Distribution of Decision Power among the Parties and Coalitions in the 44th Bulgarian Parliament as a Weighted Voting Game

Zdravko Dimitrov Slavov[1], Christina Slavova Evans[2]

[1] Varna Free University, Varna, Bulgaria, slavovibz@yahoo.com

[2] The George Washington University, Washington DC, USA, evans.christina.s@gmail.com

Abstract

Weighted voting games are a class of cooperative games that model group decision making systems in various domains, such as parliaments. One of the main challenges in a weighted voting game is to measure of player influence in decision making. This problem is fundamental in game theory and political science. In this paper we consider the 2017 Bulgarian Election and the distribution of decision power among the parties and coalitions in the 44th Bulgarian Parliament.

Keywords: Election, Bulgarian Parliament, Weighted Voting Game, Swing, Power.

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1. Introduction

We will model a parliamentary multi-winner election and the basic elements of this model are:

Firstly, a parliamentary body is to be elected.

Secondly, the parliamentary body contains a fixed number of seats.

Thirdly, a number of parties and coalitions are competing for these seats.

Fourthly, many voters are eligible to vote.

We also assume that voters have preferences on the set of political parties and coalitions, and every voter has a favorite party or coalition. The result of the election will be a parliament such that the seats are divided among a number of parties and coalitions.

In fact, eleven parties and nine coalitions were standing in the 2017 Bulgarian Election. The 240 members of the 44th Bulgarian Parliament (National Assembly, Narodno Sabranie) were elected for four-year terms by closed list proportional representation from 31 multi-member constituencies ranging in size from 4 to 16 seats. The electoral threshold was 4%.

According to the official results, 3682151 voters participated in the election, including 117668 voters who cast their ballots abroad. The number of invalid ballots was 169009 (or 4,6%). Five parties and coalitions won seats. Table 1 shows some details of the final result for 44th Bulgarian Parliament.

Table 1. The final analysis of the results for 44th Bulgarian Parliament

Parties	Voters	% of voters	Seats	% of seats
GERB	1147245	39,80	95	39,59
BSP	955214	33,14	80	33,33
UP	318512	11,05	27	11,25

DPS	315786	10,96	26	10,83
Volya	145636	5,05	12	5,00
	2882393	100,00	240	100,00

The aim of this paper is to investigate the distribution of decision power among the parties and coalitions in the 44th Bulgarian Parliament using the simple majority rule by quota 121 (more than $1/2$).

2. The Background

Game Theory is a useful tool for modeling strategic situations. This theory has been extensively used in many disciplines, including political science. Voting games date back at least to John von Neumann and Oskar Morgenstern in their monumental book "Theory of Games and Economic Behavior" published in 1944 [14]. Previous works on this problem were fragmentary and did not attract much attention. The book of Von Neumann and Morgenstern provided some new important developments such as the consideration of information sets and the introduction of formal definitions and decision rules.

We start our study with a consideration of key terms, definitions, and notations. Let N be a nonempty finite set of players which can be people, companies, institutions, political parties or countries, and every subset $S \subset N$ is referred to as a coalition. The set N is called the grand coalition and $\varnothing$ is called the empty coalition. We denote the collection of all coalitions by 2^N and the number of players of coalition $S \in 2^N$ by $|S|$. Let us label the players by $1, 2, \ldots, n$, $n = |N| \geq 2$.

Definition 1. A simple game in characteristic-function form is a pair $G = (N, v)$ where $N = \{1, 2, \ldots, n\}$ is a set of players and $v : 2^N \to \{0,1\}$ is the characteristic function which satisfies the following three conditions:

(1) $v(\varnothing) = 0$.

(2) $v(N) = 1$.

(3) v is monotonic, i.e. if $S \subset T \subset N$, then $v(S) \leq v(T)$. □

Two simple games $G_1 = (N_1, v_1)$ and $G_2 = (N_2, v_2)$ are called equal when $N_1 = N_2$ and $v_1 = v_2$.

In this paper we will consider a special class of simple games called weighted voting games with dichotomous voting rule - acceptance ("yes") or rejection ("no"). These games have been found to be well-suited to model economic or political bodies that exercise some kind of control. A weighted voting game is one type of simple cooperative game and it is a formalization model of coalition decision making in which decisions are made by vote [9] [11]. A weighted voting game is described by $G = [q; w_1, w_2, \ldots, w_n]$ where q is positive and $w_1, w_2, \ldots, w_n$ are nonnegative integer numbers such that $q \leq \sum_{k=1}^{n} w_k = \tau$. The set of weights $\{w_1, w_2, \ldots, w_n\}$ corresponds to the set of players $\{1, 2, \ldots, n\}$. By convention, we commonly take $w_i \geq w_j$ when $i < j$. This game has the following properties:

(1) $1 \leq q \leq \tau$.

(2) $n = |N| \geq 2$ is the number of players.

(3) $w_i \geq 0$ is the number of votes of player $i \in N$ and $w_1 \geq 1$.

(4) $w_1 \geq w_2 \geq \ldots \geq w_n$.

(5) q is the needed quota so that a coalition can win.

(6) the symbol $[q; w_1, w_2, \ldots, w_n]$ represents the weighted voting game G defined by

$$v(S) = \begin{cases} 1, & \sum_{k \in S} w_k \geq q \\ 0, & \sum_{k \in S} w_k < q \end{cases}, \text{ where } S \subset N.$$

Two weighted voting games $G_1 = [q_1; w_1^1, w_2^1, \ldots, w_m^1]$ and $G_2 = [q_2; w_1^2, w_2^2, \ldots, w_n^2]$ are equal when $m = n$, $q_1 = q_2$ and $w_i^1 = w_i^2$ for all players $i \in N$.

For any weighted voting game, the form $[q; w_1, w_2, \ldots, w_n]$ is often called a weighted representation. Obviously, one weighted voting game has many representations. For example, the following two weighted voting games $G_1 = [51; 49, 49, 2]$ and $G_2 = [2; 1, 1, 1]$ represent the same voting rule, i.e. they have the same characteristic function and each coalition of two or three players is winning.

For any coalition $S \subset N$ in game G, S is winning if $v(S) = 1$, S is losing if $v(S) = 0$, and S is blocking if S and $N \setminus S$ are both losing coalitions. The collections of all winning, all losing and all blocking coalitions in game G are denoted by $W(G)$, $L(G)$ and $B(G)$, respectively. If game G is known, we simply write W, L and B.

Of course, any simple game has winning and losing coalitions and this game is determined by the set of all winning (or losing) coalitions. We also get that $N \in W$ and $\varnothing \in L$; therefore, W and L are nonempty, $W \cap L = \varnothing$, $W \cup L = 2^N$, $B \subset L$ and $W \cap B = \varnothing$. Observe that a coalition having a winning sub-coalition is also winning, a sub-coalition of a losing coalition is also losing, and the complement of a blocking coalition is also blocking. It is easy to show that B can be either empty or nonempty. From $\varnothing \in L$, $\varnothing \notin B$ and $B \subset L$ it follows that $L \setminus B$ is nonempty. Sometimes, coalitions of $L \setminus B$ are called strictly losing.

First, for any player $i \in N$, the collection of all winning coalitions including i is denoted by W_+^i and the collection of all winning coalitions excluding i is denoted by W_-^i. Clearly, if $S \in W_-^i$, then $S \cup \{i\} \in W_+^i$; therefore, we obtain the inequality $|W_-^i| \leq |W_+^i|$. We also have that $W_+^i \cap W_-^i = \varnothing$, $W_+^i \cup W_-^i = W$ and $|W_-^i| \leq \frac{1}{2}|W| \leq |W_+^i|$.

Next, for any player $i \in N$, the collection of all losing coalitions including i is denoted by L_+^i and the collection of all losing coalitions excluding i is denoted by L_-^i, $L_+^i \cap L_-^i = \varnothing$ and $L_+^i \cup L_-^i = L$. From $S \in L_+^i$ it follows that $S \setminus \{i\} \in L_-^i$; therefore, we get that $|L_+^i| \leq \frac{1}{2}|L| \leq |L_-^i|$.

Finally, for any player $i \in N$, the collection of all blocking coalitions including i is denoted by B_+^i and the collection of all blocking coalitions excluding i is denoted by B_-^i. In this case we obtain $B_+^i \cap B_-^i = \varnothing$ and $B_+^i \cup B_-^i = B$.

For any coalition $S \in W$, S is called a minimal winning coalition if $S \setminus \{i\}$ is not winning for all $i \in S$. The collection of all minimal winning coalitions is denoted by MW for a known game or $MW(G)$ for any game G. For any player $i \in N$, the collection of all minimal winning coalitions including i is denoted by MW_+^i and the collection of all minimal winning coalitions excluding i is denoted by MW_-^i.

It is easy to prove that MW and W are finite sets, $MW \subset W$ and MW is nonempty. Clearly, we have that $MW_+^i \cap MW_-^i = \varnothing$, $MW_+^i \cup MW_-^i = MW$, $MW_+^i \subset W_+^i$ and $MW_-^i \subset W_-^i$ for all $i \in N$.

For any coalition $S \in L$, S is called a maximal losing coalition if $S \cup \{i\}$ is not losing for all $i \in N \setminus S$. The collection of all maximal losing coalitions is denoted by ML. For any player $i \in N$, the collection of all maximal losing coalitions including i is denoted by ML_+^i and the collection of all maximal losing coalitions excluding i is denoted by ML_-^i.

By analogy, ML and L are finite sets, $ML \subset L$ and ML is nonempty, and $ML_+^i \cap ML_-^i = \varnothing$, $ML_+^i \cup ML_-^i = ML$, $ML_+^i \subset L_+^i$ and $ML_-^i \subset L_-^i$ for all $i \in N$.

The set of minimal winning coalitions determines a simple game uniquely. When $MW(G_1) = MW(G_2)$ we call that games G_1 and G_2 are equivalent.

A player who does not belong to any minimal winning coalition is called a dummy, i.e. player $i \in N$ is a dummy if $i \notin S$ for all $S \in MW$. A player who belongs to all minimal winning coalitions is called a veto player or vetoer, i.e. player $i \in N$ has the capacity to veto if $i \in S$ for all $S \in MW$. A player $i \in N$ is a dictator if $\{i\}$ is a winning coalition.

In voting power theory, a dummy player has no decision power, a veto player can block every decision and a dictator has all of the decision power. Formally, for any player $i \in N$, i being a dictator is equivalent to $\{i\} \in MW$, i being a veto player is equivalent to $i \in \bigcap_{S \in MW} S$ (or $i \in \bigcap_{S \in W} S$) and i being a dummy is equivalent to $i \notin \bigcup_{S \in MW} S$.

Now we will consider two examples.

Example 1. In the Scottish Parliament in 2009 there were 5 political parties: 47 representatives of the Scottish National Party, 46 of the Labor Party, 17 of the Conservative Party, 16 of the Liberal Democrats, and 2 of the Scottish Green Party. Typically, all representatives from a party vote as a block. The quota is 65. In this case we obtain weighted voting game $SP = [65; 47, 46, 17, 16, 2]$. □

Example 2. The Finish Parliament with 200 seats uses three different rules: a simple majority by quota 101 (more than $1/2$), a qualified majority by quota 134 (more than $2/3$), and in some special cases by quota 167 (more than $5/6$) [8]. □

Now, let us analyze the sum $v(S)+v(N \setminus S)$ for $S \subset N$. Clearly, $0 \le v(S) \le 1$ imply inequalities $0 \le v(S)+v(N \setminus S) \le 2$.

Definition 2. A weighted voting game is called proper if $v(S)+v(N \setminus S) \le 1$ for all $S \subset N$. □

Note that a weighted voting game being proper is equivalent to the complement of a winning coalition is not winning. This means that in a proper game both coalitions S and $N \setminus S$ cannot be winning. In this context, if S is winning, then $N \setminus S$ is losing, but the converse statement is not always true.

In what follows, we will study proper games only.

Definition 3. A proper game is called decisive if $v(S)+v(N \setminus S)=1$ for all $S \subset N$. □

It is easy to prove that a proper game being decisive is equivalent to it having no blocking coalition. For any coalition $S \subset N$ in a decisive game, S being winning is equivalent to $N \setminus S$ being losing.

Theorem 1. (a) For any proper game G the following equations and inequalities are true:

(a) $1 \le |L| = |W| + |B|$, $2|W| + |B| = 2^n$, $1 \le |MW| \le |W| \le 2^{n-1} \le |L|$ and $|B| < |L|$.

(b) Game G being decisive is equivalent to $|W| = |L| = 2^{n-1}$.

Proof. (a) Game G is proper and $S \in L$ imply that either $N \setminus S \in W$ or $N \setminus S \in L$, i.e. either $N \setminus S \in W$ or $S \in B$. Let $P = N \setminus S \subset N$ and define a one-to-one correspondence between $S \in L \setminus B$ and $P = N \setminus S \in W$. We know that $B \cap W = \varnothing$; therefore, as a result we get $|L| = |W| + |B|$. We also know that L is nonempty, i.e. $1 \le |L|$.

Observe that from the above results $|W| + |L| = 2^n$ and $|L| = |W| + |B|$ we obtain $2|W| + |B| = 2^n$.

From $|L| = |W| + |B|$ and $|B| \ge 0$ it follows $|W| \le |L|$. We also know that $|W| + |L| = 2^n$; therefore, $|W| \le 2^{n-1} \le |L|$. But $MW \subset W$ and MW is nonempty, i.e. $1 \le |MW| \le |W| \le 2^{n-1} \le |L|$.

It is clear that from $|L| = |W| + |B|$ and $1 \le |W|$ it follows $|B| < |L|$.

(b) First, let game G be decisive. For each $S \subset N$ we have known that $S \in W$ is equivalent to $N \setminus S \in L$, i.e. $|W| = |L|$. From $|W| + |L| = 2^n$ it follows that $|W| = |L| = 2^{n-1}$.

Second, let us assume that $|W| = |L| = 2^{n-1}$. According to (a) we obtain that B is empty. Thus, we find that game G is decisive.

The theorem is proven.

Note that the set of all winning coalitions and the set of all minimal winning coalitions in weighted voting game $[q; w_1, w_2, \ldots, w_n]$ are the same as the set of all winning coalitions and the set of all minimal winning coalitions in weighted voting game $[\lambda q; \lambda w_1, \lambda w_2, \ldots, \lambda w_n]$ for every positive integer number λ. As a result, we obtain that weighted voting game $G = [q; w_1, w_2, \ldots, w_n]$ is equivalent to game

$G_\lambda = [\lambda q;\, \lambda w_1,\, \lambda w_2,\ldots,\, \lambda w_n]$. For integer number $\lambda > 1$, the two distinct representations $[q;\, w_1,\, w_2,\ldots,\, w_n]$ and $[\lambda q;\, \lambda w_1,\, \lambda w_2,\ldots,\, \lambda w_n]$ are equivalent. It follows that the number of representations of a weighted voting game is infinitive. This means that G and G_λ are equal as two simple games.

For any proper game G, a pair of players $i, j \in N$ is called symmetric or i and j are symmetric (shortly $i \approx j$) if and only if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all coalitions $S \in N \setminus \{i, j\}$. Proper game G is symmetric if every pair of players is symmetric.

In this interpretation, if $w_i = w_j$, then players i and j are symmetric, but the converse statement is not true. For example, see game $G = [30; 14, 13, 11, 10]$. The players of G have different weights but every pair of players is symmetric. Hence, symmetric does not imply that all players have equal weights, but symmetric implies that all players are granted equal impact on collective decisions.

It is interesting to note that game $[30; 14, 13, 11, 10]$ is equivalent to game $[3; 1, 1, 1, 1]$ and game $[3; 1, 1, 1, 1]$ is symmetric. It is easy to prove that if G_1 and G_2 are equivalent games and G_1 is symmetric, then G_2 is also symmetric because the set of all minimal winning coalitions of game G_1 is equal to the set of all minimal winning coalitions of game G_2.

For any proper game G and $i, j \in N$, player $i \in N$ is said to be more desirable than $j \in N$ (shortly $i \succeq j$) if and only if $v(S \cup \{i\}) \geq v(S \cup \{j\})$ for all coalitions $S \in N \setminus \{i, j\}$. The relation $\succeq$ is known as the desirability relation. It is known that all weighted voting games have desirability relation $\succeq$ on N, and $1 \succeq 2 \succeq \ldots \succeq n$ because we assume that $w_1 \geq w_2 \geq \ldots \geq w_n$.

By analogy, for any proper game G and $i, j \in N$, player $i \in N$ is said to be strictly more desirable than $j \in N$ (shortly $i \succ j$) if and only if $v(S \cup \{i\}) \geq v(S \cup \{j\})$ for all coalitions $S \in N \setminus \{i, j\}$ and there exists a coalition $T \in N \setminus \{i, j\}$ such that $v(T \cup \{i\}) > v(T \cup \{j\})$.

It is easy to show that for a proper game and $i, j \in N$:

(1) $i \succeq j$ is equivalent to $i \succ j$ or $i \approx j$.

(2) $i \succ j$ is equivalent to $i \succeq j$ and not $i \approx j$.

(3) $i \approx j$ is equivalent to $i \succeq j$ and $j \succeq i$.

Consider a pair of symmetric players $i, j \in N$. It follows that if $i \in S \subset N$ and $j \notin S$, then $v(S) = v(S \cup \{j\} \setminus \{i\})$. Now, it is easy to see that $|W_+^i| = |W_+^j|$, $|L_+^i| = |L_+^j|$ and $|B_+^i| = |B_+^j|$.

Definition 4. For $i \in N$ and $S \in W_+^i$, player i is called a negative swing member of S (critical or pivotal) if $S \setminus \{i\}$ is not winning. For any player $i \in N$, the collection of all winning coalitions including i as a negative swing number is denoted by W_s^i. For $i \in N$ and $S \in L_-^i$, player i is called a positive swing member of S (critical or pivotal) if $S \cup \{i\}$ is not losing. For any player $i \in N$, the collection of all losing coalitions including i as a positive swing number is denoted by L_s^i. $\square$

It is often said that $\left|W_s^i\right|$ and $\left|L_s^i\right|$ are the number of swings of player $i \in N$.

Note that each member of a minimal winning coalition is a negative swing player, each member of a maximal losing coalition is a positive swing player, a winning coalition may have a negative swing member and a losing coalition may have a positive swing member.

It is easy to show that each positive swing for player $i \in N$ corresponds to a pair of coalitions $(S, S \cup \{i\}) \in L_-^i \times W_+^i$ such that S is losing and $S \cup \{i\}$ is winning, and each negative swing for player $i \in N$ corresponds to a pair of coalitions $(S \setminus \{i\}, S) \in L_-^i \times W_+^i$ such that $S \setminus \{i\}$ is losing and S is winning. In the first case we say that i is a swing player for the pair $(S, S \cup \{i\})$, but in the second case we say that that i is a swing player for the pair $(S \setminus \{i\}, S)$.

It follows that $MW_+^i \subset W_s^i \subset W_+^i$ and $ML_-^i \subset L_s^i \subset L_-^i$ for all $i \in N$.

Theorem 2 [4]. For any proper game $\left|W_s^i\right| = \left|L_s^i\right|$ for all $i \in N$.

3. Power Indices of Players

The concept of decision power of the players in games is well-known. Weighted voting games use mathematical models to analyze the distribution of decision power of the players. Power indices measure the power of players and can be used to determine their payoffs.

The most popular indices in political science are the Banzhaf power indices (1965) and the Coleman power indices (1971). These power indices are based on the concept of swing. More precisely, they are based on the number of coalitions in which the player is swing. It has been demonstrated that different indices reflect specific conditions in the voting body. If all players of this body are voting independently, then the political power of the players is estimated by the Banzhaf power indices and the Coleman power indices.

3.1. The Banzhaf power indices

The absolute Banzhaf power index was introduced by the American jurist and law professor John Banzhaf Ⅲ in 1965 [1]. It concerns the number of times each player $i \in N$ could change a coalition from losing to winning and it requires that we know the number of negative swings for each player i. For each player $i \in N$, the absolute Banzhaf index is denoted by η_i and it equals the number of negative swings for this player, i.e. $\eta_i = \left|W_s^i\right|$ for all $i \in N$.

The normalized Banzhaf power index is the vector $\vec{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$, given by

$$\beta_i = \frac{\eta_i}{\sum_{k=1}^{n} \eta_k} \text{ for } i = 1, 2, \ldots, n.$$

The Penrose-Banzhaf index is defined by $b_i = \dfrac{\eta_i}{2^{n-1}}$ for $i \in N$. The Banzhaf index was originally created in 1946 by Leonel Penrose, but was reintroduced by John Banzhaf in 1965.

In [6] and [7], it is proven that for any player $i \in N$, $\eta_i = \left|W_s^i\right| = \left|L_s^i\right| = \left|W_+^i\right| - \left|W_-^i\right|$.

It is important to note that the Banzhaf power index is monotonic with respect to the weights when we are evaluating the power, i.e. for $i, j \in N$, $\eta_i = \eta_j$ when $w_i = w_j$, and $\eta_i \geq \eta_j$ when $w_i > w_j$. We also get that for any proper game, player $i \in N$ being a dummy is equivalent to $\eta_i = 0$, see also [12] and [13].

3.2. The Coleman power indices

In [5] Coleman considers two different power indices of the players in a game, see also [2]. For any player $i \in N$, they are defined as follows.

(a) The preventive power index $P_i = \frac{|W_s^i|}{|W|}$. This index can be interpreted as the probability of player $i \in N$ to block the decision making.

(b) The initiative power index $I_i = \frac{|L_s^i|}{|L|}$. This index can be interpreted as the probability of player $i \in N$ to initiative action.

It is easy to show that $0 \leq P_i \leq 1$, $0 \leq I_i \leq 1$, $\eta_i = P_i|W| = I_i|L|$ and $I_i \leq P_i$ for all $i \in N$. It is important to note that a game being decisive is equivalent to $P_i = I_i$ for all $i \in N$ (or $\sum_{i=1}^{n} I_i = \sum_{i=1}^{n} P_i$). For more information see Theorems 1 and 2.

For any player $i \in N$, both indices P_i and I_i achieve their lower bound of 0 if and only if player i is a dummy; index P_i achieves its upper bound of 1 if and only if i is a veto player; and index I_i achieves its upper bound of 1 if and only if player i is a dictator [2] [3].

In [6], the authors prove that for any non-dummy player $i \in N$ the Penrose-Banzhaf index b_i is the harmonic mean of the two Coleman indices P_i and I_i, i.e. $\frac{2}{b_i} = \frac{1}{P_i} + \frac{1}{I_i}$ when $b_i, P_i, I_i \neq 0$. It is important to point out that the Coleman power indexes are monotonic with respect to the weights when we are evaluating the power, i.e. for two different players $i, j \in N$, $P_i = P_j$ and $I_i = I_j$ when $w_i = w_j$, and $P_i \geq P_j$ and $I_i \geq I_j$ when $w_i > w_j$.

4. Distribution of Decision Power

As discussed above, the 44[th] Bulgarian Parliament with 240 seats uses a simple majority rule by quota 121. We will calculate the power indices of the players. Let us assume that all representatives from a party or a coalition vote as a block. Thus, we obtain a weighted voting game $BP = [121; 95, 80, 27, 26, 12]$ when $n = 5$ and $\tau = 240$.

We will focus our attention on the game $BP = [121; 95, 80, 27, 26, 12]$ and we find that:

$$W = \begin{Bmatrix} \{\underline{1},\underline{2}\},\{\underline{1},\underline{3}\},\{\underline{1},\underline{4}\},\{\underline{1},2,3\},\{\underline{1},2,4\},\{\underline{1},2,5\},\{\underline{1},3,4\},\{\underline{1},3,5\},\{\underline{1},4,5\}, \\ \{\underline{2},\underline{3},\underline{4}\},\{1,2,3,4\},\{\underline{1},2,3,5\},\{\underline{1},2,4,5\},\{\underline{1},3,4,5\},\{2,3,4,5\}\},\{1,2,3,4,5\} \end{Bmatrix}, \quad |W| = 16.$$

It is necessary to note that underlining the critical players to make it easier to count. As a result, we obtain.

$\left|W_+^1\right| = 14$, $\left|W_+^2\right| = 10$, $\left|W_+^3\right| = 10$, $\left|W_+^4\right| = 10$, $\left|W_+^5\right| = 8$.

$MW = \{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\}\}$, $\left|MW\right| = 4$.

$\left|MW_+^1\right| = 3$, $\left|MW_+^2\right| = 2$, $\left|MW_+^3\right| = 2$, $\left|MW_+^4\right| = 2$, $\left|MW_+^5\right| = 0$.

$\left|W_s^1\right| = 12$, $\left|W_s^2\right| = 2$, $\left|W_s^3\right| = 2$, $\left|W_s^4\right| = 2$, $\left|W_s^5\right| = 0$.

$\left|L\right| = 16$. $\left|B\right| = 0$.

We obtain that this game is decisive (see also Theorem 1), player 5 is a dummy and this game is equivalent to the game $BP1 = [3; 2, 1, 1, 1, 0]$ because $W(BP) = W(BP1)$. Table 2 shows the power indices of players in the game $BP = [121; 95, 80, 27, 26, 12]$. Thus, we also get that $1 \phi 2 \approx 3 \approx 4 \phi 5$.

Table 2. Voting power analysis of game $BP = [121; 95, 80, 27, 26, 12]$

Parties	% of voters	% of seats	η_i	β_i	b_i	P_i	I_i
GERB	39,80	39,59	12	0,67	0,750	0,750	0,750
BSP	33,14	33,33	2	0,11	0,125	0,125	0,125
UP	11,05	11,25	2	0,11	0,125	0,125	0,125
DPS	10,96	10,83	2	0,11	0,125	0,125	0,125
Volya	5,05	5,00	0	0,00	0,000	0,000	0,000
	100,00	100,00	18	1,00	1,125	1,125	1,125

5. The Power of the Parliament as a Whole

In section 3 we considered the decision power of the players in a weighted voting game. Now we will consider the power of the collectivity to act. Here we use the Coleman collective index and it is defined by $A = \frac{\left|W\right|}{2^n}$. It is computed as the share of the set of winning coalitions in the set of all coalitions. For more information see [5] and [10]. According to Theorem 1 we get that $\frac{1}{2^n} \le A \le \frac{1}{2}$ and a weighted voting game being decisive is equivalent to $A = \frac{1}{2}$.

In contrast to player's indices the Coleman collective index is not defined for individuals but for the voting body as a whole. Hence, the power of the collectivity to act is very important for game theory and political science.

For our study the power index of the collectivity to act are $A(BP) = \frac{16}{32} = \frac{1}{2}$ when $\frac{q}{\tau} = \frac{121}{240}$.

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