{bmatrix} A & b \\ b^* & a \end{bmatrix}.$$

Then, the following statements are equivalent:

(i). $\mathcal{A}$ *is positive semidefinite.*

(ii). $\det \mathcal{A} = (\det A)(a - b^* A^{-1} b)$.

See [2] for the proof.
If the off-diagonal entries of A are all nonnegative, then, A is copositive if and only if A is positive semidefinite.

Theorem 1.15 ***(The Binet-Cauchy formula)****. Let A and B be matrices of size $n \times m$ and $m \times n$, respectively, and $n = \leq m$. Then*

$$\det AB = \sum_{1 \leq k_1 < k_2 < \ldots < k_n \leq m} A_{k_1 \ldots k_n} B^{k_1 \ldots k_n},$$

where $A_{k_1 \ldots k_n}$ is the minor obtained from the columns of A whose numbers are $k_1, \ldots, k_n$ and $B^{k_1, \ldots, k_n}$ is the minor obtained from the rows of B whose numbers are $k_1, \ldots, k_n$.

See [11] for the proof.

2 Choi Matrix of 2−positive Map ϕ from $\mathcal{M}_n$ to $\mathcal{M}_{n+1}$

The matrix C_ϕ denotes the Choi's matrix of ϕ. By the Jamiolkowski-Choi isomorphism correspondence $\phi \to C_\phi$, ϕ is positive if and only if C_ϕ is block-positive, and ϕ is completely positive if and only if C_ϕ is positive. Complete positivity is determined by applying the Choi matrix operator entries $C_\phi = [\phi(A_{ij})] = (I \otimes \phi)(\sum_{ij} E_{ij} \otimes A_{ij}) = \sum_{ij} E_{ij} \otimes \phi(A_{ij}) \in \mathbb{C}^{nm\times nm}$, where $E_{ij} \in \mathbb{C}^{n\times n}$ is the matrix with 1 in the ij-th entry and zeros elsewhere.
This study is motivated by Majewski and Marciniak [7] in their quest to decompose positive maps between $\mathcal{M}_2$ to $\mathcal{M}_n$ a sum of a positive and copositive maps. We have given the version of the Choi matrix in [7], [8] and [9] a new look. The Choi matrix of the maps $\phi_{(\mu,c_1,\ldots,c_{n-1})}$ which is 2-positive is partitioned in the following manner.
Let the $\phi : \mathcal{M}_n \to \mathcal{M}_{n+1}$ be a linear positive map where $n \geq 1, 2, 3 \ldots$. We define the choi matrix for these linear maps as a block matrix of the form,

$$\left[\begin{array}{cccc|ccccc||cccc|ccccc}
a_{11} & 0 & \cdots & 0 & c_{11} & c_{12} & \cdots & \cdots & c_{1k} & 0 & \cdots & \cdots & 0 & y_{11} & y_{12} & \cdots & \cdots & y_{1n} \\
0 & a_{22} & \ddots & \vdots & \vdots & \vdots & & & \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots & & & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots & \vdots & & & \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & & & \vdots \\
0 & \cdots & 0 & a_{nn} & c_{n1} & c_{n2} & \cdots & \cdots & c_{nk} & 0 & \cdots & \cdots & 0 & y_{n1} & y_{n2} & \cdots & \cdots & y_{nk} \\
\hline
\bar{c}_{11} & \cdots & \cdots & \bar{c}_{1n} & b_{11} & 0 & \cdots & \cdots & 0 & \bar{z}_{11} & \cdots & \cdots & \bar{z}_{1n} & t_{11} & t_{12} & \cdots & \cdots & t_{1k} \\
\bar{c}_{21} & \cdots & \cdots & \bar{c}_{2n} & 0 & b_{22} & \ddots & & \vdots & \bar{z}_{12} & \cdots & \cdots & \bar{z}_{2n} & t_{21} & t_{22} & \cdots & \cdots & t_{2k} \\
\vdots & & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \ddots & & \vdots \\
\vdots & & & \vdots & \vdots & & \ddots & \ddots & 0 & \vdots & & & \vdots & \vdots & \vdots & & \ddots & \vdots \\
\bar{c}_{k1} & \cdots & \cdots & \bar{c}_{kn} & 0 & \cdots & \cdots & 0 & b_{kk} & \bar{z}_{k1} & \cdots & \cdots & \bar{z}_{kn} & t_{k1} & t_{k2} & \cdots & \cdots & t_{kk} \\
\hline\hline
0 & \cdots & \cdots & 0 & z_{11} & z_{12} & \cdots & \cdots & z_{1k} & d_{11} & 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & & \vdots & \vdots & \vdots & & & \vdots & 0 & d_{22} & \ddots & \vdots & \vdots & \vdots & & & \vdots \\
\vdots & & \ddots & \vdots & \vdots & \vdots & & & \vdots & \vdots & \ddots & \ddots & 0 & \vdots & \vdots & & & \vdots \\
0 & \cdots & \cdots & 0 & z_{n1} & z_{n2} & \cdots & \cdots & z_{nk} & 0 & \cdots & 0 & d_{nn} & 0 & 0 & \cdots & \cdots & 0 \\
\hline
\bar{y}_{11} & \cdots & \cdots & \bar{y}_{1n} & \bar{t}_{11} & \bar{t}_{12} & \cdots & \cdots & \bar{t}_{1k} & 0 & \cdots & \cdots & 0 & u_{11} & u_{12} & \cdots & \cdots & u_{1k} \\
\bar{y}_{21} & \cdots & \cdots & \bar{y}_{2n} & \bar{t}_{21} & \bar{t}_{22} & \cdots & \cdots & \bar{t}_{2k} & 0 & \cdots & \cdots & 0 & u_{21} & u_{22} & \cdots & \cdots & u_{2k} \\
\vdots & & & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & & & \vdots & \vdots & \vdots & \ddots & & \vdots \\
\vdots & & & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots & & & \vdots & \vdots & \vdots & & \ddots & \vdots \\
\bar{y}_{k1} & \cdots & \cdots & \bar{y}_{kn} & \bar{t}_{k1} & \bar{t}_{k2} & \cdots & \cdots & \bar{t}_{kk} & 0 & \cdots & \cdots & 0 & u_{k1} & u_{k2} & \cdots & \cdots & u_{kk}
\end{array}\right]$$

which we represent as:

$$C_\phi = \left[\begin{array}{cc|cc}
A_n & C_{n\times k} & 0_n & Y_{n\times k} \\
C^*_{k\times n} & B_k & Z^*_{k\times n} & T_k \\
\hline
0_n & Z_{n\times k} & D_n & 0_{n\times k} \\
Y^*_{k\times n} & T^*_k & 0_{k\times n} & U_k
\end{array}\right] \tag{1}$$

where $A, B, D \in \mathcal{M}_n$ are positive diagonal matrices. $U \geq 0 \in \mathcal{M}_k$, $T \in \mathcal{M}_k$ not necessarily positive and $C, Y, Z \in \mathcal{M}_{n\times k}$ where $k \geq 2$.

Lemma 2.1 *Let A and B be positive diagonal matrix of order n and k respectively. Then $M = \begin{bmatrix} A & C \\ C^* & B \end{bmatrix}$ is a positive matrix of order $n + k$ satisfying the matrix inequality $C^*AC \leq (\det A)B$.*

Proof. Let $M = \begin{bmatrix} A & c \\ c^* & b \end{bmatrix}$ where $A \in M_{n-1}$, $c \in \mathbb{C}^{n-1}$ and $b \in \mathbb{R}$. Since A is a positive diagonal matrix and by Theorem 1.14 and Theorem 1.12,

$$\det M = (\det A)(b - c^*A^{-1}c) = (\det A)(b - c^* \frac{A}{(\det A)} c)$$

but $\det M$ is positive therefore $b - c^* \frac{A}{(\det A)} c \geq 0$. Thus, $c^* A c \leq (\det A) b$.

Let $M = \begin{bmatrix} A & C \\ C^* & B \end{bmatrix}$ where $A \in M_{n-2}$, $C \in M_{2,n-2}$ and $B \in M_2$. By Theorem 1.14, Theorem 1.12 and Theorem 1.15.

$$\det M = (\det A) \det(B - C^* A^{-1} C) = (\det A)(B - C^* \frac{A}{(\det A)} C) \geq 0$$

therefore $C^* A C \leq (\det A) B$.

Next let $n = k$. Because A_{n-2} is invertible. By Theorem 1.14 and Theorem 1.12, $M \geq 0$ if and only if $B - C^* A^{-1} C \geq 0$. That is,

$$B - C^* A^{-1} C = B - C^* \frac{A}{(\det A)} C \geq 0.$$

Now let $n < k$, writing M in form

$$\begin{bmatrix} A_n & C_{n\times k} \\ C^*_{k\times n} & B_k \end{bmatrix}.$$

A is a diagonal matrix, By Theorem 1.14, Theorem 1.12 and Theorem 1.15.

$$\begin{aligned} \det M &\geq (\det A_n). \det(B_k - C^*_{k\times n} A_n^{-1} C_{n\times k}) \\ &= (\det A_n). \det(B_k - C^*_{k\times n} \frac{A_n}{(\det A_n)} C_{n\times k}) \geq 0 \end{aligned}$$

which holds if and only if $C^*_{k\times n} A_n C_{n\times k} \leq (\det A) B_k$.

□

Lemma 2.2 *Let A be a C^*-algebra, if $M = \begin{bmatrix} 0 & y \\ z & x \end{bmatrix} \in \mathcal{M}_2(\mathbb{C})$ is positive, then either $x = 0$ or $z = 0$, and $y \geq 0$.*

Proof. We may assume A is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, then there is a vector in $(v_1, v_2) \in \mathcal{H}$ such that $\langle M v_1, v_2 \rangle \geq 0$. That is,

$$\begin{aligned} \langle M v_1, v_2 \rangle &= (v_1, v_2) \begin{bmatrix} 0 & y \\ z & x \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= v_2 z v_1 + v_1 y v_2 + v_2 x v_2 \\ &= Re\langle z v_2, v_1 \rangle + Re\langle y v_1, v_2 \rangle + \langle x v_2, v_2 \rangle \geq 0, \end{aligned}$$

holds only if either $x = 0$ or $z = 0$, and $y \geq 0$

□

Proposition 2.3 *Let $\phi : \mathcal{M}_n \to \mathcal{M}_{n+1}$ be a 2-positive map with the Choi matrix defined by 1. Then the following conditions hold.*

(i). A, B and D are positive diagonal matrices while U are positive matrices.

(ii). If $\det A = 0$, then $C = 0$ and if $\det A > 0$ then $C^* A C \leq (\det A) B$.

(iii). If $\det A = 0$, then $Y = 0$ and if $\det A > 0$ then $Y^* A Y \leq (\det A) U$.

(iv). The matrix $\begin{bmatrix} B & T \\ T^* & U \end{bmatrix} \in \mathcal{M}_2(\mathcal{M}_k)$ *is block positive.*

Remark 2.4 *The case of where $A = a \geq 0$, $D = 0$ as real numbers and C, Y and vectors has been proved in [7, Proposition 2.1]. In this study we have given a general case where A and D are square matrices.*

Proof. Assume that ϕ is a 2-positive linear map. Then the Choi matrix,

$$C_\phi = \left[\begin{array}{c|c} \phi(E_{11}) & \phi(E_{12}) \\ \hline \phi(E_{21}) & \phi(E_{22}) \end{array}\right] \geq 0.$$

Applying Lemma 2.1 and block positivity of the Choi matrix the block diagonal entries,

$$\phi(E_{11}) = \begin{bmatrix} A & C \\ C^* & B \end{bmatrix} \text{ and } \phi(E_{22}) = \begin{bmatrix} D & 0 \\ 0 & U \end{bmatrix} \text{ are positive matrices.}$$

Let $\det A = 0$, then by Lemma 2.2 $\phi(E_{11}) \geq 0$ if and only if $C = 0$. However, if $\det A \neq 0$. Then by Lemma 2.1 $C^*AC \leq (\det A)B$. It is clear that $\phi(E_{22}) \geq 0$ since $0 \leq (\det D)U$.

For the positivity of the Choi matrix C_ϕ. We need to show also that the submatrix $\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ is positive. By the argument in Theorem 1.12 and Theorem 1.14 we have that $(\det A)D \geq 0$. Alternatively $(\det A)D = \det(AD) = (\det A)(\det D) \geq 0$, because A and D are diagonal matrices of the same order.

For the block matrix $\begin{bmatrix} B & T \\ T^* & U \end{bmatrix} \in \mathcal{M}_2 \otimes \mathcal{M}_k$, by Lemma 2.1, $T^*BT \leq (\det B)U$.

□

Here we give examples with the maps $\phi_{(\mu,\alpha)}$ and $\phi_{(\mu,\alpha,c_2)}$ from their Choi matrices.

Example 2.5 *The linear positive map $\phi_{(\mu,\alpha)}$ is 2-positive when $\mu > 0$ and $\alpha \geq 0$ for all $r \in \mathbb{R}^+$. The Choi matrix is clearly 2-positive by block positivity of $C_{\phi_{(\mu,\alpha)}}$ as the block diagonals are positive matrices. To show the conditions for positivity, the Choi matrix is represented as;*

$$C_{\phi_{(\alpha,\mu)}} = \left[\begin{array}{c|cc||c|cc} \mu^{-r} & 0 & 0 & 0 & 0 & -\mu \\ \hline 0 & \mu^{-r} & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 \\ \hline\hline 0 & 0 & 0 & \alpha & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \mu^{-r} & 0 \\ -\mu & 0 & 0 & 0 & 0 & \mu^{-r} \end{array}\right].$$

We have that $\det A = \mu^{-r}$ is undefined when $\mu = 0$ implying $\mu > 0$. $\alpha \geq 0$ since C is a zero vector as $(\det A)B \geq 0$.

$$\begin{aligned} (\det A)U - Y^*AY &= \mu^{-r}\begin{bmatrix} \mu^{-r} & 0 \\ 0 & \mu^{-r} \end{bmatrix} - \begin{bmatrix} 0 \\ -\mu \end{bmatrix} \mu^{-r} \begin{bmatrix} 0 & -\mu \end{bmatrix} \\ &= \begin{bmatrix} \mu^{-2r} & 0 \\ 0 & \mu^{-2r} - \mu^{-r+2} \end{bmatrix} \end{aligned}$$

which holds when $\mu^{-2r} - \mu^{-r+2} \geq 0$. As $r \to 0$ we have that

$$1 - \mu^2 = (1-\mu)(1+\mu) \geq 0$$

which holds when $0 < \mu \leq 1$.
Since Z is zero vector,

$$(\det D)B - Z^*DZ \;=\; \alpha \begin{bmatrix} \mu^{-r} & 0 \\ 0 & \alpha \end{bmatrix} \geq 0.$$

Finally, the matrix $T = 0$ implying $U - TB^{-1}T^* = U \geq 0$. Therefore the matrix $\begin{bmatrix} B & T \\ T^* & U \end{bmatrix}$ is a positive block matrix.

Remark 2.6 *This criterion works not only for 2-positive maps from $\mathcal{M}_2$ to $\mathcal{M}_{n+1}$ [10], [8] and [9] but also 2-positive maps from $\mathcal{M}_{2k}$ to $\mathcal{M}_{2k+1}$ since $\mathcal{M}_{2k} \to \mathcal{M}_{2k+1} \subset \mathcal{M}_n \to \mathcal{M}_{n+1}$ for all $n \in \{1, 2, \ldots\}$ and $k = 2n+1$. We note that if the order of the Choi matrix is an even integer, then the map is 2−positive. Though the example show that a map ϕ is 2-positive, it does not explicitly give the conditions for positivity of the maps as shown in the next example.*

Example 2.7 *The map $\phi_{(\mu,\alpha,\kappa)}$ is 2-positive map and has the Choi matrix when $\alpha, \kappa \geq 0$ and $0 < \mu \leq 1$,*

$$C_{\phi_{(\mu,\alpha,c_2)}} = \left[\begin{array}{ccc|ccc||ccc|ccc}
\mu^{-r} & 0 & 0 & 0 & 0 & -\alpha & 0 & 0 & 0 & 0 & 0 & -\mu \\
0 & \kappa & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\alpha & 0 & 0 & 0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & -\kappa & 0 \\
\hline\hline
0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \kappa & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \kappa & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\kappa & 0 & 0 & 0 & 0 & \mu^{-r} & 0 \\
-\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^{-r}
\end{array}\right].$$

We observe that A, B and U are positive matrices for $\mu > 0$ and $\alpha, \kappa \geq 0$, to be exact they are diagonal matrices. Since $A > 0$,

$$\begin{aligned}
(\det A)B - C^*AC &= \mu^{-2r}\kappa \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \mu^{-r} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\alpha & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu^{-r} & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \mu^{-r} \end{bmatrix} \begin{bmatrix} 0 & 0 & -\alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \alpha\kappa\mu^{-2r} & 0 & 0 \\ 0 & \alpha\kappa\mu^{-2r} & 0 \\ 0 & 0 & \kappa\mu^{-3r} - \alpha^2\mu^{-r} \end{bmatrix}.
\end{aligned}$$

This holds when

$$\kappa\mu^{-2r} \geq \alpha^2. \tag{2}$$

$$\begin{aligned}
(\det A)U - Y^*AY &= \mu^{-2r}\kappa \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \mu^{-r} & 0 \\ 0 & 0 & \mu^{-r} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\mu & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu^{-r} & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \mu^{-r} \end{bmatrix} \begin{bmatrix} 0 & 0 & -\mu \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \alpha\kappa\mu^{-2r} & 0 & 0 \\ 0 & \kappa\mu^{-3r} & 0 \\ 0 & 0 & \kappa\mu^{-3r} - \mu^{2-r} \end{bmatrix}.
\end{aligned}$$

This is positive when

$$\kappa\mu^{-2r} \geq \mu^2. \tag{3}$$

From the inequalities 2 and 3 as $r \to 0$ *we get that* $\kappa \geq \mu^2$ *and* $\kappa \geq c_1^2$.
Finally,

$$\begin{aligned} U - T^*B^{-1}T &= \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \mu^{-r} & 0 \\ 0 & 0 & \mu^{-r} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\kappa \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \mu^{-r} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\kappa & 0 \end{bmatrix} \\ &= \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \mu^{-r} - \frac{\kappa^2}{\alpha^2} & 0 \\ 0 & 0 & \mu^{-r} \end{bmatrix} \end{aligned}$$

is positive if

$$\mu^{-r} - \frac{\kappa^2}{\alpha^2} \geq 0. \tag{4}$$

From the inequalities 2, 3 and 4, $\alpha \neq 0$, $\kappa \neq 0$ *and* $\mu > \alpha$. *Thus,* $\alpha, \kappa \in [0,1]$ *and* $\mu \geq 0$.

3 Complete Positivity of Linear Positive Maps ϕ from $\mathcal{M}_n$ to $\mathcal{M}_{n+1}$

A positive map ϕ is completely positive if and only if it is k-positive. Since our map ϕ from $\mathcal{M}_n$ to $\mathcal{M}_{n+1}$ is 2-positive, we look at the conditions for complete positivity and complete copositivity of this map.

Proposition 3.1 *Let* $\phi : \mathcal{M}_n \to \mathcal{M}_{n+1}$ *be a 2-positive map with the Choi matrix of the form, 1.* ϕ *is completely positive if the following conditions hold.*

(i). $Z = 0$.

(ii). $C^*AC \leq (\det A)B$.

(iii). $(\det D)U \geq 0$.

(iv) $Y^*AY \leq (\det A)U$.

(v) the block matrix $\begin{bmatrix} B & T \\ T^* & U \end{bmatrix}$ *is positive.*

Proof. Let L_1 be a linear subspace generated by the vector e_1 and let L_2 be the subspace spanned by $e_2, e_3, \ldots, e_{n+1}$ so that $\mathbb{C}^{n+1} = L_1 \oplus L_2$. A vector $v \in \mathbb{C}^{n+1}$ can therefore be uniquely decomposed to $v = v^1 + v^2$ where $v^i \in L_i, i = 1, 2$. The Choi matrices 1 are interpreted as operators. $B, T, U : L_2 \to L_2$, $C, Y, Z : L_2 \to L_1$, and $A, D : L_1 \to L_1$. From [17], for any $v_1, v_2 \in \mathbb{C}^{n+1}$ the positivity of the Choi matrices 1 is given by the inequality,

$$\langle v_1, \begin{bmatrix} A & C \\ C^* & B \end{bmatrix} v_1\rangle + \langle v_2, \begin{bmatrix} D & 0 \\ 0 & U \end{bmatrix} v_2\rangle + \langle v_1, \begin{bmatrix} 0 & Y \\ Z^* & T \end{bmatrix} v_2\rangle + \langle v_2, \begin{bmatrix} 0 & Z \\ Y^* & T^* \end{bmatrix} v_1\rangle \geq 0.$$

which is equivalent to

$$\begin{gathered} \langle v_1^{(1)}, Av_1^{(1)}\rangle + \langle v_1^{(2)}, Bv_1^{(2)}\rangle + \langle v_2^{(1)}, Dv_2^{(1)}\rangle + \langle v_2^{(2)}, Uv_2^{(2)}\rangle \\ +2Re\langle v_1^{(1)}, Cv_1^{(2)}\rangle + 2Re\langle v_1^{(1)}, Yv_2^{(1)}\rangle + 2Re\langle v_2^{(1)}, Zv_1^{(2)}\rangle + 2Re\langle v_1^{(2)}, Tv_2^{(2)}\rangle \geq 0 \end{gathered}$$

where $v_j = v_j^{(1)} + v_j^{(2)}$ for $j = 1, 2$, and $v_1^{(1)}, v_2^{(1)} \in L_1$ and $v_1^{(2)}, v_2^{(2)} \in L_2$.
Assume that $v_1^{(1)} = v_2^{(2)} = 0$ with $v_1^{(2)}$ an arbitrary element in L_2. This gives

$$\langle v_1^{(2)}, Bv_1^{(2)}\rangle + 2Re\langle v_2^{(1)}, Zv_1^{(2)}\rangle + \langle v_2^{(1)}, Dv_2^{(1)}\rangle \geq 0.$$

Letting $v_2^{(1)} = -\alpha Z v_1^{(2)}$ for some $\alpha \geq 0$,

$$\langle v_1^{(2)}, Bv_1^{(2)}\rangle + 2Re\langle -\alpha Z v_1^{(2)}, Zv_1^{(2)}\rangle + \langle -\alpha Z v_1^{(2)}, -D\alpha Z v_1^{(2)}\rangle \geq 0.$$

This simplify to

$$\langle v_1^{(2)}, Bv_1^{(2)}\rangle - \alpha||Zv_1^{(2)}||^2 + \alpha^2||Z||^2\langle v_1^{(2)}, Dv_1^{(2)}\rangle \geq 0$$

which holds for any $v_2^{(1)} \in L_2$ and $\alpha > 0$ only for $Z = 0$.
Assume that ϕ is a 2-positive linear map. Then the Choi matrix,

$$C_\phi = \left[\begin{array}{c|c} \phi(E_{11}) & \phi(E_{12}) \\ \hline \phi(E_{21}) & \phi(E_{22}) \end{array}\right] \geq 0.$$

Applying Lemma 2.1 and block positivity of the Choi matrix the block diagonal entries,

$$\phi(E_{11}) = \begin{bmatrix} A & C \\ C^* & B \end{bmatrix} \text{ and } \phi(E_{22}) = \begin{bmatrix} D & 0 \\ 0 & U \end{bmatrix} \text{ are positive matrices.}$$

Let $\det A = 0$, then by Lemma 2.2, $\phi(E_{11}) \geq 0$ if and only if $C = 0$. However, if $\det A \neq 0$, then by Lemma. 2.1 $C^*AC \leq (\det A)B$. It is clear that $\phi(E_{22}) \geq 0$ since $0 \leq (\det D)U$.
For the positivity of the Choi matrix C_ϕ, we need to show also that the submatrices $\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ is positive. By the argument in Lemma 2.1. $0 \leq (\det A)D$. Alternatively $(\det A)D = \det(AD) = (\det A)(\det D) \geq 0$, because A and D are diagonal matrices.
By Lemma 2.1, $T^*BT \leq (\det B)U$, so $\begin{bmatrix} B & T \\ T^* & U \end{bmatrix} \in \mathcal{M}_2 \otimes \mathcal{M}_k$ is positive.

□

Proposition 3.2 *Let $\phi : \mathcal{M}_n \to \mathcal{M}_{n+1}$ be a 2-positive map with the Choi matrix of the form, 1. ϕ is completely copositive if the following conditions hold.*

(i). $Y = 0$.

(ii). $C^*AC \leq (\det A)B$.

(iii). $(\det D)U \geq 0$.

(iv) $Z^*AZ \leq (\det A)U$.

(v) if B is invertible, then $TB^{-1}T^* = U$.

Remark 3.3 *The transposition in this case imply the Partial transpose with the transpose of the Choi matrix as*

$$\left[\begin{array}{cc|cc} A & C & 0 & Z \\ C^* & B & Y^* & T^* \\ \hline 0 & Y & D & 0 \\ Z^* & T & 0 & U \end{array}\right].$$

We show the proof of (i) since the other parts of the proof follows from the proof of Theorem 3.1.

Proof. Let L_1 be a linear subspace generated by the vector e_1 and let L_2 be the subspace spanned by $e_2, e_3, \ldots, e_{n+1}$ so that $\mathbb{C}^{n+1} = L_1 \oplus L_2$. A vector $v \in \mathbb{C}^{n+1}$ can therefore be uniquely decomposed to $v = v^1 + v^2$ where $v^i \in L_i, i = 1, 2$. The Choi matrices 1 are interpreted as operators. $B, T, U : L_2 \to L_2$, $C, Y, Z : L_2 \to L_1$, and $A, D : L_1 \to L_1$. From [17], for any $v_1, v_2 \in \mathbb{C}^{n+1}$ the positivity of the Choi matrices 1 is given by the inequality,

$$\langle v_1, \begin{bmatrix} A & C \\ C^* & B \end{bmatrix} v_1\rangle + \langle v_2, \begin{bmatrix} D & 0 \\ 0 & U \end{bmatrix} v_2\rangle + \langle v_1, \begin{bmatrix} 0 & Z \\ Y^* & T \end{bmatrix} v_2\rangle + \langle v_2, \begin{bmatrix} 0 & Y \\ Z^* & T^* \end{bmatrix} v_1\rangle \geq 0.$$

which is equivalent to

$$\langle v_1^{(1)}, Av_1^{(1)}\rangle + \langle v_1^{(2)}, Bv_1^{(2)}\rangle + \langle v_2^{(1)}, Dv_2^{(1)}\rangle + \langle v_2^{(2)}, Uv_2^{(2)}\rangle$$
$$+2Re\langle v_1^{(1)}, Cv_1^{(2)}\rangle + 2Re\langle v_1^{(1)}, Zv_2^{(1)}\rangle + 2Re\langle v_2^{(1)}, Yv_1^{(2)}\rangle + 2Re\langle v_1^{(2)}, Tv_2^{(2)}\rangle \geq 0$$

where $v_j = v_j^{(1)} + v_j^{(2)}$ for $j = 1, 2$, and $v_1^{(1)}, v_2^{(1)} \in L_1$ and $v_1^{(2)}, v_2^{(2)} \in L_2$.

Assume that $v_1^{(1)} = v_2^{(2)} = 0$ and $v_1^{(2)}$ an arbitrary element in L_2. This gives

$$\langle v_1^{(2)}, Bv_1^{(2)}\rangle + 2Re\langle v_2^{(1)}, Yv_1^{(2)}\rangle + \langle v_2^{(1)}, Dv_2^{(1)}\rangle \geq 0.$$

Letting $v_2^{(1)} = -\alpha Z v_1^{(2)}$ for some $\alpha \geq 0$

$$\langle v_1^{(2)}, Bv_1^{(2)}\rangle + 2Re\langle -\alpha Y v_1^{(2)}, Yv_1^{(2)}\rangle + \langle -\alpha Y v_1^{(2)}, -D\alpha Z v_1^{(2)}\rangle \geq .0$$

This simplify to

$$\langle v_1^{(2)}, Bv_1^{(2)}\rangle - \alpha ||Yv_1^{(2)}||^2 + \alpha^2 ||Y||^2 \langle v_1^{(2)}, Dv_1^{(2)}\rangle \geq 0$$

which holds for any $v_2^{(1)} \in L_2$ and $\alpha > 0$ only for $Y = 0$.

□

4 Conclusion

In [7], [8] and [9] the authors looked at Choi matrices where $A > 0$ and $D = 0$ are scalars with matrices $C, Y, Z \in \mathcal{M}_{1\times k}$ for positive maps between $\mathcal{M}_2$ and $\mathcal{M}_n$ with $n \geq 2$. Note the operation under goes a partial transposition of the Choi matrix. The first part of the proof in Proposition 3.1 and Proposition 3.2 follow from [7, Lemma 2.3]. In our case A and D are positive diagonal square matrices. The conditions show the positivity of the Choi matrix 1 which implies complete positivity of ϕ. The conditions (i) through to (v) are necessary for the positivity of the Choi matrix as shown in Proposition 2.3.

Conflicts of Interest

Author declares no conflict of interest.

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