Earlier and Recent Results on Convex Mappings and Convex Optimization

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Abstract

The main purpose of this review-paper is to recall and partially prove earlier, as well as recent results on convex optimization, published by the author in the last decades. Examples are given along the article. Some of these results have been published recently. Most of theorems have a clear geometric meaning. Minimum norm elements are characterized in normed vector spaces framework. Distanced convex subsets and related parallel hyperplanes preserving the distance are also discussed. The convex involved objective-mappings are real valued or take values in an order-complete vector lattice. On the other side, an optimization problem related to Markov moment problem is solved in the end.

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1 Introduction

As it is well known, convex optimization is more convenient to realize than optimization of arbitrary mappings. Hahn-Banach type results and their consequences (such as Caratheodory's theorem, Krein Milman theorem, and existence of subgradients of convex functions) make convex optimization much easier than other optimum problems. The reader can study the monographs [1]-[11] for the background on convexity and functional analysis. In the last years, various generalizations and new results have been proved and applied in the works [12]-[24]. Our references are far from being complete. It covers only a part of. convex optimization. There are basic recently published results on constrained optimization, where the objective function is a (nonconvex) polynomial and the constraints are defined by means of polynomials too (the variable runs over a basic semi-algebraic set). Significant results in this area have been proved by M. Putinar, J.B. Lasserre, J.W. Helton and many other authors. Most of their publications are available online. We restrict ourselves to convex optimization and inequalities. On the other side, most of the results in the present work remain valid for vector-valued convex mappings. Applications of Krein Milman theorem to the moment problem can be found in [17], [23]. Optimization problems related to the moment problem are contained in [16], [24] and the references there. The rest of the paper is organized as follows. Section 2 improves the main result of [19], passing from convex real valued functions to convex operators having their ranges contained in an arbitrary order-complete vector lattice. A characterization of finite dimensional bounded convex subsets is deduced. Section 3 is devoted to characterizations of elements of minimum norms of convex closed subsets not containing the origin, in normed vector spaces. Related results are also discussed [22]. In Section 4, aspects of optimization in finite dimensional spaces are pointed out (finding minimum length of a surrounding curve and minimum area of a surrounding surface). In Section 5, a constrained optimization problem related to Markov moment problem [24] is briefly discussed. Section 6 concludes the paper.

2 On convex operators defined on bounded convex subsets of $\mathbf{R}^n$

In this section, we improve the basic result from [19].

Theorem 2.1.

Let X is an arbitrary real vector space, $B \subset X$ *a finite dimensional convex bounded subset, Y and order complete vector lattice and* $\varphi : B \to Y$ *a convex operator. Then there exists* $y_0 \in Y$ *such that* $\varphi(x) \geq y_0$ *for all* $x \in B$.

Proof. Since *B* is finite dimensional and convex, its relative interior $ri(B)$ is nonempty. Recall that by $ri(B)$ one denotes the interior of *B* with respect to the topology induced on B by that on the (finite dimensional) linear variety generated by *B*. Now by [20], φ is subdifferentiable at any point from $ri(B)$. Let $b_0 \in ri(B)$ and *h* a subgradient of φ at b_0, that is an affine operator $h : X \to Y$ such that $h(b_0) = \varphi(b_0)$ and $h(x) \leq \varphi(x)$ for all $x \in B$. On the other hand, let $x_1, \ldots, x_{p+1}$ $p+1$ be affine independent points in the linear variety generated by *B*, such that $B \subset \mathrm{co}\{x_1, \ldots, x_{p+1}\}$ (here *p* is the linear dimension of the linear variety generated by *B*). Such a system of points does exist thanks to the fact that *B* is finite dimensional and bounded. Now the following relations hold

$$\varphi(x) \geq h(x) = h\left(\sum_{j=1}^{p+1} \alpha_j x_j\right) = \sum_{j=1}^{p+1} \alpha_j h(x_j) \geq \left(\sum_{j=1}^{n} \alpha_j\right) \inf\ \{h(x_j);\, j \in \{1, \ldots, p+1\}\} =$$
$$= \inf\ \{h(x_j);\, j \in \{1, \ldots, p+1\}\} := y_0$$

where

$$x = \sum_{j=1}^{p+1} \alpha_j x_j,\ \alpha_j \geq 0,\ \ j \in \{1, \ldots, p+1\},\ \sum_{j=1}^{p+1} \alpha_j = 1.$$

This concludes the proof. □

Remark 2.1. The preceding proof furnishes a constructive method of finding a lower bound y_0 for $\varphi(B)$ .Since the convex bounded subset Bcan be approximated by convex polytopes, it results that a lower bound for $h(B)$ can be approximated by the lower bounds of h on these convex polytopes. On the other side, .computing the latter lower bound might be a difficult task, since the approximating polytopes might have many extreme points. Hence the volume of computations increases, the aim being to obtain a better approximation for $inf(h(B))$.

Remark 2.2. A concrete example of an order-complete Banach lattice (which is also a commutative real Banach algebra) of self-adjoint operators can be found in [8] (see also [7]).

In the next theorem, we show that the only convex subsets $B \subset X$ such that any convex real function on B is bounded from below are the finite dimensional convex bounded subsets.

Theorem 2.2.

Let X be an arbitrary real infinite dimensional vector space and $B \subset X$ *a convex subset, such that any convex real function defined on B is bounded from below. Then B is contained in a finite dimensional subspace of X and is bounded there.*

Proof. Let x^* be an arbitrary linear functional in the algebraic dual X^* of *X*. Then x^* and $-x^*$ are convex, and, by hypothesis, both of them are bounded from below on *B*. Thus $x^*(B)$ is bounded in **R**. Hence *B* is weakly bounded in X, endowed with the weak topology corresponding to the dual pair (X, X^*).. Let us endow X with the finest locally convex topology which is compatible with this dual pair (X, X^*).. By [4, Corollary 2, section 3.2, Chapter IV], we derive that B is bounded in the latter topology. Application of [4, exercise 7, Chapter II], leads to the fact that B is contained in a finite dimensional subspace and bounded there. This concludes the proof. □

3 Elements of minimum norm and related results

In the sequel, we present the characterization of an element of minimum norm in terms of linear continuous forms of norm one, related to the distance function, in arbitrary normed linear spaces. Related geometric aspects are briefly discussed.

Lemma 3.1.

Let X be a real normed vector space, $H = \{f = \alpha\}$ a closed hyperplane in X and $x_0 \in X$. Then the distance $d(x_0, H)$ is given by formula:

$$d(x_0, H) = \frac{|f(x_0) - \alpha|}{\| f \|}. \quad (3.1)$$

Theorem 3.1.

Let X be a real normed vector space, $A \subset X$ a closed convex subset not containing the origin and $a_0 \in A$. The following statements are equivalent:

(a) $\| a_0 \| = \inf \{\| a \|; \ a \in A\}$;

(b) *there exists* $f \in X^*$, *such that*

$$\| f \| = 1, \quad \| a_0 \| = f(a_0) \le f(a), \quad \forall a \in A;$$

(c) *there exists a closed homogeneous hyperplane H such that*

$$d(H, A) = \| a_0 \|.$$

Proof. Let B be the open ball of radius $\| a_0 \|$, centered at the origin. From (a) we infer that the intersection of B with A is empty. From the Hahn-Banach theorem, we infer that there exists a hyperplane $H = \{f = \alpha\}$, $f \in X^* \setminus \{0\}$, which separates B from A. Scaling by a suitable constant, we can assume that $\alpha = \| a_0 \|$. Hence we have:

$$B \subset \{f < \| a_0 \|\}, \quad A \subset \{f \ge \| a_0 \|\}. \quad (3.2)$$

If $\| x \| < 1$, then we can write

$$\| \| a_0 \| \cdot x \| < \| a_0 \| \Rightarrow f(\| a_0 \| \cdot x) < \| a_0 \|.$$

This leads to $\| f \| \le 1$. Since $a_0 \in A$, from the second inclusion (2.2) we infer that

$$f(a_0) \geq \| a_0 \| \Rightarrow f\left(\frac{a_0}{\| a_0 \|}\right) \geq 1 \Rightarrow \| f \| \geq 1.$$

The conclusion is:

$$\| f \| = 1, \quad \| a_0 \| \leq f(a_0) \leq \| f \| \cdot \| a_0 \| = \| a_0 \| \Rightarrow f(a) \geq \| a_0 \| = f(a_0), \quad \forall a \in A,$$

where we have used (3.2) once more. Now, the proof of (a) $\Rightarrow$ (b) is complete.

(b) $\Rightarrow$ (c). Let f be a functional verifying (b), $H = f^{-1}(\{0\})$. From this and also using Lemma 3.1, one obtains for any $a \in A$:

$$d(a,H) = \frac{| f(a) - 0 |}{\| f \|} = f(a) \geq \| a_0 \|, \quad \forall a \in A \Rightarrow d(A,H) \geq \| a_0 \| .$$

On the other hand,

$$d(A,H) \leq d(a_0,H) = f(a_0) = \| a_0 \| .$$

Comparing the preceding relations, we conclude that $d(A,H) = \| a_0 \|$.

(c) $\Rightarrow$ (a). This implication is almost obvious:

$$\| a_0 \| = d(A,H) = d(H,A) \leq d(0,A) \Rightarrow \| a_0 \| = \inf \ \{\| a \|; \ a \in A\}.$$

Now the proof of the theorem is complete. □

Corollary 3.1.

Let X be a real Hilbert space, $A \subset X$ *a closed convex subset not containing the origin and* $a_0 \in A$.

The following statements are equivalent:

(a) $\| a_0 \| = \inf \ \{\| a \|; \ a \in A\}$;

(b) $\| a_0 \|^2 \leq \langle a_0, a \rangle, \quad \forall a \in A$;

(c) $d(\{a_0\}^{\perp}, A) = \| a_0 \|$.

Proof. (a) $\Rightarrow$ (b). From the corresponding implication of Theorem 3.1, there exists $f \in X^*$ such that

$$\| f \| = 1, \quad \| a_0 \| = f(a_0) \leq f(a), \quad \forall a \in A \Rightarrow$$
$$\exists u \in X, \quad f = \langle u, \cdot \rangle, \quad \| u \| = 1, \ \langle u, a_0 \rangle = \| a_0 \| = \| a_0 \| \cdot \| u \| .$$

It follows that in Cauchy-Schwarz-Buniakovski inequality occurs equality, so that we must have:

$$u = \lambda a_0 \Rightarrow \| a_0 \| = \langle u, a_0 \rangle = \lambda \langle a_0, a_0 \rangle \leq \langle u, a \rangle = \lambda \langle a_0, a \rangle \Rightarrow$$

$$\lambda = \frac{1}{\| a_0 \|}, \quad \| a_0 \|^2 \le \langle a_0, a \rangle \quad \forall a \in A.$$

The implication (b) $\Rightarrow$ (c) follows from the corresponding implication of Theorem 3.1, taking

$$H = \{a_0\}^{\perp} = \{u\}^{\perp}.$$

The implication (c) $\Rightarrow$ (a) also follows from the corresponding implication of the preceding theorem, applied for the same hyperplane $H = \{a_0\}^{\perp}$. □

We recall the following related earlier results.

Theorem 3.2.

Let X be a normed vector space, $C \subset X$ *a closed convex cone,*

$$x_0 \in X \setminus C, \quad d_0 = d(x_0, C).$$

Then there exists a linear continuous functional $f \in X^*$, such that

$$\| f \| \le 1, \quad f(u) \le 0 \ \forall u \in C, \quad f(x_0) = d_0.$$

Proof. Let $p: X \to R$ be the functional defined by $p(x) = d(x, C) = \inf\{\| x - y \|; \ y \in C\}$. Obviously we have $p(x_0) = d_0$, We prove that p is sublinear:

$$p(\lambda x) = \inf\{\| \lambda x - y \|; \ y \in C\} = \inf\left\{\left\| \lambda x - \lambda \cdot \frac{1}{\lambda} y \right\|; \ y \in C\right\} =$$

$$\lambda \inf\{\| x - y \|; u \in C\}, \quad \forall \lambda > 0, \ \lambda = 0 \Rightarrow p(\lambda x) = p(0) = 0 = \lambda p(x).$$

On the other hand, for all $x_1, x_2 \in X$, $y_1, y_2 \in C$ one has:

$$\begin{aligned} &\| x_1 - y_1 \| + \| x_2 - y_2 \| \ge \| (x_1 + x_2) - (y_1 + y_2) \| \ge p(x_1 + x_2) \Rightarrow \\ &\inf\{\| x_1 - y_1 \|; y_1 \in C\} + \inf\{\| x_2 - y_2 \|; y_2 \in C\} \ge p(x_1 + x_2) \Rightarrow \\ &p(x_1 + x_2) \le p(x_1) + p(x_2), \quad \forall x_1, x_2 \in X. \end{aligned}$$

Let $X_0 = \{\alpha x_0; \alpha \in R\}$, $f_0 : X_0 \to R$, $f_0(\alpha x_0) = \alpha d_0$, $\forall \alpha \in R$. Then f_0 is linear and for $\alpha \ge 0$ we have

$$f_0(\alpha \cdot x_0) = \alpha \cdot d_0 = \alpha p(x_0) = p(\alpha \cdot x_0),$$

while for all $\alpha > 0$, the following remark holds

$$f_0(\alpha \cdot x_0) = \alpha \cdot d_0 < 0 \le p(\alpha \cdot x_0).$$

The conclusion is that f_0 is bounded from above by *p* on the one-dimensional subspace X_0. From the Hahn-Banach theorem, we infer that there exists a linear extension $f : X \to R$ of f_0, such that $f \le p$ on *X*. This yields

$$f(y) \leq 0 \quad \forall y \in C,\ f(x) \leq p(x) \leq d(x,0) = \| x \|, \quad \forall x \in X,$$
$$f(x_0) = d_0 = d(x_0, C) = p(x_0).$$

Thus *f* is continuous, of norm at most 1, and satisfies all the other assertion from the statement. The proof is complete. □

Remark. If X is a reflexive Banach space, then the distance $d(x_0, C)$ is attained at least at one point y_0 of *C*. In this case, we have

$$f(x_0) = \| x_0 - y_0 \| \Rightarrow f(x_0 - y_0) \geq f(x_0) = \| x_0 - y_0 \| \Rightarrow$$
$$f\left(\frac{x_0 - y_0}{\| x_0 - y_0 \|} \right) \geq 1 \Rightarrow \| f \| = 1.$$

In particular, in any Hilbert space, the linear functional from Theorem 2.2 is of norm 1.

Theorem 3.3. [22]

Let X be a normed linear space, A,B convex subsets of X *such that* $d_0 = d(A,B) > 0$. *Then there exists two closed parallel hyperplanes* H_1, H_2 *which keep the two convex subsets distanced, such that* $d(H_1, H_2) = d(A,B)$.

Proof. From the hypothesis we derive

$$d_0 = d(0, A-B) \Rightarrow B(0, d_0) \bigcap (A-B) = \Phi \Rightarrow$$
$$\exists f \in X^*,\ f(x) < d_0\ \forall x \in B(0, d_0),\ f(y) \geq d_0\ \forall y \in A-B.$$

The preceding relations further yield:

$$\| f \| \leq 1, \quad \inf f(A) = \alpha \geq d_0 + \sup f(B) = \beta \Rightarrow \alpha - \beta \geq d_0.$$

Now the closed hyperplanes we are looking for are

$$H_1 = \{ f = \alpha \}, \quad H_2 = \{ f = \beta \}.$$

Since the two hyperplanes separate the convex sets *A* and *B*, we have

$$d(H_1, H_2) \leq d(A,B) .$$

In order to prove the converse inequality, observe that the distance beween two *parallel hyperplanes* equals the distance from a point situated on a hyperplane, to the other one. This last remark, relation $\| f \| \leq 1$ and Lemma 3.1 formula (3.1), lead to:

$$d(H_1, H_2) = d(h_1, H_2) = \frac{| f(h_1) - \beta |}{\| f \|} = \frac{\alpha - \beta}{\| f \|} \geq \alpha - \beta \geq d_0 .$$

This concludes the proof. □

In Hilbert spaces, if the algebraic difference of the two convex is topologically closed, then the distance from the preceding theorem is attained at a pair of points $(a, b) \in A \times B$. If the two sets have smooth boundaries, the line joining these points is orthogonal to the tangent hyperplanes at these points, and these hyperplanes are parallel. These remarks lead to the following result, which is useful in applications. It avoids using Lagrange multipliers in determining the distance between two suitable convex sets, and the attaining points.

Corollary 3.2.

Let X be a real Hilbert space, $p: X \to \mathbb{R}$ convex and smooth, $q: X \to \mathbb{R}$ concave and smooth such that $q(x) < p(x)$ for all $x \in X$. If there are $a_1, b_1 \in X$ such that

$$d((a_1, p(a_1)),(b_1, q(b_1))) = d(\text{graph } p, \text{graph } q) > 0,$$

then there exists $\alpha \in \mathbb{R}$ s. t

$$\nabla p(a_1) = \alpha \nabla q(b_1);\ \exists \lambda \in \mathbb{R}: (\nabla p(a_1), -1) = \lambda\big(a_1 - b_1, p(a_1) - q(b_1)\big).$$

Examples

1) Let X be a separable Hilbert space, $\{e_n\}_{n\in N}$ a Hilbert base in X, and consider the hyperplane

$$H = \left\{x; \sum_{n=0}^{\infty} \alpha_n \langle x, e_n \rangle = 1\right\}, \quad (\alpha_n)_n \in (l^2), \quad \alpha_n > 0\ \forall n \in N.$$

Then it is not difficult to see that the orthogonal projection of the origin on H is the vector

$$P_H(0) = \sum_{n=0}^{\infty} \frac{\alpha_n}{\left(\sum_{j=0}^{\infty} \alpha_j^2\right)} \cdot e_n.$$

In particular, we have

$$d(0, H) = \frac{1}{\left(\sum_{n=0}^{\infty} \alpha_n^2\right)^{1/2}} = d(0, B), \quad B := H \cap \{x \in X;\ \langle x, e_n \rangle \geq 0\ \forall n \in N\}$$

It follows that we can determine the orthogonal projection of the origin on the base B of the cone of all elements with nonnegative Fourier coefficients. Note that it is a reflexive cone, because the closed unit ball is weakly compact. One can prove that the base B is unbounded, because its "intersection with the coordinate axis" are the point $y_n = \alpha_n^{-1} e_n$, $n \in N$, $\| y_n \| \to \infty$.

2) Let $X = l^1$ or $L^1([0,1])$, $B = \left\{x = (x_n)_n \in l^1;\ x_n \geq 0\ \forall n \in N,\ \sum_{n=1}^{\infty} x_n = 1\right\}$,

respectively

$$B = \left\{ \varphi \in (L^1)_+ ; \quad \int_0^1 \varphi(t)dt = 1 \right\}.$$

In both cases we have

$$d(0, B) = \| y \|_1 = 1, \quad \forall y \in B,$$

and obviously *B* is bounded, convex and closed. Hence there are an infinity of points (all the points of *B*) at which the distance to origin is attained. This happens because of the form of ball in l^1 - norm (respectively in L^1 – norm).

4 Optimization in finite dimensional spaces

In this section, minimum – length surrounding curve and minimum – area surrounding surface for a finite dimensional compact set are constructed. Let $K \subset R^2$ be a simply connected compact subset defined by

$$K = \{F \le 1\}, \ \partial K = \{F = 1\}, \ F \in C^2(R^2), \ \nabla F(x) \ne 0 \ \forall x \in K \setminus S, \tag{4.1}$$

where *S* is the set of minimum points of *F* situated in *K*. Consider the problem of surrounding $K_\varepsilon = \{x; f(x, K) \le \varepsilon\}$ by a closed path γ of minimal length. A variation of the problem is determining only "a half" of such a path. The problem has an obvious practical significance.

Theorem 4.1.

Let $C = cl\,(\text{co}\,(K_\varepsilon)) = \text{co}\,(K_\varepsilon)$ be the (closed) convex hull of subset K_ε. Then the path $\gamma = \partial C$ is the shortest curve surrounding K_ε. The concave (respectively convex) "branches" defined by this path are uniformly approximated by graphs of polynomials which preserve concavity (respectively convexity).

Proof. Due to Carathéodory's theorem [ ], we have:

$$C = \left\{ \sum_{j=1}^{3} \alpha_j x_j ; \quad x_j \in \text{Ex}\,(K_\varepsilon), \ \alpha_j \in R_+, \ \sum_{j=1}^{3} \alpha_j = 1 \right\},$$

so that $C := \text{co}\ (K_\varepsilon)$ is not only convex, but also compact (one denotes by $Ex(K)$ the set of all extreme points of K, $cl(B)$ is the topological closure of B and $co(A)$ is the convex hull of A). The geometric meaning of the above equality is that constructing C is equivalent to joining some "convex components" to K_ε, whenever the latter subset is not convex. Recall that for a finite dimensional compact subset *K* one has $\text{co}\,(K) = \text{co}\,(\text{Ex}\,(K))$. This is a consequence of Carathéodory's theorem too. To form the boundary of the convex hull, we replace the arches of "non-convexity" by line segments. Any point

$$x \in C \setminus K_\varepsilon$$

lies in a triangle having such a line segment as one of the edges, with extreme points of K_ε as the segments's ends. The third vertex of this triangle is also an extreme point in K_ε. Clearly, $\gamma = \partial C$ does not intersect the interior of $K_\varepsilon \subset C$. From the above arguments, the path γ is formed by joining the arches of ∂K_ε with the new added line segments as described above. Because these line segments represent the shortest path between the segment's ends, and because obviously γ surrounds $C \supset K_\varepsilon$, the first assertion in the statement follows. The points at which the Hessian of *F* is positive semi-definite remain unchanged when constructing the convex hull of K_ε. Consider the concave function having as graph the "upper branch" and the convex function define by the "lower branch" of γ. In order to avoid non-smooth paths, the function for the "upper branch" is approximated by the corresponding Bernstein polynomials, which preserve the concavity of any concave continuous function. Also approximating the convex function defined by the "lower branch" of γ with Bernstein polynomials, the conclusion follows. □

Remark 4.1.

The following variant of Theorem 4.1 is natural and useful in applications, due to its iterative character: the boundary of K_ε is usually a continuous piecewise smooth curve, which is uniformly approximated by polygonal lines. Given a non-convex polygon, it is easy to describe an algorithm of constructing its convex hull, simply dropping out some edges and vertices of "non-convexity". The convex hull of K_ε is approximated by such convex polygons.

The method from theorem 4.2 below works for compact subsets $K \subset R^n$ defined by means of a C^2-function on R^n, in a similar way to that from (4.1).

Theorem 4.2.

Let consider a compact subset $K \subset R^n,\ n \geq 3,\ K_\varepsilon = \{x; d(x,K) \leq \varepsilon\}$. *Then* $\partial(\mathrm{co}\,(K_\varepsilon))$ *has a minimum surface area among hypersurfaces surrounding* K_ε.

Proof.

One uses Carathéodory's theorem. The *n* – 1 dimensional simplexes which compose part of $\partial(\mathrm{co}\,(K_\varepsilon))$ have smaller surface – area than any other hypersurfaces containing their vertices. The other part of $\partial(\mathrm{co}\,(K_\varepsilon))$, where the Hessian of *F* is positive semidefinite, is part of ∂K_ε as well. If the boundary of $co(K_\varepsilon)$ is not smooth, it can be approximated by smooth hypersurfaces. This concludes the proof. □

5 A constrained optimization problem related to Markov moment problem

This Section starts by recalling briefly one of the earlier extension type results [21] and, on the other hand, by formulating one main problem due to Douglas Todd Norris' PhD Thesis, entitled "Optimal Solutions to the L_∞ Moment Problem with Lattice Bounds" [16], directed by Professor Emeritus Robert Kent Goodrich. The latter work suggested us the results of this section. One proves a result in a general setting, motivated by a similar problem to that considered in [16] (theorem 6.2 from below). A constrained related optimization problem in infinite dimensional spaces is solved too. The next result refers to the abstract moment problem [21], and is based on constrained extension theorems for linear operators. It was recalled in Chapter 1 from above and will be applied in the sequel. The results of this section were published in [24].

Theorem 5.1. (see [21])

Let $\tilde{X}$ *be a preordered vector space with its positive cone* $\tilde{X}_+$, *Y an order complete vector lattice,* $\{x_j\}_{j\in J} \subset \tilde{X}$, $\{y_j\}_{j\in J} \subset Y$ *given families,* $U_1, U_2 \in L(\tilde{X}, Y)$ *two linear operators. The following statements are equivalent*

(a) *there exists a linear operator* $U \in L(\tilde{X}, Y)$ *such that*

$$U_1(x) \le U(x) \le U_2(x), \quad \forall x \in \tilde{X}_+, \quad U(x_j) = y_j, \quad \forall j \in J;$$

(b) *for any finite subset* $J_0 \subset J$ *and any* $\{\lambda_j\}_{j\in J_0} \subset R$, *we have*:

$$\left(\sum_{j\in J_0} \lambda_j x_j = \varphi_2 - \varphi_1,\ \varphi_1, \varphi_2 \in \tilde{X}_+\right) \Rightarrow \sum_{j\in J_0} \lambda_j y_j \le U_2(\varphi_2) - U_1(\varphi_1).$$

In particular, using the latter theorem, one obtains a necessary and sufficient condition for the existence of a feasible solution (see theorem 5.2 from below). Under such condition, the existence of an optimal feasible solution follows too. On the other hand, the uniqueness and the construction of the optimal solution seems to be not obtained easily by such general methods. Therefore, we focus mainly on the existence problem. For other aspects of such problems on an optimal solution (uniqueness or non – uniqueness, construction of a unique solution, etc.), see [16]. In the latter work, one considers the following primal problem (P)

$$v = \inf\left\{\|\varphi\|_\infty : \varphi \in L^\infty_\mu(X),\ \int_X \varphi\varphi_j d\mu = b_j,\ \ j = 1,2,\dots,n,\ \ 0 \le \alpha \le \varphi \le \beta\right\}$$

where α, β are in $L^\infty_\mu(X)$, $\{\varphi_j\}_{j=1}^n$ is a subset of $L^1_\mu(X)$ and $b = (b_1, b_2, \dots, b_n)^T \in \mathbf{R}^n$. The function φ is unknown, and in general it is not determined by a finite number of moments. The next theorem generalizes some of the above existence – type results for a feasible solution. Here (X, S) is a measure space endowed with a σ – finite positive measure μ, and S is the σ – algebra of all measurable subsets of X.

Theorem 5.2.

Let $p \in [1, \infty)$ *and q be the conjugate of p. Let* $\{\varphi_j\}_{j\in J}$ *be an arbitrary family of functions in* $L^p_\mu(X)$, *where the measure* μ *is* σ *– finite, and* $\{b_j\}_{j\in J}$ *a family of real numbers. Assume that* $\alpha, \beta \in L^q_\mu(X)$ *are such that* $0 \le \alpha \le \beta$. *The following statements are equivalent*:

(a) *there exists* $\varphi \in L^q_\mu(X)$ *such that* $\int_X \varphi\varphi_j d\mu = b_j$, $j \in J, 0 \le \alpha \le \varphi \le \beta$;

(b) *for any finite subset* $J_0 \subset J$ *and any* $\{\lambda_j\}_{j\in J_0} \subset \mathbb{R}$, *the following implication holds*

$$\sum_{j\in J_0} \lambda_j \varphi_j = \psi_2 - \psi_1,\ \psi_1, \psi_2 \in (L^p_\mu(X))_+ \Rightarrow \sum_{j\in J_0} \lambda_j b_j \le \int_X \psi_2 \beta\, d\mu - \int_X \psi_1 \alpha\, d\mu$$

Moreover, the set of all feasible solutions φ *(satisfying the conditions (a)) is weakly compact with respect to the dual pair* (L^p, L^q) *and the inferior*

$$v := \inf\left\{\|\varphi\|_q : \varphi \in L^q_\mu(X), \ \int_X \varphi\varphi_j d\mu = b_j, \ \ j \in J, \ 0 \le \alpha \le \varphi \le \beta\right\} \ge \|\alpha\|_q$$

is attained at least at an optimal feasible solution φ_0.

Proof. Since the implication (a) $\Rightarrow$ (b) is obvious, the next step consists in proving that (b) $\Rightarrow$ (a). Define the real valued linear positive (continuous) forms U_1, U_2 on $\tilde{X} := L^p_\mu(X)$, by

$$U_1(\varphi) := \int_X \varphi\alpha d\mu, \ \ U_2(\varphi) := \int_X \varphi\beta d\mu, \ \ \ \varphi \in \tilde{X}.$$

Then condition (b) of the present theorem coincides with condition (b) of theorem 6.1. A straightforward application of the latter theorem, leads to the existence of a linear form U on $\tilde{X}$, such that the interpolation conditions $U(\varphi_j) = b_j$, $j \in J$ are verified and

$$\int_X \psi\alpha d\mu \le U(\psi) \le \int_X \psi\beta d\mu, \ \ \ \psi \in \tilde{X}_+.$$

In particular, the linear form U is positive on $\tilde{X} = L^p_\mu(X)$, and this space is a Banach lattice (in particular, $\tilde{X}$ is a complete metric topological vector space and an ordered vector space, whose positive cone $\tilde{X}_+$ is closed and generating). It is known that on such spaces, any linear positive functional is continuous (cf. [4], ch. V, sect. 5). The conclusion is that U can be represented by means of a nonnegative function $\varphi \in L^q_\mu(X)$. From the previous relations, we derive

$$\int_X \psi\alpha d\mu \le \int_X \psi\varphi d\mu \le \int_X \psi\beta d\mu, \ \ \ \psi \in \tilde{X}_+.$$

Writing these relations for $\psi = \chi_B$, where B is an arbitrary measurable set of positive measure $\mu(B)$, one deduces

$$\int_B (\varphi - \alpha) d\mu \ge 0, \ \ \int_B (\beta - \varphi) d\mu \ge 0, \ \ \ B \in S, \ \ \mu(B) > 0.$$

Then a standard measure theory argument shows that $\alpha \le \varphi \le \beta$ a. e. This finishes the proof of (b) $\Rightarrow$ (a). To prove the last assertion of the theorem, observe that the set of all feasible solutions is weakly compact by Alaoglu's theorem (it is a weakly closed subset of the closed ball centered at the origin, of radius $\|\beta\|_q$). On the other hand, the norm of any normed linear space is lower weakly semi - continuous. The conclusion is that the norm $\|\cdot\|_q$ is weakly lower semi-continuous on the weakly (convex) and compact set described at point (a), so that it attains its minimum at a function φ_0 of this set. Hence, there exists at least one optimal feasible solution. This concludes the proof. □

Remark 5.1. If the set $\{\varphi_j\}_{j\in J}$ is total in the space $L^p_\mu(X)$, then the set of all feasible solutions is a singleton, so that there exists a unique solution.

6 Conclusions

Existence and finding lower bounds for convex mappings are discussed in the first part of this review-paper. Some of these results generalize previously published theorems. In particular, a characterization of bounded finite dimensional convex subsets is deduced. Minimum norm elements in normed vector spaces setting and related results are also pointed out. Related examples are given. Other minimization problems having geometric meaning are briefly presented. In the end, an optimization problem related to a Markov moment problem in concrete spaces of integrable functions is solved. Each result is followed by its proof, so that the results and their associated methods are presented together.

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