Exact Solution of Grid Produced Turbulence; New Approach

Takeo R. M. Nakagawa and Ai Nakagawa

Academy of Hakusan, 2-14, Meiko, Hakusan 920-2152 Japan

npo.hakusan@kjc.biglobe.ne.jp

Abstract

A new approach to the kinetic theory of grid produced turbulence in the final period of decay has been proposed. The governing equations are the two-point velocity correlation equations in which the triple-point correlations are neglected as the closure assumption, and the pressure-velocity correlations are discarded by considering the homogeneous turbulence. Without recourse to the isotropic conditions, these equations are found to be separable into a pair of Oseen type equations. As a result, the double velocity correlations are solved exactly as an initial value problem. It has been shown that the decay law for the turbulent energy in the final period becomes in the form,

$$\langle(\Delta u)^2\rangle/U^2 = 3\ [1 + 8(x/M - x_0/M)/(\tau^2 R_e)]^{-5/2},$$

where $\langle(\Delta u)^2\rangle/U^2$ is the normalized turbulent energy, U the main flow velocity, x the coordinate in the main flow direction, M the grid mesh size, x_0 the point at which the initial condition is given, τ the constant determined by each the initial condition, and R_e the Reynolds number based on U and M. The present theory shows reasonable agreement with the experiment.

Keywords: Turbulent Energy, Kinetic Theory, Double Velocity Correlation, Grid Produced Turbulence, Final Period, Exact Solution

1. Introduction

- "Turbulence" has been one of classical but mysterious problems to scientists for many years. The grid produced turbulence has been unsettled problem for over one century [9, 10]. None the less, partly for practical requirements (wind tunnel design) and partly for its theoretical tractability (homogeneous and isotropic turbulence), the research has concentrated into the grid produced turbulence. It is, however, impossible to over-emphasize the fact that mainly because of the nonlinearity of the Navier-Stokes equation, no one has ever solved any problem about turbulence being practically important.

Tsugé [13] has proposed to use the sequence of separated point velocity correlation equations, viz. the two-point velocity correlation equation, the three-point velocity correlation equation, etc., rather than conventional velocity correlation equation, say one-point velocity correlation equation [1, 2]. Though Tsugé [13] has derived the two-point correlation equations based on Klimontovich [6] formalism, it is known that those equations can be derived by using the conventional Navier-Stokes equations formally at least. It should be noted that the formal derivation of Tsugé's [13] velocity correlation equations is quite similar to that of Hinze's[4] two-point velocity correlation equations. Previously, Hinze's[4] correlation equations have been thought as "entirely intractable by mathematics" due to the six-dimensional nature under general three-dimensional flow situation. Tsugé[13], however, has demonstrated that these types of correlation equations are separable into a pair of Oseen type equation at the respective points.

Very recently, Tsugé's approach[13] has been successfully applied to soft turbulence in the Bénard convection, and a good agreement between the theory and the experiment has been reported [11].

In the present paper, using the experimental initial conditions in preference to the Loitsyanskii invariant [8], it has been demonstrated the way how to solve the double velocity correlation equation exactly.

2. The velocity correlation equations

As being stated in the previous section, the two-point velocity correlation equations have been formally derived by using the conventional Navier-Stokes equation for the general case of inhomogeneous and anisotropic turbulence [9].

The two-point velocity correlation equation has the form

$$\langle \Delta u_i(a) NS[\overset{0}{u}(b), \overset{0}{p}(b)]_l + \Delta u_l(b) NS[\overset{0}{u}(a), \overset{0}{p}(a)]_i \rangle = 0 \qquad (2.1)$$

with the following definition,

$$NS[\overset{0}{u}, \overset{0}{p}] = [\partial/\partial t + u_j \partial/\partial x_j - \nu\nabla^2] u_j + 1/\rho\, \partial p/\partial x_i, \qquad (2.2)$$

where bracket $\langle\ \rangle$ denotes an ensemble average, arguments (a) and (b) mean point a and point b , respectively, $\overset{0}{z}$ stands for instantaneous fluid-dynamic quantity, z is its ensemble average, Δz is the fluctuation given by

$$\Delta z = \overset{0}{z} - z, \qquad (2.3)$$

and u_i Eulerian velocity, t time, x_j Eulerian Cartesian coordinates, ρ density, p static pressure, ∇^2 Laplacian operator, and ν kinematic viscosity.

The solenoidal conditions of the two-point velocity correlation are

$$\partial R^{(1,1)}_{i,l}(a,b)/\partial x_i(a) = \partial R^{(1,1)}_{i,l}(a,b)/\partial x_l(b) = 0, \qquad (2.4)$$

where

$$R^{(1,1)}_{i,l}(a,b) = \langle \Delta u_i(a) \Delta u_l(b) \rangle$$

is the two-point double velocity correlation.

3. The application to grid produced turbulence

It is evident that the turbulence produced by the grid mesh is not what is so called isotropic. The grid produced turbulence has a definite spatial directivity, viz. the direction of the main flow

$$u = (U, 0, 0), \qquad (3.1)$$

while the isotropic turbulence has not, where U is the constant main flow velocity.

Now, it will be shown that the present method enables the solution for the double velocity correlations, to be obtained without introducing the isotropic condition.

In the case of the present flow (3.1), (2.1) becomes

$$\{U[\partial/\partial x_1(a) + \partial/\partial x_1(b)] - \nu[\nabla^2(a) + \nabla^2(b)]\}R^{(1,1)}_{ij}(a,b) = 0, \qquad (3.2)$$

where the time derivative terms have been neglected because a time-dependent solution for fluctuation is not to be expected under the present steady primary flow, the triple correlations are neglected by the closure condition, and pressure-velocity correlations are discarded by introducing the homogeneity assumption of turbulence. In this context, it is worth noting that the pressure-velocity correlations are shown by Batchelor [1] to be identically zero for the case of fully homogeneous turbulence. For later convenience, the non-dimensional length x, and non-dimensional double velocity correlations R_{ij} are introduced by the following re-definition,

$$x = x/M, \qquad (3.3)$$

$$R_{ij} = R_{ij}/U^2, \qquad (3.4)$$

and the Reynolds number is defined as follows,

$$R_e = MU/\nu, \qquad (3.5)$$

where M is the grid mesh size. Then, the non-dimensional versions of (3.2) is simply obtainable by replacing, in this equation,

$$U = 1, \nu = R_e^{-1}. \qquad (3.6)$$

Therefore, (3.2) becomes,

$$\{[\partial/\partial x_1(a) + \partial/\partial x_1(b)] - R_e^{-1}[\nabla^2(a) + \nabla^2(b)]\}R^{(1,1)}_{ij}(a,b) = 0, \qquad (3.7)$$

4. The solution of grid produced turbulence in the final period of decay as being well known, we will be able to express $R^{(1,1)}_{ij}(a,b)$ in (3.7) formally,

$$R^{(1,1)}_{ij}(a,b) = [R^{(1,1)}_{ij}(a,b)]_c + [R^{(1,1)}_{ij}(a,b)]_p, \quad (4.1)$$

where $[R^{(1,1)}_{ij}(a,b)]_c$ and $[R^{(1,1)}_{ij}(a,b)]_p$ are the complementary and particular solutions, respectively. However, because the right hand side of (3.7) is zero, the particular solution is also zero. The required general solution is equal to the complementary solution as

$$R^{(1,1)}_{ij}(a,b) = [R^{(1,1)}_{ij}(a,b)]_c. \quad (4.2)$$

It may be obvious that (3.7) is solvable by the method of variable separation, viz.

$$R^{(1,1)}_{ij}(a,b) = \emptyset_i(a)\emptyset_j(b), \quad (4.3)$$

and $\emptyset_s$ follows the following equation,

$$[\partial/\partial x_1 - R_e^{-1}\nabla^2 - i\lambda]\emptyset_s = 0, \quad (4.4)$$

where $i\lambda$ is the separation constant such that the general solution is expressible in the form,

$$R^{(1,1)}_{ij}(a,b) = \int \emptyset_i(a)\emptyset_j(b)\delta[\lambda(a) + \lambda(b)]d\lambda(a)\, d\lambda(b). \quad (4.5)$$

where δ is the Dirac delta function.

It is easily seen that (4.4) corresponds to the special case of the Oseen equation for waves travelling in a uniform flow with frequency λ. Such waves decay due to viscous effect and dispersion. This fact suggests that a solution of the following form should be sought,

$$\emptyset_s = \int A_s(k, \beta, \lambda)\exp(-\beta x_1 + ik_i x_i)dk_2 dk_3, \quad (4.6)$$

where coefficient β and wave number k satisfy the following dispersion relation,

$$(\beta - ik_1)^2 + R_e(\beta - ik_1) + i\lambda R_e - k_2^2 - k_3^2 = 0, \quad (4.7)$$

which assures that $\emptyset_s$ is the solution of (4.4). After (4.7) is decomposed into the real and the imaginary parts, β and λ become as the first approximation,

$$\beta \approx k^2/Re, \quad (4.8)$$

$$\lambda \approx k_1, \quad (4.9)$$

where $k^2 = k_1^2 + k_2^2 + k_3^2$. If the expression like (4.6) for point a, and b, respectively, are substituted into (4.5), we obtain the solution for the double velocity correlations,

$R^{(1,1)}{}_{ij}(a,b) =$

$$\int c_{ij}\exp\{-R_e^{-1}[k^2(a)x_1(a) + k^2(b)x_1(b)] + i[k_i(a)x_i(a) + k_i(b)x_i(b)]\}\cdot$$

$$\cdot\ \delta[k(a)+k(b)]dk(a)dk(b), \qquad (4.10)$$

where we put $c_{ij} = A_iA_j$, and use the the relations (4.8) and (4.9) as well as the condition that the double velocity correlations are homogeneous in the planes parallel with the grid plane.

For later convenience, we will rewrite (4.10) as follows,

$$R^{(1,1)}{}_{ij}(a,b) = \int c_{ij}\exp[-2k^2(x-x_0)/R_e + ik_ir_i]dk, \qquad (4.11)$$

with

$x-x_0 = [x_1(a) + x_1(b)]/2,$

$r = x(a) - x(b),$

where x_0 is the point just behind the grid at which an initial condition will be given.

There exists abundant evidence that the longitudinal double velocity Correlation(Appendix) attains a Gaussian distribution at a point behind a grid, e.g. [16], so that it may be reasonable to use a Gaussian distribution as the initial form of the longitudinal double velocity correlation function. Thus, c_{ij} can be determined as follows. Substituting the tensor expression of the double velocity correlations given by Batchelor[2] at the point $x=x_0$ into (4.11), we get

$$(f-g)r_ir_j/r^2 + g\delta_{ij} = \int c_{ij}\exp(ik_ir_i)dk, \qquad (4.12)$$

where δ_{ij} is the Kronecker delta. Moreover, applying the operator

$$(2\pi)^{-3}\int \exp(-i\overset{0}{k}_ir_i)dr,$$

to the both sides of (4.12), we obtain

$$c_{ij} = (2\pi)^{-3}\int [(f-g)r_ir_j/r^2 + g\delta_{ij}]\exp(-i\overset{0}{k}_ir_i)dr, \qquad (4.13)$$

where we use the definition of the Dirac delta function as follows,

$$(2\pi)^{-3}\int \exp[ir\cdot(k-\overset{0}{k})]dr = \delta[k-\overset{0}{k}].$$

In general, it is convenient to choose a coordinate system to integrate the right hand side of (4.13) such that the direction of $\overset{0}{k}$ coincides with the z-axis. A Gaussian distribution of the longitudinal double velocity correlation function may be expressed as follows,

$$f=\exp(-r^2/\tau^2), \qquad (4.14)$$

where τ is a constant, which value is obtained by Batchelor & Townsend[3] experimentally. Also, using the functional form f and the solenoidal conditions of the double velocity correlation,

$$g=f + (r/2)\partial f/\partial r,$$

the transverse double correlation(Appendix) becomes

$$g=(1-r^2/\tau^2)\exp(-r^2/\tau^2). \qquad (4.15)$$

Substituting (4.14) and (4.15) into (4.13), and integrating the right hand side of (4.13) with respect to r, we finally have

$$c_{ij}= \tau^5/(32\pi^{3/2})(k^2\delta_{ij}-k_ik_j)\exp(-\tau^2k^2/4). \qquad (4.16)$$

Substituting (4.16) into (4.11), the turbulent energy decay law in the final period becomes

$$<(\Delta u)^2> = 3[1 + 8(x-x_0)/(\tau^2R_e)]^{-5/2}, \qquad (4.17)$$

where $<(\Delta u)^2>$ is the turbulent energy in the final period of decay, and here we put the subscripts i=j as well as |r|=0 in (4.11) by the definition of the turbulent energy.

Eq.(4.17) shows that the turbulent energy $<(\Delta u)^2>$ depends on $(x-x_0)$, which is the distance from the point x_0, at which the initial condition is given, constant value of τ in the longitudinal double correlation function $f = \exp(- r^2/\tau^2)$, which is the initial condition, and the Reynolds number defined by $R_e =MU/\nu$.

Eq.(4.17) may be alternatively expressed as

$$<(\Delta u)^2> = <(\Delta u_1)^2>+ <(\Delta u_2)^2>+ <(\Delta u_3)^2>. \qquad (4.18)$$

In particular, if turbulence is isotropic,

$$<(\Delta u_1)^2>=<(\Delta u_2)^2>=<(\Delta u_3)^2>,$$

so that

$$<(\Delta u)^2> = 3<(\Delta u_1)^2>=3<(\Delta u_2)^2>=3<(\Delta u_3)^2>.$$

5. Result

In this section, the turbulent energy derived in the previous section will be compared with the experiment by Batchelor & Townsend[3]. For this purpose, the non-dimensional values must be explicitly expressed in terms of dimensional forms. Thus, (4.17) is rewritten as

$$<(\Delta u)^2> / U^2 = 3\,[1 + 8(x/M - x_0/M)/(\tau^2 R_e)]^{-5/2} \quad . \qquad (5.1)$$

Batchelor & Townsend [3] have reported the longitudinal double velocity correlation function f at point $x/M=x_0/M$ behind the grid as

$$f=\exp[-(r/M)^2/\tau^2], \qquad (5.2)$$

where τ is a constant value. This value τ will be obtained by fitting with data describing the shape of longitudinal double velocity correlation function f due to Batchelor & Townsend[3], where the grid mesh size M=0.159cm, main flow velocity U=620cm/s, and the Reynolds number based on M and U is 650. In Figure 1, plotted are the experimental data by Batchelor & Townsend[3] and longitudinal double velocity correlation function f in (5.2), and thus it is found $\tau=0.2$ approximately. This function f and the constant value of τ will be used as the initial condition at point $x/M=x_0/M$ just behind the grid.

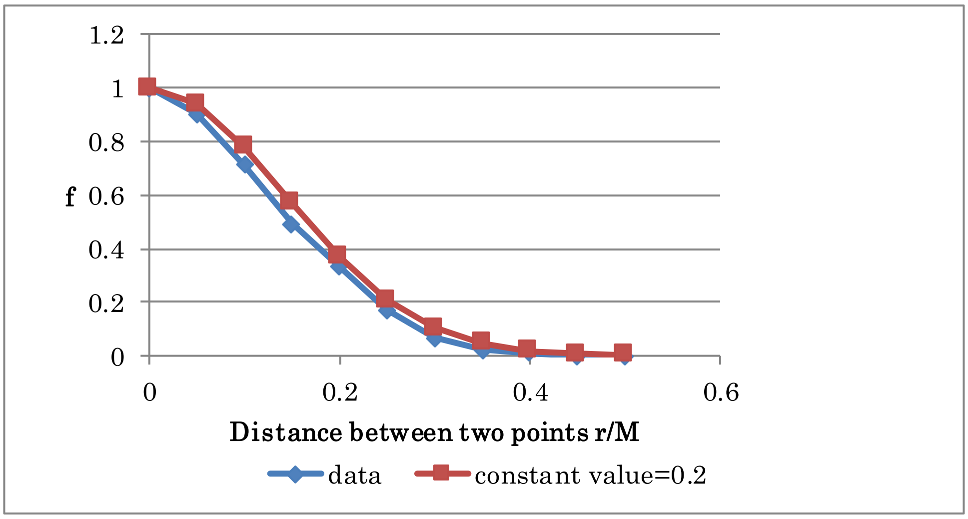

Figure 1: Longitudinal Double velocity correlation function f against distance between two points r/M.

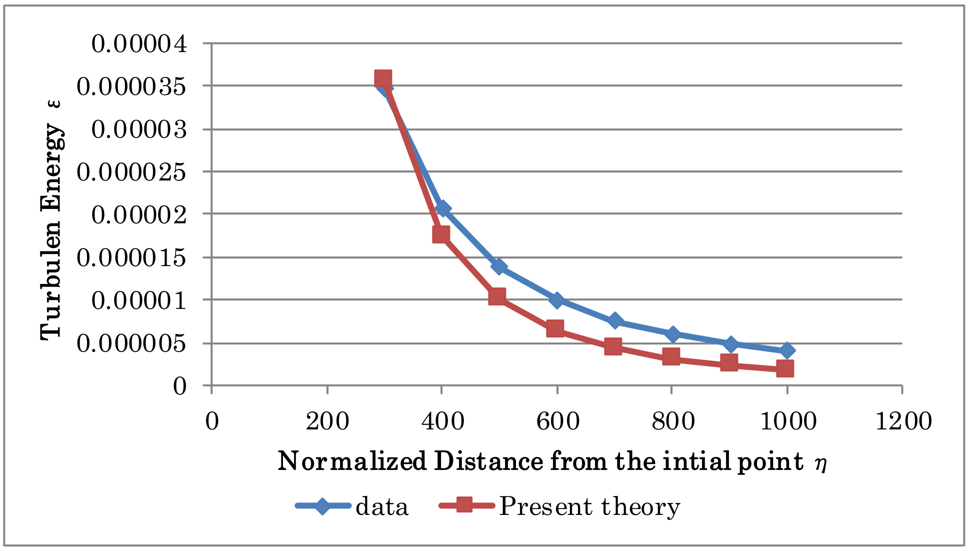

Figure 2: Turbulent energy $\varepsilon = <\Delta u)^2> / U^2$ against normalized coordinate $\eta = x/M - x_0/M$ in the main flow direction.

Shown is the comparison between theory and experiment, in Figure 2, where the ordinate is the turbulent energy $\varepsilon = <\Delta u)^2> / U^2$, and the abscissa is the distance $\eta = x/M - x_0/M$ in the main flow direction, where the origin of x_0 is just behind the grid.

For the direct comparison between the present theory and Batchelor & Townsend's experimental data[3], used is (5.1) with values of $\tau = 0.2$, U=620 cm/s, M=0.159 cm, and $R_e = 650$.

It is evident in Figure 2 that the present theory shows reasonable agreement with Batchelor & Townsend' experimental data [3] in the final period $300 < \eta < 1000$, though this theory is unable to explain their data in the initial period $\eta < 300$.

By re-examining Batchelor & Townsend's experimental data [3] carefully, it is realized that the analytical expressions for initial period of decay is found to be

$\varepsilon = \eta^{-0.93}$, while that for final period of decay is $\varepsilon = \eta^{-1.8}$. These are different from classical -1 decay law and -5/2 decay law in the initial period and in the final period respectively [1, 2, 3].

It is, however, clear that any adequate theory for the initial period of decay on the grid produced turbulence must consider the triple correlations, but this is beyond the scope of the present study, so that is left for the future.

6. Discussion

The forgoing experiments have been mainly performed with low Reynolds number, i.e. $R_e \leq 1{,}000$. Hence, we should establish the effect of the Reynolds number by conducting experiments at high Reynolds number.

It may be reasonable to infer that in the case of high Reynolds number, the initial stage of grid turbulence will persists over greater distance than that of low Reynolds number. In contrast to the final period (e.g. [3], [12]), most experimental turbulent energy decay laws in the initial period are identical; it decays as x^{-1}(e.g. [3]. [14]). Contrary to this statement, it should be noted that the present careful examination of Batchelor & Townsend's data[3] reveals it decays as $x^{-0.93}$ rather than x^{-1}.

Uberoi & Wallis[15] examined the effect of grid geometry on the turbulent energy decay, and shows that the decay law depends on the grid geometry. Whereas Ling & Wan [7] made an experimental study of weak turbulence created by a mechanically agitated grid, and found that the turbulent energy decay law depends on the velocity ratio of agitator to the mean flow.

In the present theory, the pressure-velocity correlations have been discarded, for it has been reported that they are zero for the case of fully homogeneous turbulence [1]. Note that the present theory only requires homogeneity in planes being parallel with the grid plane, but not in any plane in parallel with the main flow direction. That is, this theory is concerned with two-dimensional homogeneous turbulence in planes being parallel with the grid plane.

7. Concluding remarks

A new approach to the kinetic theory of grid produced turbulence in the final period is proposed. The governing equations are the two-point velocity correlation equations in which the triple-point correlations are neglected as the closure assumption, and the pressure-velocity correlations are discarded by the homogeneity assumption of turbulence. Without recourse to the isotropic conditions, these equations are found to be separable into a pair of Oseen type equations. As a result, the double velocity correlations are solved exactly as an initial value problem. It has been derived that the decay law for the turbulent energy in the final period in the form,

$$\langle(\Delta u)^2\rangle/U^2 = 3\ [1 + 8(x/M - x_0/M)/(\tau^2 R_e)]^{-5/2}$$

where $\langle(\Delta u)^2\rangle/U^2$ is the normalized turbulent energy, U the main flow velocity, x the coordinate in the main flow direction, M the grid mesh size, x_0 the point at which the initial condition is given, τ a constant being determined by each the initial condition, and R_e the Reynolds number based on U and M.

The present theory shows reasonable agreement with the experiment.

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Appendix: Longitudinal and transverse double velocity correlations

In this Appendix, longitudinal double velocity correlation f and transverse double velocity correlations g have been graphically introduced, where the main flow velocity is U. Figure 3 sketches f and g briefly.

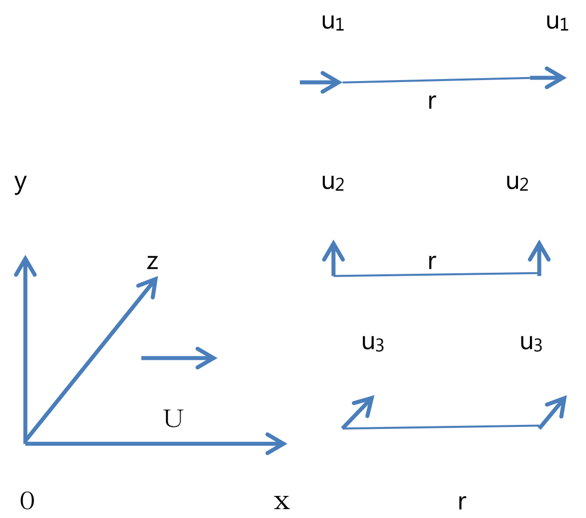

Figure 3 Sketch for explaining longitudinal and transverse double velocity correlations.

In this sketch, the right-hand coordinate system (x, y, z) is defined, and the direction the main flow U is in the x-direction. The longitudinal double velocity correlation f is the correlation between u_1 and u_1, where r is the variable distance between them. On one hand, there two transverse double velocity correlations g: One is the correlation between u_2 and u_2, while the other is the correlation between u_3 and u_3.