COMPLETELY g-$\star$-CLOSED SETS

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Abstract

In this paper, the notion of completely g-$\star$-closed sets is introduced in ideal topological spaces. Characterizations and properties of completely g-$\star$-closed sets and completely g-$\star$-open sets are given. A characterization of normal spaces is given in terms of completely g-$\star$-open sets. Also it is established that a completely g-$\star$-closed subset of an I-compact space is I-compact.

Keywords : completely g-$\star$-closed set, strongly I_g-$\star$-closed set, $\star$-g-closed set and I-compact space.

1 Introduction and preliminaries

By a space, we always mean a topological space (X, τ) with no separation properties assumed. If H$\subseteq$X, cl(H) and int(H) will, respectively, denote the closure and interior of H in (X, τ).

Example 1.1. *A subset H of a space (X, τ) is called semi-open [8] if H$\subseteq$cl(int(H)).*

Definition 1.2. *A subset H of a space (X, τ) is said to be g-closed [9] if cl(H)$\subseteq$U whenever H$\subseteq$U and U is open in X.*

An ideal I on a space (X, τ) is a nonempty collection of subsets of X which satisfies (i) A$\in I$ and B$\subseteq$A$\Rightarrow$B$\in I$ and (ii) A$\in I$ and B$\in I\Rightarrow$A$\cup$B$\in I$ [7]. Given a space (X, τ) with an ideal I on X and if $\wp$(X) is the set of all subsets of X, a set operator $(.)^\star$: $\wp$(X)$\rightarrow\wp$(X), called a local function [7] of A with respect to τ and I, is defined as follows: for A$\subseteq$X, $A^\star(I,\tau)=\{x\in X \mid U\cap A\notin I$ for every $U\in\tau(x)\}$ where $\tau(x)=\{U\in\tau \mid x\in U\}$. We will make use of the basic facts about the local functions [[6], Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator $cl^\star(.)$ for a topology $\tau^\star(I,\tau)$, called the $\star$-topology, finer than τ is defined by $cl^\star(A)=A\cup A^\star(I,\tau)$ [14]. When there is no chance for confusion, we will simply write $A^\star$ for $A^\star(I,\tau)$ and $\tau^\star$ for $\tau^\star(I,\tau)$.

If I is an ideal on X, then (X, τ, I) is called an ideal topological space.

Lemma 1.3. *[6] Let (X, τ, I) be an ideal topological space and A, B subsets of X. Then the following properties hold:*

1. $A\subseteq B\Rightarrow A^\star\subseteq B^\star$,
2. $A^\star=cl(A^\star)\subseteq cl(A)$,
3. $(A^\star)^\star\subseteq A^\star$,
4. $(A\cup B)^\star=A^\star\cup B^\star$,

5. $(A\cap B)^\star\subseteq A^\star\cap B^\star$.

Definition 1.4. *A subset H of an ideal topological space (X, τ, I) is called $\star$-closed [6] (resp. $\star$-dense in itself [5]) if $H^\star\subseteq H$ (resp. $H\subseteq H^\star$). The complement of a $\star$-closed set is called $\star$-open.*

Definition 1.5. *A subset H of an ideal topological space (X, τ, I) is called I_g-closed [2, 11] if $H^\star\subseteq U$ whenever $H\subseteq U$ and U is open in (X, τ, I).*

Definition 1.6. *[2] An ideal topological space (X, τ, I) is called T_I if every I_g-closed subset of X is $\star$-closed in X.*

Lemma 1.7. *If (X, τ, I) is a T_I space and $A\subseteq X$ is an I_g-closed set, then A is a $\star$-closed set [[11], Corollary 2.2].*

Lemma 1.8. *In an ideal topological space (X,τ,I), every g-closed set is I_g-closed but not conversely [[2], Theorem 2.1].*

Definition 1.9. *[10] A subset H of an ideal topological space (X, τ, I) is said to be*

1. *$\star$-g-closed if $cl(H)\subseteq U$ whenever $H\subseteq U$ and U is $\star$-open in (X, τ, I),*
2. *$\star$-g-open if its complement is $\star$-g-closed.*

Recall that every open set is $\star$-g-open but not conversely.

Proposition 1.10. *[1] If A is $\star$-g-closed of (X, τ, I) and B is closed in X, then $A\cap B$ is $\star$-g-closed in (X, τ, I).*

Definition 1.11. *[1] A subset A of an ideal topological space (X,τ,I) is said to be*

1. *strongly I_g-$\star$-closed if $A^\star\subseteq U$ whenever $A\subseteq U$ and U is $\star$-g-open in (X, τ, I).*
2. *strongly I_g-$\star$-open if its complement is strongly I_g-$\star$-closed.*

Theorem 1.12. *[1] In an ideal topological space (X, τ, I), for $A\subseteq X$, the following statements are equivalent.*

1. *A is strongly I_g-$\star$-closed,*
2. *$cl^\star(A)\subseteq U$ whenever $A\subseteq U$ and U is $\star$-g-open in X,*
3. *$cl^\star(A)-A$ contains no nonempty $\star$-g-closed set,*
4. *$A^\star-A$ contains no nonempty $\star$-g-closed set.*

Theorem 1.13. *[1] Let (X, τ, I) be an ideal topological space. If A and B are subsets of X such that $A\subseteq B\subseteq cl^\star(A)$ and A is strongly I_g-$\star$-closed, then B is strongly I_g-$\star$-closed.*

Definition 1.14. *An ideal I is said to be codense [3] or τ-boundary [12] if $\tau\cap I=\{\phi\}$.*

Theorem 1.15. *[1] In an ideal topological space (X,τ,I), every $\star$-closed set is strongly I_g-$\star$-closed but not conversely.*

Theorem 1.16. *[1] In an ideal topological space (X,τ,I), every strongly I_g-$\star$-closed set is I_g-closed but not conversely.*

Lemma 1.17. *Let (X, τ, I) be an ideal topological space. Then I is codense if and only if $G\subseteq G^\star$ for every semi-open set G in X [[13], Theorem 3].*

Lemma 1.18. *Let (X, τ, I) be an ideal topological space and $A\subseteq X$. If $A\subseteq A^\star$, then $A^\star=cl(A^\star)=cl(A)=cl^\star(A)$ [[13], Theorem 5].*

Definition 1.19. *A subset H of an ideal topological space (X, τ, I) is said to be I-compact [4] or compact modulo I [12] if for every open cover $\{U_\alpha \mid \alpha \in \Delta\}$ of H, there exists a finite subset Δ_0 of Δ such that $H - \cup\{U_\alpha \mid \alpha \in \Delta_0\} \in I$. The space (X, τ, I) is I-compact if X is I-compact as a subset.*

Theorem 1.20. *Let (X, τ, I) be an ideal topological space. If A is an I_g-closed subset of X, then A is I-compact [[11], Theorem 2.17].*

Definition 1.21. *[1] A subset A of an ideal topological space (X, τ, I) is called a $\star$-g-I-locally closed set (briefly, $\star$-g-I-LC) if $A = U \cap V$ where U is $\star$-g-open and V is $\star$-closed.*

2 Properties of completely g-$\star$-closed sets

Definition 2.1. *A subset A of an ideal topological space (X, τ, I) is said to be*

1. *completely g-$\star$-closed if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\star$-g-open in X,*
2. *completely g-$\star$-open if its complement is completely g-$\star$-closed.*

Theorem 2.2. *In an ideal topological space (X, τ, I), every completely g-$\star$-closed set is g-closed.*

Proof. It follows from the fact that every open set is $\star$-g-open. □

The converses of Theorem 2.2 is not true in general as shown in the following example.

Example 2.3. *Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b, c\}\}$ and $I = \{\phi, \{a\}\}$. Then completely g-$\star$-closed sets are ϕ, X, $\{a\}$, $\{b, c\}$ and g-closed sets are ϕ, X, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$. Clearly $\{b\}$ is g-closed but not completely g-$\star$-closed.*

The following Theorem gives characterizations of completely g-$\star$-closed sets.

Theorem 2.4. *In an ideal topological space (X, τ, I), for $A \subseteq X$, the following statements are equivalent.*

1. *A is completely g-$\star$-closed,*
2. *$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\star$-g-open in X,*
3. *$cl(A) - A$ contains no nonempty $\star$-g-closed set.*

Proof. (1) ⇒ (2) Let A ⊆ U where U is $\star$-g-open in X. Since A is completely g-$\star$-closed, cl(A) ⊆ U.

(2) ⇒ (3) Let F be a $\star$-g-closed subset such that F ⊆ cl(A) − A. Then F ⊆ cl(A). Also F ⊆ cl(A) − A ⊆ X − A and hence A ⊆ X − F where X − F is $\star$-g-open. By (2) cl(A) ⊆ X − F and so F ⊆ X − cl(A). Thus F ⊆ cl(A) ∩ (X − cl(A)) = ϕ.

(3) ⇒ (1) Let A ⊆ U where U is $\star$-g-open in X. Then X − U ⊆ X − A and so cl(A) ∩ (X − U) ⊆ cl(A) ∩ (X − A) = cl(A) − A. Since cl(A) is always a closed subset and X − U is $\star$-g-closed, cl(A) ∩ (X − U) is a $\star$-g-closed set contained in cl(A) − A and hence cl(A) ∩ (X − U) = ϕ by (3). Thus cl(A) ⊆ U and A is completely g-$\star$-closed.

Theorem 2.5. *Every closed set is completely g-$\star$-closed.*

Proof. Let A be closed. To prove A is completely g-$\star$-closed, let U be any $\star$-g-open subset such that A $\subseteq$ U. Since A is closed, cl(A) $\subseteq$ A $\subseteq$ U. Thus A is completely g-$\star$-closed.

The converse of Theorem 2.5 is not true in general as shown in the following example.

Example 2.6. *Let X={a, b, c, d}, τ={ϕ, X, {d}, {a, c}, {a, c, d}} and I={ϕ, {a}, {d}, {a, d}}. Then completely g-$\star$-closed sets are ϕ, X, {b}, {a, b}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {b, c, d} and closed sets are ϕ, X, {b}, {b, d}, {a, b, c}. Clearly {b, c} is completely g-$\star$-closed but not closed.*

Theorem 2.7. *In an ideal topological space (X, τ, I), $A^\star$ is always completely g-$\star$-closed for every subset A of X.*

Proof. Let $A^\star \subseteq$U where U is $\star$-g-open in X. Since $A^\star$ is closed, cl($A^\star$)$\subseteq A^\star \subseteq$U. Hence $A^\star$ is completely g-$\star$-closed.

Theorem 2.8. *Let (X, τ, I) be an ideal topological space. Then every completely g-$\star$-closed, $\star$-g-open set is closed.*

Proof. Let A be completely g-$\star$-closed and $\star$-g-open. We have A $\subseteq$ A where A is $\star$-g-open. Since A is completely g-$\star$-closed, cl(A) $\subseteq$ A. Thus A is closed.

Corollary 2.9. *If (X, τ, I) is a T_I space and A is a completely g-$\star$-closed set, then A is $\star$-closed set.*

Proof. By assumption A is completely g-$\star$-closed in (X, τ, I) and so by Theorem 2.2, A is g-closed and hence I_g-closed by Lemma 1.8. Since (X, τ, I) is a T_I-space, by Definition 1.6, A is $\star$-closed.

Corollary 2.10. *Let A be a completely g-$\star$-closed set in (X, τ, I). Then the following are equivalent.*

1. *A is a closed set,*
2. *cl(A) $-$ A is a $\star$-g-closed set.*

Proof. (1) $\Rightarrow$ (2) By (1) A is closed. Hence cl(A) $\subseteq$ A and cl(A) $-$ A $=$ ϕ which is a $\star$-g-closed set.

(2) $\Rightarrow$ (1) Since A is completely g-$\star$-closed, by Theorem 2.4(3), cl(A) $-$ A contains no non-empty $\star$-g-closed set. By assumption (2), cl(A) $-$ A is $\star$-g-closed and hence cl(A) $-$ A $=$ ϕ. Thus cl(A) $\subseteq$ A and hence A is closed.

Theorem 2.11. *In an ideal topological space (X, τ, I), every completely g-$\star$-closed set is strongly I_g-$\star$-closed.*

Proof. Let A be a completely g-$\star$-closed set. Let U be any $\star$-g-open set such that A $\subseteq$ U. Since A is completely g-$\star$-closed, cl(A) $\subseteq$ U. So, $A^\star \subseteq A \cup A^\star = cl^\star(A) \subseteq$cl(A)$\subseteq$U and thus A is strongly I_g-$\star$-closed.

The converse of Theorem 2.11 is not true in general as shown in the following example.

Example 2.12. *In Example 2.6, strongly I_g-$\star$-closed sets are ϕ, X, {a}, {b}, {d}, {a, b}, {a, d}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {b, c, d}. Clearly {a} is strongly I_g-$\star$-closed but not completely g-$\star$-closed.*

Theorem 2.13. *If (X, τ, I) is an ideal topological space and A is a $\star$-dense in itself, strongly I_g-$\star$-closed subset of X, then A is completely g-$\star$-closed.*

Proof. Let A $\subseteq$ U where U is $\star$-g-open in X. Since A is strongly I_g-$\star$-closed, $A^\star \subseteq$ U. As A is $\star$-dense in itself, by Lemma 1.18, cl(A) $=$ $A^\star$. Thus cl(A)$\subseteq$U and hence A is completely g-$\star$-closed.

Corollary 2.14. *If (X, τ, I) is any ideal topological space where $I=\{\phi\}$, then A is strongly I_g-$\star$-closed in X if and only if A is completely g-$\star$-closed in X.*

Proof. In (X, τ, I), if $I = \{\phi\}$ then $A^\star$ = cl(A) for the subset A. A is strongly I_g-$\star$-closed in X $\Leftrightarrow$ $A^\star \subseteq$ U whenever A $\subseteq$ U and U is $\star$-g-open in X $\Leftrightarrow$ cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is $\star$-g-open in X $\Leftrightarrow$ A is completely g-$\star$-closed in X.

Corollary 2.15. *In an ideal topological space (X, τ, I) where I is codense, if A is a semi-open and strongly I_g-$\star$-closed subset of X, then A is completely g-$\star$-closed.*

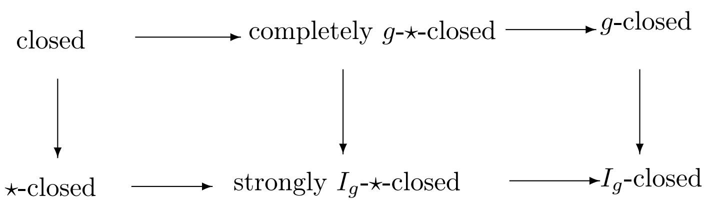
Proof. By Lemma 1.17, A is $\star$-dense in itself. By Theorem 2.13, A is completely g-$\star$-closed.

Example 2.16. *In Example 2.3, strongly I_g-$\star$-closed sets are ϕ, X, {a}, {b, c} and g-closed sets are ϕ, X, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}. Clearly {b} is g-closed but not strongly I_g-$\star$-closed.*

Example 2.17. *In Example 2.6, g-closed sets are ϕ, X, {b}, {a, b}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {b, c, d}. Clearly {a} is strongly I_g-$\star$-closed but not g-closed.*

Remark 2.18. *We see that from Examples 2.16 and 2.17, g-closed sets and strongly I_g-$\star$-closed sets are independent.*

Remark 2.19. *We have the following implications for the subsets stated above.*



Theorem 2.20. *Let (X, τ, I) be an ideal topological space and A$\subseteq$X. Then A is completely g-$\star$-closed if and only if A=F$-$N where F is closed and N contains no nonempty $\star$-g-closed set.*

Proof. If A is completely g-$\star$-closed, then by Theorem 2.4(3), N=cl(A) $-$ A contains no nonempty $\star$-g-closed set. If F=cl(A), then F is closed such that F$-$N = (A$\cup$cl(A))
$-$(cl(A)$-$A)=(A$\cup$cl(A))$\cap$(cl(A)$\cap A^c)^c$=(A$\cup$cl(A))$\cap$((cl(A))^c $\cup$ A)=(A$\cup$cl(A))$\cap$(A$\cup$
(cl(A))^c)=A$\cup$(cl(A)$\cap$(cl(A))^c) =A.

Conversely, suppose A = F$-$N where F is closed and N contains no nonempty $\star$-g-closed set. Let U be an $\star$-g-open set such that A$\subseteq$U. Then F$-$N$\subseteq$U which implies that F$\cap$(X$-$U)$\subseteq$N. Now A$\subseteq$F and cl(F)$\subseteq$F then cl(A)$\subseteq$cl(F) and so cl(A)$\cap$(X$-$U)$\subseteq$cl(F)$\cap$ (X$-$U)$\subseteq$F$\cap$(X$-$U)$\subseteq$N. Since cl(A)$\cap$(X$-$U) is $\star$-g-closed, by hypothesis cl(A)$\cap$(X$-$U) = ϕ and so cl(A)$\subseteq$U. Hence A is completely g-$\star$-closed.

Theorem 2.21. *Let (X, τ, I) be an ideal topological space. If A and B are subsets of X such that A$\subseteq$B$\subseteq$cl(A) and A is completely g-$\star$-closed, then B is completely g-$\star$-closed.*

Proof. Since A is completely g-$\star$-closed, then by Theorem 2.4(3), cl(A)$-$A contains no nonempty $\star$-g-closed set. But cl(B)$-$B$\subseteq$cl(A)$-$A and so cl(B)$-$B contains no nonempty $\star$-g-closed set. Hence B is completely g-$\star$-closed.

Corollary 2.22. *Let (X, τ, I) be an ideal topological space. If A and B are subsets of X such that A$\subseteq$B$\subseteq A^\star$ and A is strongly I_g-$\star$-closed, then A and B are completely g-$\star$-closed sets.*

Proof. Let A and B be subsets of X such that A⊆B⊆$A^\star$. Then A⊆B⊆$A^\star$⊆$A^\star$∪A = $cl^\star$(A). Since A is strongly I_g-$\star$-closed, by Theorem 1.13, B is strongly I_g-$\star$-closed. Since A⊆B⊆$A^\star$, we have $A^\star = B^\star$. Hence A ⊆ $A^\star$ and B ⊆ $B^\star$. Thus A is $\star$-dense in itself and B is $\star$-dense in itself and by Theorem 2.13, A and B are completely g-$\star$-closed.

The following Theorem gives a characterization of completely g-$\star$-open sets.

Theorem 2.23. *Let (X, τ, I) be an ideal topological space and A⊆X. Then A is completely g-⋆-open if and only if F⊆int(A) whenever F is ⋆-g-closed and F⊆A.*

Proof. Suppose A is completely g-$\star$-open. If F is $\star$-g-closed and F⊆A, then X−A⊆X−F and so cl(X−A)⊆X−F by Theorem 2.4(2). Therefore F⊆X−cl(X−A)=int(A). Hence F⊆int(A).

Conversely, suppose the condition holds. Let U be an $\star$-g-open set such that X−A⊆U. Then X−U⊆A and so X−U⊆int(A). Therefore cl(X−A)⊆U. By Theorem 2.4(2), X−A is completely g-$\star$-closed. Hence A is completely g-$\star$-open.

Corollary 2.24. *Let (X, τ, I) be an ideal topological space and A⊆X. If A is completely g-⋆-open, then F⊆int(A) whenever F is closed and F⊆A.*

The following Theorem gives a property of completely g-$\star$-closed.

Theorem 2.25. *Let (X, τ, I) be an ideal topological space and A,B⊆X. If A is completely g-⋆-open and int(A)⊆B⊆A, then B is completely g-⋆-open.*

Proof. Since int(A) ⊆ B ⊆ A, we have X − A ⊆ X − B ⊆ X − int(A) = cl(X − A). By assumption A is completely g-$\star$-open and so X − A is completely g-$\star$-closed. Hence by Theorem 2.21, X − B is completely g-$\star$-closed and B is completely g-$\star$-open.

The following Theorem gives a characterization of completely g-$\star$-closed sets in terms of completely g-$\star$-open sets.

Theorem 2.26. *Let (X, τ, I) be an ideal topological space and A⊆X. Then the following are equivalent.*

1. *A is completely g-⋆-closed,*
2. *A∪(X−cl(A)) is completely g-⋆-closed,*
3. *cl(A)−A is completely g-⋆-open.*

Proof. (1) ⇒ (2). Suppose A is completely g-$\star$-closed. If U is any $\star$-g-open set such that (A∪(X−cl(A)))⊆U, then X−U⊆X−(A∪(X−cl(A)))=$[A∪(cl(A))^c]^c$=cl(A)∩A^c=cl(A)− A. Since A is completely g-$\star$-closed, by Theorem 2.4(3), it follows that X−U=ϕ and so X=U. Since X is the only $\star$-g-open set containing A∪(X−cl(A)), clearly, A∪(X−cl(A)) is completely g-$\star$-closed.

(2) ⇒ (1). Suppose A∪(X−cl(A)) is completely g-$\star$-closed. If F is any $\star$-g-closed set such that F⊆cl(A)−A = X−(A∪(X−cl(A))), then A∪(X−cl(A))⊆X−F and X−F is $\star$-g-open. Therefore, cl(A∪(X−cl(A)))⊆X−F which implies that cl(A)⊆cl(A)∪cl(X− cl(A)) = cl(A∪(X−cl(A)))⊆X−F and so F⊆X−cl(A). Since F⊆cl(A), it follows that F=ϕ. Hence A is completely g-$\star$-closed by Theorem 2.4(3).

The equivalence of (2) and (3) follows from the fact that X−(cl(A)−A)=A∪(X−cl(A)).

Theorem 2.27. *Let (X, τ, I) be an ideal topological space. Then every subset of X is completely g-$\star$-closed if and only if every $\star$-g-open set is closed.*

Proof. Suppose every subset of X is completely g-$\star$-closed. Let U be any $\star$-g-open in X. Then U $\subseteq$ U and U is completely g-$\star$-closed by assumption implies cl(U) $\subseteq$ U. Hence U is closed.

Conversely, let A $\subseteq$ X and U be any $\star$-g-open such that A $\subseteq$ U. Since U is closed by assumption, we have cl(A) $\subseteq$ cl(U) $\subseteq$ U. Thus A is completely g-$\star$-closed.

The following Theorem gives a characterization of normal spaces in terms of completely g-$\star$-open sets.

Theorem 2.28. *Let (X, τ, I) be an ideal topological space. Then the following are equivalent.*

1. *X is normal,*
2. *For any disjoint closed sets A and B, there exist disjoint completely g-$\star$-open sets U and V such that A$\subseteq$U and B$\subseteq$V,*
3. *For any closed set A and open set V containing A, there exists a completely g-$\star$-open set U such that A$\subseteq$U$\subseteq$cl(U)$\subseteq$V.*

Proof. (1)$\Rightarrow$(2) The proof follows from the fact that every open set is completely g-$\star$-open.

(2)$\Rightarrow$(3) Suppose A is closed and V is an open set containing A. Since A and X$-$V are disjoint closed sets, there exist disjoint completely g-$\star$-open sets U and W such that A$\subseteq$U and X$-$V$\subseteq$W. Since X$-$V is $\star$-g-closed and W is completely g-$\star$-open, X$-$V$\subseteq$int(W). Then X$-$int(W)$\subseteq$V. Again U$\cap$W=ϕ which implies that U$\cap$int(W)=ϕ and so U$\subseteq$X$-$int(W). Then cl(U)$\subseteq$X$-$int(W)$\subseteq$V and thus U is the required completely g-$\star$-open set with A$\subseteq$U$\subseteq$cl(U)$\subseteq$V.

(3)$\Rightarrow$(1) Let A and B be two disjoint closed subsets of X. Then A is a closed set and X$-$B an open set containing A. By hypothesis, there exists a completely g-$\star$-open set U such that A$\subseteq$U$\subseteq$cl(U)$\subseteq$X$-$B. Since U is completely g-$\star$-open and A is $\star$-g-closed we have A$\subseteq$int(U). Hence A$\subseteq$int(U)=G and B$\subseteq$X$-$cl(U)=H. G and H are the required disjoint open sets containing A and B respectively, which proves (1).

Corollary 2.29. *Let (X, τ, I) be an ideal topological space. If A is a completely g-$\star$-closed subset of X, then A is I-compact.*

Proof. The proof follows from the fact that every completely g-$\star$-closed set is g-closed by Theorem 2.2 and hence I_g-closed by Lemma 1.8. By Theorem 1.20, A is I-compact.

3 $\star\star$-g-I-locally closed sets

Definition 3.1. *A subset A of an ideal topological space (X, τ, I) is called a $\star\star$-g-I-locally closed set (briefly, $\star\star$-g-I-LC) if A=U$\cap$V where U is $\star$-g-open and V is closed.*

Definition 3.2. *[1] A subset A of an ideal topological space (X, τ, I) is called a $\star$-g-I-locally closed set (briefly, $\star$-g-I-LC) if A=U$\cap$V where U is $\star$-g-open and V is $\star$-closed.*

Proposition 3.3. *Let (X, τ, I) be an ideal topological space and A a subset of X. Then the following hold.*

1. *If A is closed, then A is* $\star\star$*-g-I-LC-set.*
2. *If A is* $\star$*-g-open, then A is* $\star\star$*-g-I-LC-set.*
3. *If A is a* $\star\star$*-g-I-LC-set, then A is a* $\star$*-g-I-LC-set.*

The converses of Proposition 3.3 need not be true as shown in the following Examples.

Example 3.4.
1. *In Example 2.6,* $\star\star$*-g-I-LC-sets are* ϕ*, X, {b}, {c}, {d}, {a, c}, {b, d}, {c, d}, {a, b, c}, {a, c, d} and closed sets are* ϕ*, X, {b}, {b, d}, {a, b, c}. Clearly {c} is a* $\star\star$*-g-I-LC-set but it is not closed.*
2. *In Example 2.6,* $\star$*-g-open sets are* ϕ*, X, {c}, {d}, {a, c}, {c, d}, {a, c, d}. Clearly {b} is a* $\star\star$*-g-I-LC-set but it is not* $\star$*-g-open.*

Example 3.5. *In Example 2.6,* $\star$*-g-I-LC-sets are* ϕ*, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}. Clearly {a} is a* $\star$*-g-I-LC-set but it is not a* $\star\star$*-g-I-LC-set.*

Theorem 3.6. *Let (X,* τ*, I) be an ideal topological space. If A is a* $\star\star$*-g-I-LC-set and B is a closed set, then A∩B is a* $\star\star$*-g-I-LC-set.*

Proof. Let B be closed, then A∩B=(U∩V)∩B=U∩(V∩B), where V∩B is closed. Hence A∩B is a $\star\star$-*g*-*I*-LC-set.

Theorem 3.7. *A subset of an ideal topological space (X,* τ*, I) is closed if and only if it is* $\star\star$*-g-I-LC and completely g-*$\star$*-closed.*

Proof. Necessity is trivial. We prove only sufficiency. Let A be $\star\star$-*g*-*I*-LC-set and completely *g*-$\star$-closed set. Since A is $\star\star$-*g*-*I*-LC, A=U∩V, where U is $\star$-*g*-open and V is closed. So, we have A=U∩V⊆U. Since A is completely *g*-$\star$-closed, cl(A) ⊆ U. Also since A = U∩V⊆V and V is closed, we have cl(A) ⊆ V. Consequently, cl(A) ⊆U∩V = A and hence A is closed.

Remark 3.8. *The notions of* $\star\star$*-g-I-LC-set and completely g-*$\star$*-closed set are independent.*

Example 3.9. *In Example 2.6, clearly {c} is a* $\star\star$*-g-I-LC-set but not completely g-*$\star$*-closed.*

Example 3.10. *In Example 2.6, clearly {b, c} is completely g-*$\star$*-closed but not a* $\star\star$*-g-I-LC-set.*

4 Decomposition of continuity

Definition 4.1. *A function* $f : (X, \tau, I) \to (Y, \sigma)$ *is said to be* $\star\star$*-g-I-LC-continuous (resp. completely g-*$\star$*-continuous) if* $f^{-1}(A)$ *is* $\star\star$*-g-I-LC-set (resp. strongly* I_g*-*$\star$*-closed) in (X,* τ*, I) for every closed set A of (Y,* σ*).*

Theorem 4.2. *A function* $f : (X, \tau, I) \to (Y, \sigma)$ *is continuous if and only if it is* $\star\star$*-g-I-LC-continuous and completely g-*$\star$*-continuous.*

Proof. It is an immediate consequence of Theorem 3.7.

Conclusions

By researching generalizations of closed sets in various fields in general topology, some generalizations have been introduced and they turn out to be useful in the study of digital topology. Therefore, all functions defined in this paper will have many possibilities of applications in digital topology and computer graphics.

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