Determining the stability condition of a predator-prey interaction with a prescribed delay in the system

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Abstract:

This study attempts to model a real life situation involving delay differential equation, in particular the predator-prey interaction. A delay of τ =0.01 is prescribed into the system to determine whether the system will be stable or otherwise. The results show that when an insignificant delay is introduced, its stability returns to that of an ordinary differential equation, but when the delay is significant, it results into solutions of infinite roots.

Keywords: Delay, Differential Equation, Stability, Predator-Prey Interaction

1. Introduction

Delay Differential Equation (DDE) model arise from studying population dynamics of some species as we denote $x(t)$ as the population size at time, t. Let b and d denote the birth rate and the death rate respectively, on the time interval $[t, t+\Delta t]$, where $\Delta t \to 0$ the

$$x(t+\Delta t) - x(t) = bx(t)\Delta t - dx(t)\Delta t \tag{1}$$

Dividing (1) by Δ t and letting $\Delta t \to 0$, we obtain

$$\frac{dx}{dt} = (b-d)x = \tau x \tag{2}$$

where $r = b - d$ is the intrinsic growth rate of the population. The solution of equation (2) with an initial population $x(0) = x_0$ is given by

$$x(t) = x_0 e^{rt} \tag{3}$$

The function (3) represents the traditional exponential growth if r > 0 or decay if r < 0. Such a population growth may be valid for short period as it cannot go forever. Taking that fact that resources are limited, we define the logistic equation:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) \tag{4}$$

where $r > 0$ the intrinsic growth is rate and $k > 0$ is the carrying capacity of population. In the model (4), when x is small, the population grows as in the Malthus model (2), when x is large, the members of the species compete with each other for the limited resources.

Solving equation (4) by separating the variables, and incorporating the initial condition $x(0) = x_0$, we obtain

$$x(t) = \frac{x_0 k}{x_0 - (x_0 - k)e^{-rl}} \tag{5}$$

If $x_0 < k$, the population grows, approaching k asymptotically as $t \to \infty$. The population remains in time at $x = k$. In fact, $x = k$ is called an equilibrium of equation (4). Thus, the equilibrium $x = k$ of the logistic equation (4) is globally stable, that is

$$\lim_{t \to \infty} x(t) = k \tag{6}$$

for the solution of $x(t)$ of (4) with initial value $x(0) = x_0$.

In the above logistic model (4), it is assumed that the growth rate of a population at any time, t depends on the relative number of individuals at that time.

The delayed logistic equation obtained by assuming that it takes τ units of time before hatching is given by

$$\frac{dx}{dt} = rx(t)\left(1 - \frac{x(t-\tau)}{k}\right) \tag{7}$$

where r and k have the same meaning as in the logistic equation (4), $r > 0$ is a constant.

A delay differential equation that relates an unknown function to its derivatives and at least one of the terms appearing in the equation has an argument that is shifted by some fixed value. If there are multiple delayed terms, there may also be a corresponding number of multiple fixed shifts. Let us consider a linear first order differential delay equation:

$$\dot{x}(t) = -x(t-\tau) \tag{8}$$

When $\tau = 0$, the DDE reduces to an Ordinary Differential Equation (ODE). Note that (8) is ordinary since x is only a function of one independent variable t, but since we will only be considering ordinary equations we use the term ODE to describe a non-delayed differential equation. The ODE defined by $\tau = 0$ requires only one point to define an initial value problem (IVP):

$$\dot{x} = x, \quad x(t_0) = x_0 \tag{9}$$

In contrast to the IVP (9), defining an IVP for the DDE (8) now requires an entire function rather than a discrete point as the initial condition. The history function, $\phi(t)$, needs to be defined on the interval $[t_0, t_0 + \tau]$. Thus we define formally;

$$\dot{x}(t) = x(t-\tau), \; x(t) = \phi(t) \qquad \forall t \in [t_0, t_0 + \tau] \tag{10}$$

Equation (10) is of the more general type considered as a first order DDE.

$$\dot{u}(t) = f(u(t), u(t-\tau)), \tau > 0; \; u(t) = g(t), \, t \in (0, \tau] \tag{11}$$

We are interested in solving the IVP (10) in a specified interval. However, we need to know that the solution exist and under what conditions. Thus, existence and uniqueness theorem for delay differential equation comes to play. Thus, the ultimate aim of this study is to model a real life situation involving delay differential equation, in particular the predator-prey interaction. The essence is determine if the system will be stable or not when a delay of τ =0.01 is prescribed into the system.

Delay differential equation is one of the trending aspects of differential equations that include time delay terms in differential equations. The delays or lags can represent gestation periods, incubation periods, transport delays or can simply lump complicated biological processes together, accounting only for the time required for these processes to occur. Such models have advantage of combining a simple, initiative derivation with a wide variety of possible regimes for a single system. On the negative side, these models hide much of the detailed workings on complex biological systems, and it is sometimes precisely these details which are of interest. Mbah (2002a) used delay models to describe several aspects of infectious disease dynamics and primary infection. He used an analytical method of solution to the generalized mathematical model to study insulin-dependent diabetes mellitus.

The works of Cushing (1977), Gopadsang (1992), Kuang (1993) and Macdonald (1978) in this order showed that time delays of one type or another have been incorporated into biological models to represent source generation times, maturation periods, feeding times and reaction times. Olgac et al. (1997) used delay differential equation to model active vibration and noise. Further, delays appeared also in the study of chemostat (Zhao, 1995), circadian rhythms and neural networks (Campbell et al., 2004).

Jin et al. (2017) analyzed the stability characteristics of periodic DDEs with multiple time-periodic delays. Stability charts were produced for two typical examples of time-periodic DDEs about milling chatter, including the variable-spindle speed milling system with one-time-periodic delay and variable pitch cutter milling system with multiple delays. The simulations showed that the results gained by the proposed method are in close agreement with those existing in the past literature. The study indicated the effectiveness of their method in terms of time-periodic DDEs with multiple time-periodic delays. Moreover, for milling processes, the proposed method further provided a generalized algorithm, which possessed a good capability to predict the stability lobes for milling operations with variable pitch cutter or variable-spindle speed. More studies on stability of predator-prey model are found in (Liu and Fam 2017; Ashine et al., 2018).

Nurul (2012) studied the dynamical behavior of fish and mussel population in a fish farm where external food is supplied. The ecosystem of the fish farm was represented by a set of nonlinear differential equations involving nutrient (food), fish and mussel. He analyzed for the direction of Hopf-bifurcation, stability of the Hopf-bifurcating periodic orbits, and the period of the periodic orbits by using Poincare' normal form and center manifold theory. He performed numerical simulation to support the analytical results.

Xu and Li (2012) investigated a class of Beddington-DeAngelis functional response predator-prey model with two delays. They derived the conditions for the local stability and the existence of Hopf bifurcation at the positive equilibrium of the system. Further, explicit formulae for determining the stability and the direction of the Hopf bifurcation periodic solutions bifurcating from Hopf bifurcation were obtained by using the normal form theory and center manifold theory. A similar study was carried out by (Xia et al., 2017).

Suebcharoen (2017) studied the behaviour of a predator-prey with switching and stage-structure for predator. He determined the bounded positive solution, equilibria, and stabilities for the system of delay differential equation. The study showed that when the delay was chosen as a bifurcation parameter, the positive equilibrium was destabilized through a Hopf bifurcation.

Liu and Xin (2011) used the Homotopy perturbation method to solve nonlinear fractional partial differential equations arising in predator-prey biological population dynamics system. They gave numerical solutions and showed that some properties exhibit biologically reasonable dependence on the parameter values.

Mohr et al. (2014) introduced a general framework for age-structured predator-prey systems. In their study, individuals were distinguished into juvenile and adult classes, and several possible interactions were considered. The initial system of partial differential equations is reduced to a system of (neutral) delay differential equations with one or two delays. In addition, physically correct models for predator-prey with delay were provided.

Kundu and Maitra (2016) studied a multi-team delayed predator-prey model having two preys and one predator species and the time delay for gestation of the predator. The mathematical properties of the model were derived. By the assumption that the prey teams may help each other the effect of the rate of cooperation on the stability of the predator-prey model was observed. In a related study, Kundu and Maitra (2018) considered a delayed three species (two preys and one predator) predator-prey model where there is cooperation among the preys during predation. They studied the essential mathematical features of the proposed model to show the permanence of the system. The sufficient condition for global stability of the positive equilibrium was obtained by constructing a Lyapunov function.

Pratama et al. (2019) developed mathematical models with structural stages from small and adult predators. The predator function of adult predators follows the Holling I response function according to the characteristics in the ecosystem. Further, an analysis of the equilibrium value and stability of the interior equilibrium was carried out. Also, the analysis of the stability of interior equilibrium values with system linearization method at the value of equilibrium was performed. Four equilibrium values were realized. The stability characteristics of equilibrium values obtained using the Routh-Hurwitz criteria. It was observed that the population dynamics of prey always experience a significant increase. Thus, as the amount of immature to mature efficiency gets smaller, the growth of the population of prey will be faster. For the immature population and mature population, it was observed that the growth is linear and does not undergo significant changes.

2. Formulation of main model and methodology

Predator-prey models are useful and often used in the environmental science field because they allow researchers to both observe the dynamics of animal populations and make predictions as to how they will develop over time (Hussein, 2010). In this section, we formulate and study a model of a delay differential equation describing predator-prey interaction in an eco-system. Let $x(t)$ be the population at time t of a species of fish called prey and $y(t)$ be the population of another variety called predator which live off the prey. The assumption is that without the predator present, the prey will increase at a rate proportional to $x(t)$ and that the feeding action of the predator reduces the growth rate of the prey by an amount proportional to the product $x(t)\, y(t)$ we thus have

$$\dot{x}(t) = a_1 x(t) - b_1 x(t) y(t) \tag{12}$$

If the predator eats the prey and breeds at a rate proportional to its number and the amount of food available then

$$\dot{y}(t) = -a_2 y(t) + b_2 x(t) y(t) \tag{13}$$

where a_1, a_2, b_1 and b_2 are positive constants. The system of equations (12) and (13) bears the least expected from it. A more realistic model assumes that birth rate of the prey will diminish as $x(t)$ grows because of over-crowding and shortage of available food. In this work, it is assumed that there is a time delay of period τ for the predator to respond to changes in the sizes of x and y. Thus;

$$\begin{cases} \dot{x}(t) = a_1\left(1 - \dfrac{x(t)}{K}\right)x(t) - b_1 x(t) y(t) \\ \dot{y}(t) = -a_2 y(t) + b_2 x(t-\tau) y(t-\tau) \end{cases} \tag{14}$$

where K is a positive constant called the carrying capacity.

3. Stability Analysis of the Jacobian

Considering the systems of equation below:

$$\dot{x} = a_1\left(1 - \frac{x(t)}{p}\right)x(t) - b_1 x(t) y(t) \tag{15}$$

$$\dot{y} = -a_2 y(t) + b_2 x(t-\tau) y(t-\tau) \tag{16}$$

By the Taylor series and in consideration that $\tau = 0.01$, we have

$$x(t-\tau) = x(t) - \dot{x}(t)\tau + \ddot{x}(t)\frac{\tau^2}{2} - \cdots \tag{17}$$

$$y(t-\tau) = y(t) - \dot{y}(t)\tau + \ddot{y}(t)\frac{\tau^2}{2} - \cdots \tag{18}$$

Hence,

$$x(t-\tau) y(t-\tau) = \left(x(t) - \dot{x}(t)\tau\right) + \left(y(t) - \dot{y}(t)\tau\right) \tag{19}$$

$$= x(t) y(t) - x(t)\dot{y}(t)\tau - \dot{x}(t) y(t)\tau + \dot{x}(t)\dot{y}(t)\tau^2 \tag{20}$$

$$\cong x(t)y(t)-x(t)\dot{y}(t)\tau-\dot{x}(t)y(t)\tau \tag{21}$$

Thus,

$$\dot{y}(t)=a_2y(t)+b_2\left(x(t)y(t)-x(t)\dot{y}(t)\tau-\dot{x}(t)y(t)\tau\right) \tag{22}$$

$$\dot{y}(t)+b_2x(t)\dot{y}(t)\tau=-a_2y(t)+b_2x(t)y(t)-b_2\dot{x}(t)y(t)\tau \tag{23}$$

$$\left(1+b_2\tau x(t)\right)\dot{y}(t)=-a_2y(t)+b_2x(t)y(t)-b_2\left(a_1\left(1-\frac{x(t)}{p}\right)x(t)-b_1x(t)y(t)\right)y(t)\tau \tag{24}$$

$$=-a_2y(t)+b_2x(t)y(t)-a_1b_2\left(1-\frac{x(t)}{p}\right)x(t)y(t)\tau+b_1b_2x(t)y^2(t)\tau \tag{25}$$

$$=-a_2y(t)+b_2x(t)y(t)+b_1b_2x(t)y^2(t)\tau-a_1b_2\left(1-\frac{x(t)}{p}\right)x(t)y(t)\tau \tag{26}$$

$$\dot{y}(t)=\frac{1}{1+\tau b_2x(t)}\left(-a_2y(t)+\left(1+\tau b_1y(t)\right)b_2x(t)y(t)-\tau a_1b_2\left(1-\frac{x(t)}{p}\right)x(t)y(t)\right) \tag{27}$$

Hence, the approximated equations become:

$$\dot{x}(t)=a_1\left(1-\frac{x(t)}{p}\right)x(t)-b_1x(t)y(t) \tag{28}$$

$$\dot{y}(t)=\frac{1}{1+\tau b_2x(t)}\left(-a_2y(t)+\left(1+\tau b_1y(t)\right)b_2x(t)y(t)-\tau a_1b_2\left(1-\frac{x(t)}{p}\right)x(t)y(t)\right) \tag{29}$$

At equilibrium

$$\dot{x}(t)=\dot{y}(t)=0 \tag{30}$$

$$x(t)\left(a_1\left(1-\frac{x(t)}{p}\right)-b_1y(t)\right)=0 \tag{31}$$

$$\Rightarrow x(t)=0,\ y(t)=\frac{a_1}{b_1}\left(1-\frac{x(t)}{p}\right) \tag{32}$$

$$-a_2+\left(1+\tau b_1y(t)\right)b_2x(t)-\tau a_1b_2\left(1-\frac{x(t)}{p}\right)x(t)=0 \tag{33}$$

$$\Rightarrow y(t)=0,\ -a_2+\left(1+\tau b_1y(t)\right)b_2x(t)-\tau a_1b_2\left(1-\frac{x(t)}{p}\right)x(t)=0 \tag{34}$$

$$-a_2+\left(1+\tau b_1\frac{a_1}{b_1}\left(1-\frac{x(t)}{p}\right)\right)b_2x(t)-\tau a_1b_2\left(1-\frac{x(t)}{p}\right)x(t)=0 \tag{35}$$

$$\tau a_1b_2\left(1-\frac{x(t)}{p}\right)x(t)-\tau a_1b_2\left(1-\frac{x(t)}{p}\right)x(t)+b_2x(t)-a_2=0 \tag{36}$$

$$b_2x(t)-a_2=0 \tag{37}$$

$$x(t)=\frac{a_2}{b_2},\ \therefore\ y(t)=\frac{a_1}{b_1}\left(1-\frac{a_2}{b_2p}\right) \tag{38}$$

Hence, the equilibrium points are $(0,0)$ and $\left(\frac{a_2}{b_2},\frac{a_1}{b_1}\left(1-\frac{a_2}{b_2p}\right)\right)$

Let

$$f(x,y)=a_1\left(1-\frac{x(t)}{p}\right)x(t)-b_1x(t)y(t) \tag{39}$$

$$g(x,y)=-a_2y(t)+\left(1+\tau b_1y(t)\right)b_2x(t)y(t)-\tau a_1b_2\left(1-\frac{x(t)}{p}\right)x(t)y(t) \tag{40}$$

so that the given system is

$$\left.\begin{aligned}\dot{y}(t)&=0\\ \dot{x}(t)&=0\end{aligned}\right\} \tag{41}$$

The Jacobian of the above system of differential equations is written as

$$J=\begin{bmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{bmatrix} \tag{42}$$

where

$$\frac{\partial f}{\partial x}=a_1-2\frac{x(t)}{p}-b_1y(t),\qquad \frac{\partial f}{\partial y}=-b_1x(t) \tag{43}$$

$$\frac{\partial g}{\partial x}=b_2y(t)+\tau b_1b_2y^2(t)-\tau a_1b_2y(t)+2\tau a_1b_2x(t)y(t) \tag{44}$$

$$\frac{\partial g}{\partial y}=-a_2+b_2x(t)+2\tau a_1b_2x(t)y(t)-\tau b_1b_2x(t)+\tau a_1b_2x^2(t) \tag{45}$$

On substituting the equilibrium point $(0,0)$,we have

$$\left.\begin{aligned}\frac{\partial f}{\partial x}&=a_1, & \frac{\partial f}{\partial y}&=0\\ \frac{\partial g}{\partial x}&=0, & \frac{\partial g}{\partial y}&=-a_2\end{aligned}\right\} \tag{46}$$

The Jacobian of the above is now given as

$$J=\begin{bmatrix}a_1 & 0\\ 0 & -a_2\end{bmatrix} \tag{47}$$

The associated characteristic equation is

$$\lambda^2 - (a_1 - a_2)\lambda - a_1 a_2 = 0 \tag{48}$$

Hence, the origin $(0,0)$ is asymptotically stable provided that

$$Tr(J) = a_1 - a_2 < 0 \text{ and } \det(J) = -a_1 a_2 > 0 \tag{49}$$

For the non-trival equilibrium point $\left(\frac{a_2}{b_2}, \frac{a_1}{b_1}\left(1 - \frac{a_2}{b_2 p}\right)\right)$, we have

$$\frac{\partial f}{\partial x} = \frac{a_2}{pb_2}(a_1 - 2), \qquad \frac{\partial f}{\partial y} = \frac{-a_2 b_1}{b_2} \tag{50}$$

$$\frac{\partial g}{\partial x} = \frac{a_1}{b_1}\left(1 - \frac{a_2}{pb_2}\right)\left(b_2 - \frac{\tau a_1 a_2}{p} + 2\tau a_1 a_2\right) \tag{51}$$

$$\frac{\partial g}{\partial y} = \tau a_1\left(3a_2 - b_2 - \frac{2a_2^2}{pb_2}\right) \tag{52}$$

$$J = \begin{vmatrix} \frac{a_2}{pb_2}(a_1 - 2) & \frac{-a_2 b_1}{b_2} \\ \frac{a_1}{b_1}\left(1 - \frac{a_2}{pb_2}\right)\left(b_2 - \frac{\tau a_1 a_2}{p} + 2\tau a_1 a_2\right) & \tau a_1\left(3a_2 - b_2 - \frac{2a_2^2}{pb_2}\right) \end{vmatrix} \tag{53}$$

$$\lambda^2 - \lambda\left(\tau a_1\left(3a_2 - b_2 - \frac{2a_2^2}{pb_2}\right) + \frac{a_2}{pb_2}(a_1 - 2)\right) + \frac{\tau a_1 a_2}{pb_2}(a_1 - 2)a_1\left(3a_2 - b_2 - \frac{2a_2^2}{pb_2}\right) + \frac{a_2 a_1}{b_2}\left(1 - \frac{a_2}{pb_2}\right)\left(b_2 - \frac{\tau a_1 a_2}{p} + 2\tau a_1 a_2\right) = 0 \tag{54}$$

Comparing with $\lambda^2 - Tr(J)\lambda + \det(J) = 0$, we find that

$$Tr(J) = \tau a_1\left(3a_2 - b_2 - \frac{2a_2^2}{pb_2}\right) + \frac{a_2}{pb_2}(a_1 - 2) \tag{55}$$

$$\det(J) = \frac{\tau a_1 a_2}{pb_2}(a_1 - 2)a_1\left(3a_2 - b_2 - \frac{2a_2^2}{pb_2}\right) + \frac{a_2 a_1}{b_2}\left(1 - \frac{a_2}{pb_2}\right)\left(b_2 - \frac{\tau a_1 a_2}{p} + 2\tau a_1 a_2\right) \tag{56}$$

From (55) and (56), we shall have asymptotic stability provided $Tr(J) < 0$ and $\det(J) > 0$. This therefore demonstrates that the system is stable.

4. Conclusion

In this paper, a life example of the Predator-Prey interaction was modeled and controls were induced to the system to bring the system to equilibrium so as to avoid near periodic outbreaks of Predator population beyond its equilibrium. We prescribed a delay $\tau = 0.01$ to the differential system and thereafter determined the predators and the prey at this prevailing delay. It was observed that when the delay is insignificant, its stability returns to that of an ordinary differential equation. However, when the delay is significant, its stability results into solutions of infinite roots.

This work recommends that adequate research be carried out in biological processes in an ecosystem with emphasis on prescribed delay to the system and control effort should be made in the system through the harvesting strategy to keep the ecosystem stable as this will be beneficial to fish farmers, poultry farmers and others in animal husbandry.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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