Conformable Derivative and Fractal Derivative of Functions on the Interval [0,1]

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Abstract

Recently, a calculus-based fractals, called F$^\alpha$-calculus, has been developed which involve F$^\alpha$-integral, conjugate to the Riemann integral, and F$^\alpha$-derivative, conjugate to ordinary derivative, of orders α, $0 < \alpha < 1$, where α is dimension of F. In F$^\alpha$-calculus the staircase function has a special role. In this paper we obtain fractal Taylor series for fractal elementary functions, sine, cosine, exponential function, etc. and then we compare the graph of these fractal functions with their counterparts in standard calculus on the interval [0,1]. Then, the main part of the paper is discussed about the transition from continuous state to discrete state when we do fractal differentiation in which characteristic function $\chi_C(x)$ appears. Since F^α-derivative is local, we compare it with conformable derivative which is also local. Moreover, fractal differential equations for fractal sine, cosine, sine hyperbolic, cosine hyperbolic and exponential functions are deduced. We also represent Pythagorean trigonometric identity for sin and cosine, and hyperbolic sine and hyperbolic cosine in F$^\alpha$-calculus, respectively.

Keywords: Conformable derivative, Fractal calculus, Fractal dimension, F$^\alpha$-derivative.

MSC 2010: 26A33; 28A80.

1 Introduction

In ordinary calculus, we deal with discontinuity, lack of continuity, in some points or intervals. There are also some situations where a derivative of a function fails to exist. Discontinuity and non-differentiability are two common problems in ordinary calculus. On the other hand, we observe fractals [1, 2] which are continuous or discontinuous, and usually nowhere differentiable.

Fractals are often so irregular that defining smooth, differentiable structures on them seems very difficult. To study fractals some remarkable approaches have been used. They include fractal geometry, analysis on fractals, Harmonic analysis on fractals and in the past few years fractal calculus [5–7]. To apply the methods of ordinary calculus on fractals are powerless or inapplicable. Weierstrass function or Cantor staircase function are examples of this.

In mathematics, as an interesting example of scaling function, the Weierstrass function [3] which is continuous everywhere but differentiable nowhere was defined as a Fourier series:

$$f(x) = \sum_{n=0}^{\infty} a^n (\cos b^n \pi x), \tag{1}$$

where $0 < a < 1, b$ is a positive odd integer, and $ab > 1 + \frac{3}{2}\pi$.

Cantor staircase function [4] is any real valued function $F(x)$ on [0,1] which is monotone increasing and satisfies

- $F(0) = 0$,
- $F(\frac{x}{3}) = \frac{F(x)}{2}$,
- $F(1 - x) = 1 - F(x)$.

In [5,6] a new calculus based on fractal subsets of the real line is formulated which involves an integral of order $\alpha, 0 < \alpha < 1$, called F^α-integral and a derivative of order $\alpha, 0 < \alpha < 1$, called F^α-derivative. This enables us to differentiate functions, like the Cantor staircase, "changing" only on a fractal set. The F^α-derivative is local, unlike the classical fractional derivative. Several results inF^α-calculus are analogous to corresponding results in ordinary calculus, such as the Leibniz rule, fundamental theorems, etc. They generalize their work in $\mathbf{R}^n$ [7] so that this time a new calculus on fractal curves, such as the von Koch curve, is formulated. A Riemann-like integral along a fractal curve F, called F^α-integral, is defined where α is the dimension of F. A derivative along the fractal curve called F^α-derivative, is also defined. In this formulation, the mass function, a measure-like algorithmic quantity on the curves, plays a special role. The main concepts in fractal calculus are flag function, mass function, integral staircase function, set of change of a staircase function, compact set, α-perfect set, and characteristic function, see Fig. 1. Fractal calculus has found many applications in physics and engineering [8–13].

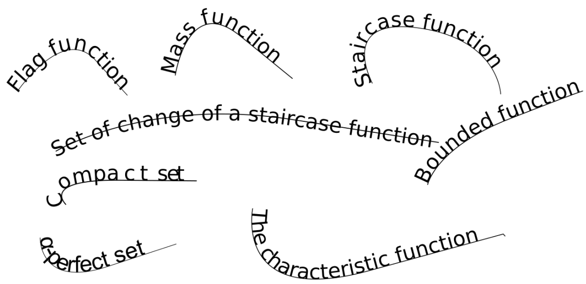

Figure 1: The main concepts in F^α-calculus

If F is an α-perfect set then the F^α-derivative of f at x is

$$\mathcal{D}_F^\alpha(f(x)) = \left\{ \begin{array}{ll} F - \lim_{y\to x} \frac{f(y)-f(x)}{S_F^\alpha(y)-S_F^\alpha(x)} & \text{if } x \in F \\ 0 & \text{othervise,} \end{array} \right\} \tag{2}$$

if limit exist. The α-perfect sets are sets having properties necessary to define F^α-derivative. As the first order derivative, the F^α-derivative is a limit of a quotient. But here the limit is F-limit, and the denominator is the difference in the values of the staircase function S_F^α at two points. Moreover, intuitively speaking, F is typically the set of change of the function, and α is typically the γ-dimension of F.
Limit based (local fractional derivative) [14]
Given a function $f : [0, \infty) \to \mathbb{R}$

$$T_\alpha(f)(t) = \lim_{\epsilon\to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}.$$

for all $t > 0, \alpha \in (0, 1)$. If f is α-differentiable in some $(0, a), a > 0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exist, then define $f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t)$. An outstanding feature in this definition is the relationship between the conformable derivative and the first- order ordinary derivative which is expressed by [14,15]

$$T_\alpha f(x) = x^{1-\alpha} \acute{f}(x) \qquad 0 < \alpha \leq 1 \quad . \tag{3}$$

If $\alpha = 1$, then $T_1 f(x) = \acute{f}(x)$, which means that the conformable derivative is a generalization of the first integer-order derivative. It should be pointed out that the term $x^{1-\alpha}$ in the above definition is not essential and it is just one kind of conformable derivative.

Conformable derivative of certain functions

1. $T^\alpha x^p = px^{p-\alpha}$ for all $p \in \mathbb{R}$.
2. $T^\alpha(1) = 0$
3. $T^\alpha(e^{cx}) = cx^{1-\alpha}e^{cx},\ c \in \mathbb{R}$
4. $T^\alpha(\sin bx) = bx^{1-\alpha}\cos bx,\ b \in \mathbb{R}$

2 Main Results

2.1 Fractal sine function $\sin(S_F^\alpha(x))$

First of all, we define fractal sine function by replacing argument x with $S_F^\alpha(x)$ in $\sin x$. So we have fractal sine function as $\sin(S_F^\alpha(x))$. This function on the interval [0,1] and for Cantor set can be represented as its fractal Taylor series. A fractal Taylor series can be written for functions on Fractal curve F as follow

$$h(S_F^\alpha(x)) = \sum_{n=0}^{\infty} \frac{[S_F^\alpha(x) - S_F^\alpha(\acute{x})]^n}{n!} (D_F^\alpha)^n h(S_F^\alpha(\acute{x})). \tag{4}$$

Let $h(S_F^\alpha(x)) = \sin(S_F^\alpha(x))$. For $S_F^\alpha(\acute{x}) = 0, \sin(S_F^\alpha(\acute{x}))|_{\acute{x}=0} = 0$

$$D_F^\alpha \sin(S_F^\alpha(\acute{x}))|_{\acute{x}=0} = \cos(S_F^\alpha(\acute{x}))\chi_F(\acute{x})|_{\acute{x}=0} = 1.$$

$$(D_F^\alpha)^2 \sin(S_F^\alpha(\acute{x}))|_{\acute{x}=0} = D_F^\alpha \cos(S_F^\alpha(\acute{x}))|_{\acute{x}=0} = 0.$$

So for Cantor set $F = C$ we have

$$\sin(S_C^\alpha(x)) = S_C^\alpha(x) - \frac{(S_C^\alpha(x))^3}{3!} + \frac{(S_C^\alpha(x))^5}{5!} - +... \quad . \tag{5}$$

Now that we obtain fractal Taylor series for fractal sine function, we compare graph of $\sin 2\pi x$ with $\sin(2\pi S_C^\alpha(x))$.

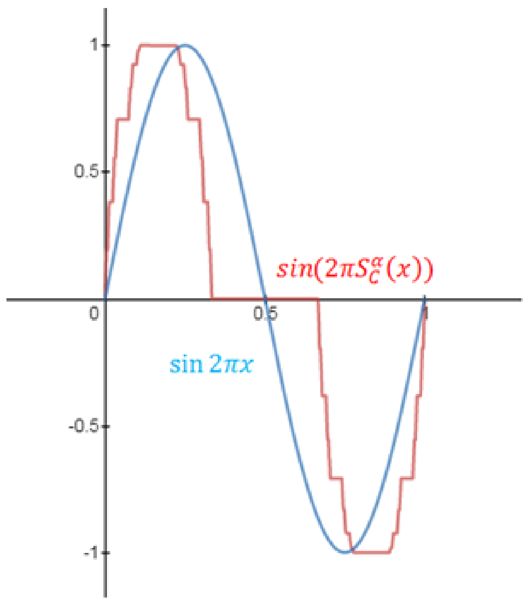

Figure 2: Comparing graph of $\sin 2\pi x$ and $\sin(2\pi S_C^\alpha(x))$.

The values of $\sin x$ and $\sin(S_C^\alpha(x))$ have been shown in Table 1. at $x = 0, 0.1, ..., 0.9$.

2.2 Fractal cosine function $\cos(S_F^\alpha(x))$

The fractal cosine function is defined as similarly as fractal sine function such that for $F = C$

$$\cos(S_C^\alpha(x)) = 1 - \frac{(S_C^\alpha(x))^2}{2!} + \frac{(S_C^\alpha(x))^4}{4!} - +... \quad . \tag{6}$$

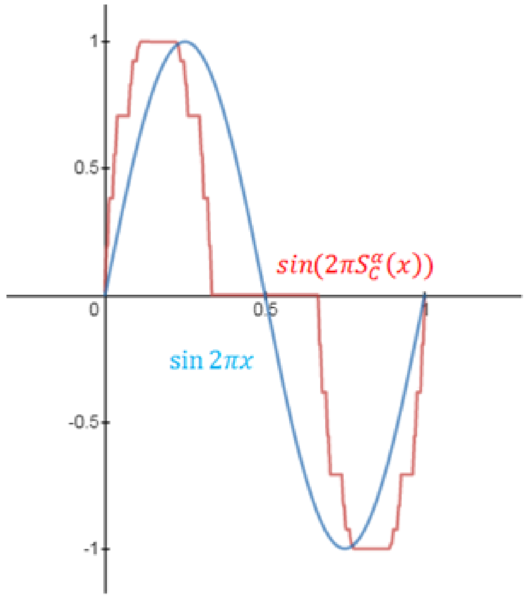

Figure 3: Comparing graph of $\cos 2\pi x$ and $\cos(2\pi S_C^\alpha(x))$.

2.3 Fractal exponential function $e^{S_C^\alpha(x)}$

The fractal Taylor series for the fractal exponential function $e^{S_C^\alpha(x)}$ at $S_C^\alpha(0) = 0$ is deduced as

$$e^{S_C^\alpha(x)} = \sum_{n=0}^{\infty} \frac{(S_C^\alpha(x))^n}{n!}. \tag{7}$$

After obtaining fractal Taylor series for fractal exponential function, we may compare both graph of $\exp x$ and $e^{S_C^\alpha(x)}$

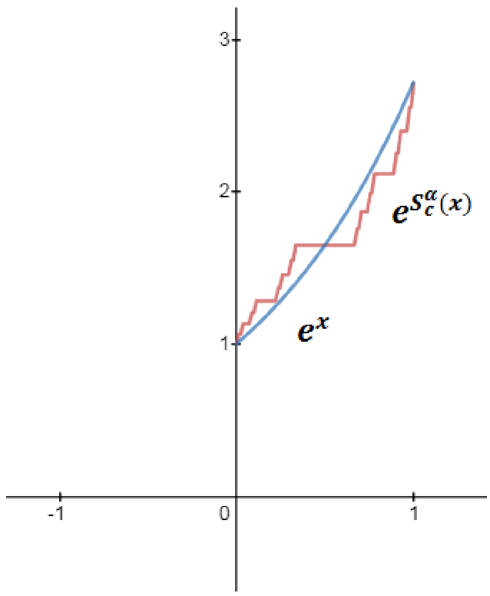

Figure 4: Comparing graph of e^x and $e^{S_C^\alpha(x)}$.

together (Fig. 4).

2.4 Fractal sinh function

It is possible to express fractal $\sinh(S_C^\alpha(x))$ function as Taylor series in which just odd exponents have appeared:

$$\sinh(S_C^\alpha(x)) = S_C^\alpha(x) + \frac{(S_C^\alpha(x))^3}{3!} + \frac{(S_C^\alpha(x))^5}{5!} + \dots \quad . \tag{8}$$

Interestingly, we compare the graph of staircase function $S_C^\alpha(x)$ with $\sinh(S_C^\alpha(x))$ (Fig. 5).

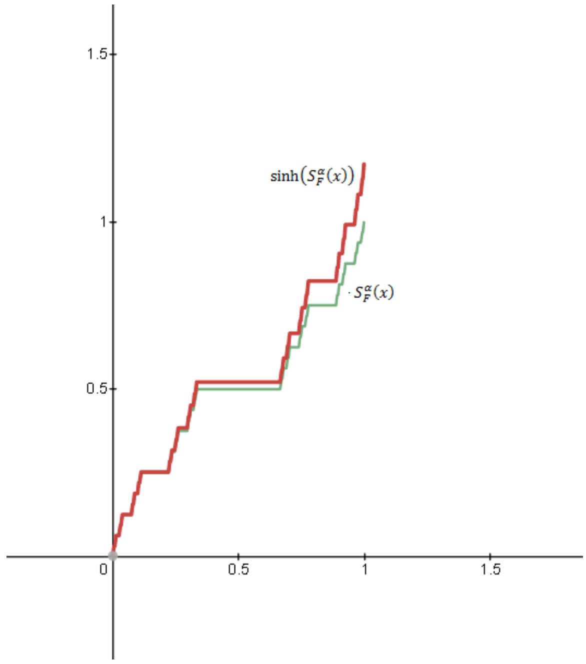

Figure 5: Comparing graph of staircase function $S_C^\alpha(x)$ and $\sinh(S_C^\alpha(x))$ when $F = C$.

2.5 Fractal cosh function

The function $\cosh(S_C^\alpha(x))$ has a Taylor series expression with only even exponents for $S_C^\alpha(x)$ (Fig. 6).

$$\cosh(S_F^\alpha(x)) = 1 + \frac{(S_F^\alpha(x))^2}{2!} + \frac{(S_F^\alpha(x))^4}{4!} - + \dots \quad . \tag{9}$$

2.6 Transition from continuity state to discrete state

Using Eq. 3, we calculate conformable derivative of $\sin x$ and $\exp(x)$ (namely $T^{0.63}(\sin x)$ and $T^{0.63}(e^x)$ such that we have

$$T^{0.63}(\sin x) = x^{0.37} \cos x. \tag{10}$$

And

$$T^{0.63}(e^x) = x^{0.37} e^x. \tag{11}$$

When drawing functions $\sin x, \exp x, x^{0.37}\cos x$, and $x^{0.37}e^x$, all of them namely functions and their derivatives are continuous on the interval [0,1]. In fractal calculus, instead of variable x we have staircase function $S_C^\alpha(x)$. As the same way, instead of $\sin x$ we have $\sin(S_C^{0.63}(x))$, etc. Thus, for fractal derivatives of $\sin(S_C^{0.63}(x))$ and $e^{S_C^{0.63}(x)}$ we obtain

$$D_C^{0.63}(\sin(S_C^{0.63}(x))) = \cos(S_C^{0.63}(x))\chi_C(x). \tag{12}$$

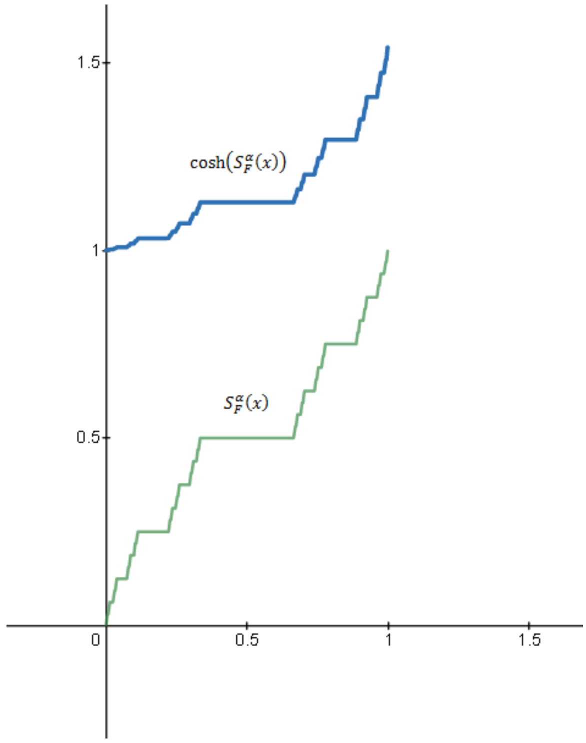

Figure 6: Comparing graph of staircase function $S_C^\alpha(x)$ and $\cosh(S_C^\alpha(x))$ when $F = C$.

And

$$D_C^{0.63}(e^{S_C^{0.63}(x)}) = e^{S_C^{0.63}(x)}\chi_C(x). \tag{13}$$

Note that characteristic function $\chi_C(x)$ is in essence a discrete function. In graphs of functions $D_C^{0.63}(\sin(S_C^{0.63}(x)))$ and $D_C^{0.63}(e^{S_C^{0.63}(x)})$, because of factor $\chi_C(x)$, there exist discontinuity (Fig. 8). So, although the fractal functions $\sin(S_C^{0.63}(x))$ and $e^{S_C^{0.63}(x)}$ are continuous, but their fractal derivatives become discrete. For this reason, the title of this section has been chosen "transition from continuity state to discrete state". On an impossible assumption, if there was not factor $\chi_{C_1}(x)$ we had their graphs in Fig. 7 while these graphs were still continuous.
Let $C_1 = [0, 1]$, $\quad D_{C_1}^{0.63}\sin(S_{C_1}^{0.63}(x)) = \cos(S_{C_1}^{0.63}(x))\chi_{C_1}(x)$
Let $C_2 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $\quad D_{C_2}^{0.63}\sin(S_{C_2}^{0.63}(x)) = \cos(S_{C_2}^{0.63}(x))\chi_{C_2}(x)$
Let $C_3 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, \frac{9}{9}]$, $\quad D_{C_3}^{0.63}\sin(S_{C_3}^{0.63}(x)) = \cos(S_{C_3}^{0.63}(x))\chi_{C_3}(x)$
$C = \cap_{n=1}^{\infty} C_n$.

The following Table 1. provides the data needed to plot all functions (both elementary functions and fractal elementary functions) and their derivatives (both conformable derivative and fractal derivative). The values of two elementary functions and their conformable derivatives have been calculated for 11 points $x = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.99, 0.999$ Furthermore, in Table 1. for two fractal elementary functions and their fractal derivatives these calculations have been done for these 11 points.

Table 1. Comparing values of $\sin x$,$\sin(S_C^\alpha(x))$ and e^x, $e^{S_C^\alpha(x)}$ together, where fractal dimension $\alpha = \frac{\ln 2}{\ln 3} = 0.63$ for Cantor set .

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.99	0.999
$\sin x$	0.099	0.198	0.295	0.389	0.479	0.564	0.644	0.717	0.783	0.836	0.840
$\sin(S_C^{0.63}(x))$	0.198	0.247	0.389	0.479	0.479	0.479	0.564	0.681	0.717	0.815	0.836
$T^{0.63}(\sin x) = x^{0.37}\cos x$	0.424	0.540	0.611	0.656	0.679	0.683	0.670	0.641	0.597	0.546	0.540
$D_C^{0.63}\sin(S_C^{0.63}(x)) = \cos(S_C^{0.63}(x))\chi_C(x)$	0.980	0	0.920	0	0	0	0.825	0	0.696	0	0
e^x	1.105	1.221	1.349	1.491	1.648	1.822	2.013	2.225	2.459	2.691	2.715
$e^{S_C^{0.63}(x)}$	1.221	1.284	1.492	1.648	1.648	1.648	1.821	2.117	2.225	2.593	2.691
$T^{0.63}(e^x) = x^{0.37}e^x$	0.471	0.673	0.864	1.062	1.275	1.508	1.764	2.049	2.365	2.681	2.714
$D_C^{0.63}(e^{S_C^{0.63}(x)}) = e^{S_C^{0.63}(x)}\chi_C(x)$	1.221	0	1.492	0	0	0	1.821	0	2.225	0	0

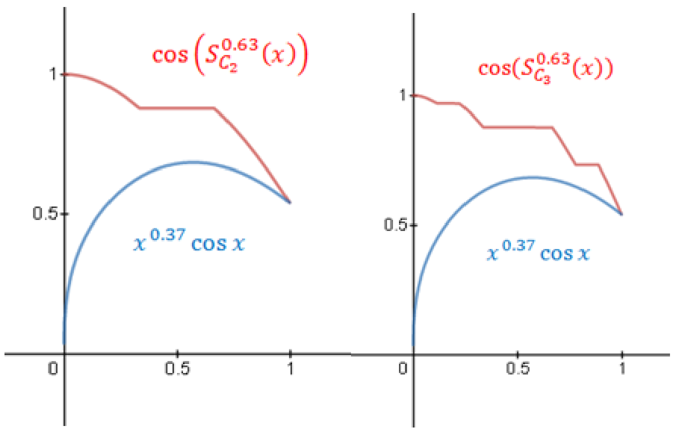

Figure 7: Comparing the result of conformable derivative, $x^{0.37}\cos x$, with fractal derivative, $\cos(S_C^{0.63}(x))$, in continuous case for the first iteration(C_2- left) and second iteration(C_3- right).

It is important to note that the values of the staircase function are undefined at points $x = 0$ and $x = 1$.

3 F^α-differential equations

It can be verified that both $\sin(S_F^\alpha(x))$ and $\cos((S_F^\alpha(x)))$ on the interval [0,1] for $F = C$ satisfy in the following fractal differential equation

$$(D_C^\alpha)^2\sin(S_C^\alpha(x)) + \sin(S_C^\alpha(x))\chi_C(x) = 0. \tag{14}$$

$$(D_C^\alpha)^2\cos(S_C^\alpha(x)) + \cos(S_C^\alpha(x))\chi_C(x) = 0. \tag{15}$$

and similarly $\sinh(S_C^\alpha(x))$ and $\cosh((S_C^\alpha(x)))$ satisfy in the following fractal differential equation.

$$(D_C^\alpha)^2\sinh(S_C^\alpha(x)) - \sinh(S_C^\alpha(x))\chi_C(x) = 0. \tag{16}$$

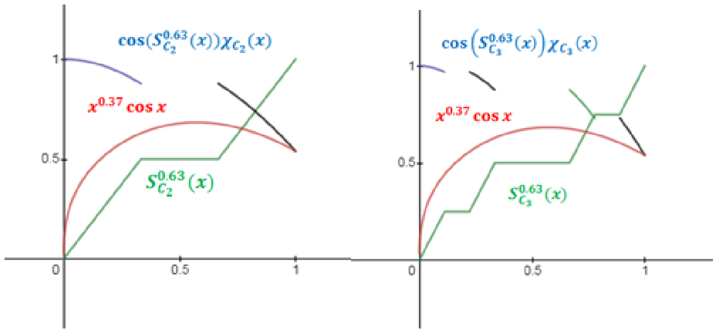

Figure 8: Comparing the result of conformable derivative, $x^{0.37}\cos x$, with fractal derivative, $\cos(S_C^{0.63}(x))\chi_C(x)$, in discontinuous case for the first iteration(C_2, left) and second iteration(C_3, right). The stair case functions have also been drawn (green graphs) to determine ranges in which discontinuities are created.

$$(D_C^\alpha)^2\cosh(S_C^\alpha(x)) - \cosh(S_C^\alpha(x))\chi_C(x) = 0. \tag{17}$$

It can also be verified that fractal exponential function $e^{S_C^\alpha(x)}$ satisfies in the following fractal differential equation

$$(D_C^\alpha)e^{S_C^\alpha(x)} - e^{S_C^\alpha(x)}\chi_C(x) = 0. \tag{18}$$

4 Fractal trigonometric identities

There is conjugacy graphically even for trigonometric identities in ordinary calculus and F^α-calculus which leads to relations such as The Pythagorean trigonometric identity for (sine and cosine) & (sinh and cosh) in F^α-calculus. (see Fig. 9). It should be noted that these identities have no validity at points $x = \{0, 1\}$ while we have not verified them analytically in this manuscript.

$$[\sin(S_F^\alpha(x))]^2 + [\cos(S_F^\alpha(x))]^2 = 1. \tag{19}$$

$$[\cosh(S_F^\alpha(x))]^2 - [\sinh(S_F^\alpha(x))]^2 = 1. \tag{20}$$

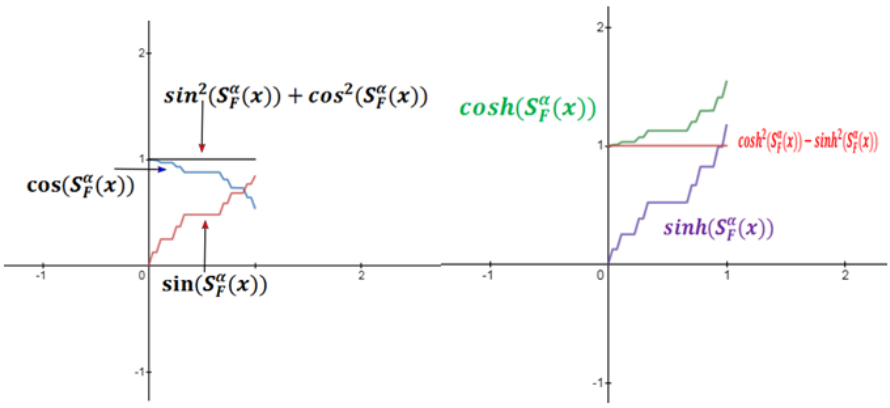

Figure 9: Pythagorean trigonometric identities for (sine and cosine, left) & (hyperbolic sine and hyperbolic cosine, right) with argument $S_F^\alpha(x)$ where $F = C$.

References

[1] Mandelbrot, B.B. *The Fractal Geometry of Nature*. W. H. Freeman and Company, 1977.

[2] Falconer, K. *Fractal Geometry: Mathematical Foundations and Applications*. Wiley; 2^{nd} ed., 2007.

[3] Nicoletti, D. Properties of the Weierstrass function in the time and frequency domains. *Chaos, Solitons & Fractals*, 5(1):1-8, 1995.

[4] Chalice, D. R. A Characterization of the Cantor Function. *The American Mathematical Monthly*, 98(3):255-258, 1991.

[5] Parvate, A. and Gangal, A.D. Calculus on fractal subsets of real line - I: formulation. *Fractals*, 17(01):53-81, 2009.

[6] Parvate, A. and Gangal, A.D. Calculus on fractal subsets of Real Line II: conjugacy with ordinary calculus. *Fractals*, 19(03):271-290, 2011.

[7] Parvate, A., Satin, S., and Gangal, A.D. Caclulus on fractal curves in R^n *Fractals*, 19(01):15-27, 2011.

[8] Golmankhaneh, A. K., C. Tunç, Sumudu transform in fractal calculus, *Applied Mathematics and Computation.*, 350:386-401, 2019.

[9] Golmankhaneh, A. K., Statistical Mechanics Involving Fractal Temperature, *Fractal Fract*, 3 (2019), 20; https://doi.org/10.3390/fractalfract3020020.

[10] Golmankhaneh, A. K., Balankin, A. S., Sub- and super-diffusion on Cantor sets: Beyond the paradox. *Physics Letters A*, 382(14):960-967, 2018.

[11] Golmankhaneh, A. K., Baleanu, D., Calculus on fractals. *Fractional Dynamics*, Sciendo Migration, 2015. Chapter:307-332.

[12] Jafari, F. K., Asgari, M. S. and Pishkoo, A., Fractal calculus for fractal materials, *Fractal Fract*, 3 (2019), 8; https://doi.org/10.3390/fractalfract3010008.

[13] Golmankhaneh, A. K., Baleanu, D., Diffraction from fractal grating Cantor sets. *Journal of Modern Optics*, 63(14):1364-1369, 2016.

[14] Khalil, R., Al Horani, M., Yousef, A., Sababheh, M., A new definition of fractional derivative, *Journal of Computational and Applied Mathematics*, 264 (2014), 65-70.

[15] Zhao, D., Luo, M., General conformable fractional derivative and its physical interpretation, *Calcolo*, 54 (2017) 903-917, DOI 10.1007/s10092-017-0213-8.