Exponent of Convergence of Solutions to Linear Differential Equations in the Unit Disc

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Abstract

In this paper we investigate the $[p,q]$-exponent of convergence of $f^{(i)}-\varphi$ where $f \not\equiv 0$ is a solution of linear differential equation with analytic or meromorphic coefficients with finite $[p,q]$-order in the unit disc and φ is a small function of f. By this investigation we can deduce the value distribution of the fixed points of $f^{(i)}$ by taking $\varphi(z)=z$. We will see the similarities and differences between $T(r,f)$ and $M(r,f)$.

Indexing terms/Keywords: Linear Differential Equations, Exponent of Convergence, [p,q]-Order, Growth of Solutions, Unit Disc.

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1 Introduction

The study of the growth and oscillation of solution of the linear differential equation

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+...+A_1(z)f'+A_0(z)f=0,$$

where the coefficients $A_j(z)(j=0,1,...,k-1)$ are meromorphic functions in the complex plane $\mathbb{C}$ and in the unit disc $D=\{z\in\mathbb{C}:|z|<1\}$ has been prospered greathly by making use the Nevanlinna value distribution theory of a meromorphic function (see [9], [16], [21]). Active research in this field was started by H. Wittich [27] and his students in the 1950's and 1960's. After that many authors have investigated the complex differential equation and in the most cases, the order of growth of solutions is infinite, (see e.g. [2, 8, 12, 15, 19]). So, to express more precisely the growth when the order is infinite, anothor notions have been introduced as hyper order, iterated order and $[p,q]$-order. For the unity of notations, we introduce here the concepts of $[p,q]$-order and $[p,q]$-type in the unit disc similar to the complex plane (see e.g. [13, 14, 17]). For $p\geq q\geq 1$ integers, the $[p,q]$-order of meromorphic function $f(z)$ in D is defined by

$$\sigma_{[p,q]}(f)=\limsup_{r\to 1^-}\frac{\log_p^+ T(r,f)}{\log_q\left(\frac{1}{1-r}\right)},$$

where $\log_1^+(x)=\log^+(x)=\max\{\log x,0\}$, $\log_{n+1}^+(x)=\log^+\log_n^+(x)$ and $T(r,f)$ is the Nevanlinna characteristic function of f. For an analytic function $f(z)$ in D, we have also

$$\sigma_{M,[p,q]}(f)=\limsup_{r\to 1^-}\frac{\log_{p+1}^+ M(r,f)}{\log_q\left(\frac{1}{1-r}\right)},$$

where $M(r,f)=\max_{|z|=r}|f(z)|$. For the relationship between $\sigma_{[p,q]}(f)$ and $\sigma_{M,[p,q]}(f)$ we have the following proposition.

Proposition 1 *[1] Let $p\geq q\geq 1$ integers and f is an analytic function $f(z)$ in D, then*
(1) if $p=q$, we have

$$\sigma_{[p,q]}(f)\leq\sigma_{M,[p,q]}(f)\leq 1+\sigma_{[p,q]}(f),$$

(2) and if $p>q$, we have

$$\sigma_{[p,q]}(f)=\sigma_{M,[p,q]}(f).$$

The $[p,q]$−type of a meromorphic function $f(z)$ in D with $0<\sigma_{[p,q]}(f)=\sigma<\infty$ is defined by

$$\tau_{[p,q]}(f)=\limsup_{r\to 1^-}\frac{\log_{p-1}^{+}T(r,f)}{\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}};$$

and if f is an analytic function f in D with $0<\sigma_{M,[p,q]}(f)=\sigma<\infty$ we have also

$$\tau_{M,[p,q]}(f)=\limsup_{r\to 1^-}\frac{\log_{p}^{+}M(r,f)}{\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}}.$$

We will use the notation $\lambda_{[p,q]}(f)$ to denote the $[p,q]$- exponent of convergence of the zero-sequence of meromorphic function $f(z)$ and $\overline{\lambda}_{[p,q]}(f)$ to denote the $[p,q]$- exponent of convergence of distinct zero-sequence of $f(z)$, which are defined as the following:

$$\lambda_{[p,q]}(f)=\limsup_{r\to 1^-}\frac{\log_p N\left(r,\frac{1}{f}\right)}{\log_q\left(\frac{1}{1-r}\right)}\ \text{ and }\ \overline{\lambda}_{[p,q]}(f)=\limsup_{r\to 1^-}\frac{\log_p \overline{N}\left(r,\frac{1}{f}\right)}{\log_q\left(\frac{1}{1-r}\right)}.$$

2 Statement of results

Xu, Tu and Zheng investigated the relationship between small functions and derivatives of solutions of higher order differential equations:

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\ldots+A_1(z)f'+A_0(z)f=0, \tag{1}$$

where $A_j(z)$ are entire or meromorphic functions in the complex plane, and obtained the following result.

Theorem 2 *[20] Let $A_j(z)$ $j=0,1,\ldots,k-1$ be entire functions with finite order and satisfy one of the following conditions:*
(i) $\max\left\{\sigma\left(A_j\right):j=1,2,\ldots,k-1\right\}<\sigma(A_0)<\infty$;
(ii) $0<\sigma(A_{k-1})=\ldots=\sigma(A_1)=\sigma(A_0)<\infty$ and $\max\left\{\tau\left(A_j\right):j=1,2,\ldots,k-1\right\}=\tau_1<\tau(A_0)=\tau$;
then for every solution $f\not\equiv 0$ of (1) and for any entire function $\varphi(z)\not\equiv 0$ satisfying $\sigma_2(\varphi)<\sigma(A_0)$, we have

$$\overline{\lambda}_2\left(f^{(i)}-\varphi\right)=\lambda_2\left(f^{(i)}-\varphi\right)=\sigma_2(f)=\sigma(A_0)\quad(i\in\mathbb{N}).$$

In 2013, Latreuch and Belaidi established the following results.

Theorem 3 *[17] Let $p \geq q \geq 1$ be integers and let $A_j(z)$ $j = 0, 1, ..., k-1$ be analytic functions in the unit disc D satisfying*

$$\max\left\{\sigma_{[p,q]}\left(A_j\right) : j = 1, 2, ..., k-1\right\} < \sigma_{[p,q]}(A_0).$$

If $f \not\equiv 0$ is a solution of (1), then $\sigma_{[p,q]}(f) = \infty$ and

$$\sigma_{[p,q]}(A_0) \leq \sigma_{[p+1,q]}(f) \leq \max\left\{\sigma_{M,[p,q]}\left(A_j\right) : j = 0, 1, ..., k-1\right\}.$$

Furthermore, if $p > q$, then

$$\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0).$$

Theorem 4 *[17] Let $p \geq q \geq 1$ be integers. Suppose that $A_j(z)$ $j = 0, 1, ..., k-1$ satisfy the hypotheses of Theorem 3, and let $\varphi(z) \not\equiv 0$ be analytic function in D such that $\sigma_{[p,q]}(\varphi) < \infty$. Then, every solution $f \not\equiv 0$ of (1) satisfies*

$$\overline{\lambda}_{[p,q]}(f-\varphi) = \lambda_{[p,q]}(f-\varphi) = \sigma_{[p,q]}(f) = \infty$$

and

$$\overline{\lambda}_{[p+1,q]}(f-\varphi) = \lambda_{[p+1,q]}(f-\varphi) = \sigma_{[p+1,q]}(f).$$

In this paper, we will continue this investigation for $f^{(i)} - \varphi$ for every $i \in \mathbb{N}$ and we will study also the case when we have $\max\left\{\sigma_{[p,q]}\left(A_j\right) : j = 1, 2, ..., k-1\right\} = \sigma_{[p,q]}(A_0)$. We will see the similarities and differences between $M(r, f)$ and $T(r, f)$ in this investigation.

Theorem 5 *Let $p \geq q \geq 1$ be integers and let $A_j(z)$ $j = 0, 1, ..., k-1$ be analytic functions in the unit disc D satisfying one of the following conditions:*
(1) $\max\left\{\sigma_{M,[p,q]}\left(A_j\right) : j = 1, 2, ..., k-1\right\} < \sigma_{M,[p,q]}(A_0) < \infty$;
(2) $\max\left\{\sigma_{M,[p,q]}\left(A_j\right) : j = 1, 2, ..., k-1\right\} \leq \sigma_{M,[p,q]}(A_0) < \infty$; *and*
$\max\left\{\tau_{M,[p,q]}\left(A_j\right) : \sigma_{M,[p,q]}\left(A_j\right) = \sigma_{M,[p,q]}(A_0)\right\} < \tau_{M,n}(A_0) < \infty.$
Then, for every solution $f \not\equiv 0$ of (1) and for any analytic function $\varphi(z) \not\equiv 0$ in the unit disc D satisfying $\sigma_{M,[p+1,q]}(\varphi) < \sigma_{M,[p,q]}(A_0)$ we have

$$\overline{\lambda}_{[p+1,q]}\left(f^{(i)} - \varphi\right) = \lambda_{[p+1,q]}\left(f^{(i)} - \varphi\right) = \sigma_{M,[p+1,q]}(f) = \sigma_{M,[p,q]}(A_0), \tag{2}$$

$(i \in \mathbb{N})$, *where $f^{(0)} = f$.*

Now we investigate the case when the coefficients of (1) are meromorphic in the unit disc and obtain the following result.

Theorem 6 *Let $p > q \geq 1$ be integers and let $A_j(z)$ $j = 0, 1, ..., k-1$ be meromorphic functions in the unit disc D satisfying one of the following conditions with $\delta(\infty, A_0) > 0$:*
(1) $\max\left\{\sigma_{[p,q]}\left(A_j\right) : j = 1, 2, ..., k-1\right\} < \sigma_{[p,q]}(A_0) < \infty$;
(2) $\max\left\{\sigma_{[p,q]}\left(A_j\right) : j = 1, 2, ..., k-1\right\} \leq \sigma_{[p,q]}(A_0) < \infty$; *and*
$\max\left\{\tau_{[p,q]}\left(A_j\right) : \sigma_{[p,q]}\left(A_j\right) = \sigma_{[p,q]}(A_0)\right\} < \tau_{[p,q]}(A_0) < \infty.$
Then, for every solution $f \not\equiv 0$ of (1) and for any meromorphic function $\varphi(z) \not\equiv 0$ in the unit disc D satisfying $\sigma_{[p+1,q]}(\varphi) < \sigma_{[p,q]}(A_0)$ we have

$$\overline{\lambda}_{[p+1,q]}\left(f^{(i)} - \varphi\right) = \lambda_{[p+1,q]}\left(f^{(i)} - \varphi\right) = \sigma_{[p+1,q]}(f) \geq \sigma_{[p,q]}(A_0), \; (i \in \mathbb{N}) \tag{3}$$

where $f^{(0)} = f$.

It remains the case $p = q \geq 1$ for meromorphic coefficients and also for analytic coefficients by making use $[p,q]$-order of $T(r,f)$. We can investigate these cases as follows.

Theorem 7 *Let $p \geq 1$ be an integer and let $A_j(z)$ $j = 0,1,...,k-1$ be analytic functions in the unit disc D satisfying one of the following conditions:*
(1) $\max\left\{\sigma_{[p,p]}\left(A_j\right) : j = 1,2,...,k-1\right\} < \sigma_{[p,p]}(A_0) < \infty$;
(2) $\sum_{j\in J} \tau_{[p,p]}\left(A_j\right) < \tau_{[p,p]}(A_0) < \infty$; where $J = \left\{j \neq 0 : \sigma_{[p,p]}\left(A_j\right) = \sigma_{[p,p]}(A_0)\right\}$ and $\sigma_{[p,p]}\left(A_j\right) < \sigma_{[p,p]}(A_0)$ for $j \notin J$.
Then, for every solution $f \not\equiv 0$ of (1) and for any analytic function $\varphi(z) \not\equiv 0$ in the unit disc D satisfying $\sigma_{[p+1,p]}(\varphi) < \sigma_{[p,p]}(A_0)$ we have

$$\sigma_{[p,p]}(A_0) \leq \overline{\lambda}_{[p+1,p]}\left(f^{(i)} - \varphi\right) = \lambda_{[p+1,p]}\left(f^{(i)} - \varphi\right) = \sigma_{[p+1,p]}(f) \leq \alpha_M, \tag{4}$$

$(i \in \mathbb{N})$, where $\alpha_M = \max\left\{\sigma_{M,[p,p]}\left(A_j\right) : j = 0,1,...,k-1\right\}$.

The condition (2) with the particular case $p = 1$ has been investigated recently in [3, Thm 1.8] to prove that $\sigma(A_0) \leq \sigma_2(f) \leq \alpha_M$.

Adding the condition $\delta(\infty, A_0) > 0$ in Theorem 7, we get the following corollary concerning the meromorphic case.

Corollary 8 *Let $p \geq 1$ be an integer and let $A_j(z)$ $j = 0,1,...,k-1$ be meromorphic functions in the unit disc D satisfying (1) or (2) of Theorem 7 with $\delta(\infty, A_0) > 0$. Then, for every solution $f \not\equiv 0$ of (1) and for any meromorphic function $\varphi(z) \not\equiv 0$ in the unit disc D satisfying $\sigma_{[p+1,p]}(\varphi) < \sigma_{[p,p]}(A_0)$ we have*

$$\overline{\lambda}_{[p+1,p]}\left(f^{(i)} - \varphi\right) = \lambda_{[p+1,p]}\left(f^{(i)} - \varphi\right) = \sigma_{[p+1,p]}(f) \geq \sigma_{[p,p]}(A_0),\ (i \in \mathbb{N}).$$

3 Preliminaries lemmas

Throughout this paper, we use the following notations that are not necessarily the same at each occurrence:

$E \subset (0,1)$ is a set of finite logarithmic measure, that is $\int_E \frac{dr}{1-r} < \infty$.

$F \subset (0,1)$ is a set of infinite logarithmic measure, that is $\int_F \frac{dr}{1-r} = \infty$.

$c > 0,\ \varepsilon > 0,\ \sigma \geq 0,\ \sigma_1 \geq 0,\ \tau \geq 0,\ \tau_1 \geq 0$, are real constants.

Lemma 9 *[20] Assume that $f \not\equiv 0$ is a solution of (1). Set $g = f - \varphi$; then g satisfies the equation*

$$g^{(k)} + A_{k-1}g^{(k-1)} + ... + A_0 g = -\left[\varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + ... + A_0\varphi\right]. \tag{5}$$

Lemma 10 *[20] Assume that $f \not\equiv 0$ is a solution of (1). Set $g_i = f^{(i)} - \varphi$, $(i \in \mathbb{N} - \{0\})$; then g_i satisfies the equation*

$$g_i^{(k)} + U_{k-1}^i g_i^{(k-1)} + ... + U_0^i g_i = -\left[\varphi^{(k)} + U_{k-1}^i \varphi^{(k-1)} + ... + U_0^i \varphi\right], \tag{6}$$

where

$$U_j^i = \left(U_{j+1}^{i-1}\right)' + U_j^{i-1} - \frac{\left(U_0^{i-1}\right)'}{U_0^{i-1}} U_{j+1}^{i-1}, \tag{7}$$

$j = 0, 1, ..., k-1$, $U_j^0 = A_j$ *and* $U_k^i \equiv 1$.

Lemma 11 *Let* $h : (0,1) \to (c, \infty)$ *be monotone increasing function such that*

$$\limsup_{r\to 1^-} \frac{\log_p h(r)}{\log_q \frac{1}{1-r}} = \alpha, \tag{8}$$

(α *is finite or infinite value); then there exists a set* $F \subset (0,1)$ *with infinite logarithmic measure such that for all* $r \in F$*, we have*

$$\lim_{r\to 1^-} \frac{\log_p h(r)}{\log_q \frac{1}{1-r}} = \alpha.$$

Proof. By (8), there exists an increasing sequence $\{r_m\} \to 1^-$ when $m \to \infty$, satisfying $1 - \left(1 - \frac{1}{m}\right)(1 - r_m) < r_{m+1}$ and

$$\lim_{r_m\to 1^-} \frac{\log_p h(r_m)}{\log_q \frac{1}{1-r_m}} = \alpha.$$

Then, there exists m_0 such that for all $m \geq m_0$ and $r \in I_m = \left[r_m, 1 - \left(1 - \frac{1}{m}\right)(1 - r_m)\right]$, we have

$$\frac{\log_p h(r_m)}{\log_q 1/\left[\left(1 - \frac{1}{m}\right)(1 - r_m)\right]} \leq \frac{\log_p h(r)}{\log_q 1/(1-r)} \leq \frac{\log_p h\left(1 - \left(1 - \frac{1}{m}\right)(1 - r_m)\right)}{\log_q 1/(1 - r_m)}. \tag{9}$$

The limit of both sides of (9), when $r_m \to 1^-$, is equal to α; so for $r \in I_m$, we have

$$\lim_{r\to 1^-} \frac{\log_p h(r_m)}{\log_q \frac{1}{1-r_m}} = \alpha.$$

Set $F = \bigcup_{m=m_0}^{\infty} I_m$. Then

$$m_l(F) = \sum_{m=m_0}^{\infty} \int_{I_m} \frac{dr}{1-r} = \sum_{m=m_0}^{\infty} \log\left(\frac{m}{m-1}\right) = \infty.$$

■

Lemma 12 *[6, Theorem 3.1] Let* f *be a meromorphic function in the unit disc* D *such that* $f^{(j)}$ *does not vanish identically. Let* $\varepsilon > 0$ *be a constant;* k *and* j *be integers satisfying* $k > j \geq 0$ *and* $d \in (0,1)$*. Then we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq \left(\left(\frac{1}{1-|z|}\right)^{(2+\varepsilon)} \max\left\{\log\frac{1}{1-|z|}, T(s(|z|), f)\right\}\right)^{k-j}, \ |z| \notin E,$$

where $s(|z|) = 1 - d(1 - |z|)$.

Lemma 13 *[17] Let* $p \geq q \geq 1$ *be integers. If* $A_j(z)$ $j = 0, 1, ..., k-1$ *are analytic functions in the unit disc* D *of finite* $[p,q]$*-order, then every solution* $f \not\equiv 0$ *of (1) satisfies*

$$\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) \leq \max\left\{\sigma_{M,[p,q]}\left(A_j\right) : j = 0, 1, ..., k-1\right\}.$$

Lemma 14 *Let $f(z)$ be an analytic function in the unit disc D with $\sigma_{M,[p,q]}(f)=\sigma$, $\tau_{M,[p,q]}(f)=\tau$, $0<\sigma<\infty$, $0<\tau<\infty$, then for any given $0<\beta<\tau$, there exists a set $F\subset(0,1)$ that has infinite logarithmic measure such that for all $r\in F$ we have*

$$M(r,f)>\exp_p\left\{\beta\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right\}.$$

Proof. By the definition of $\tau_{M,[p,q]}(f)=\tau$, there exists an increasing sequence $\{r_m\}\to 1^-$ satisfying $1-\left(1-\frac{1}{m}\right)(1-r_m)<r_{m+1}$ and

$$\lim_{m\to\infty}\frac{\log_p M(r_m,f)}{\left(\log_{q-1}\frac{1}{1-r_m}\right)^{\sigma}}=\tau.$$

Then, there exists m_0 such that for all $m\geq m_0$ and for a given ε, we have

$$\log_p M(r_m,f)>(\tau-\varepsilon)\left(\log_{q-1}\frac{1}{1-r_m}\right)^{\sigma}. \tag{10}$$

For a given β $(0<\beta<\tau-\varepsilon)$, Then, there exists m_1 such that for all $m\geq m_1$, we have

$$\left(\log_{q-1}\frac{1-\frac{1}{m}}{1-r}\right)^{\sigma}>\left(\frac{\beta}{\tau-\varepsilon}\right)\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}. \tag{11}$$

By (10) and (11), for all $m\geq\max\{m_0,m_1\}$ and for $r\in\left[r_m,1-\left(1-\frac{1}{m}\right)(1-r_m)\right]$, we have

$$\begin{aligned}\log_p M(r,f) &\geq \log_p M(r_m,f)>(\tau-\varepsilon)\left(\log_{q-1}\frac{1}{1-r_m}\right)^{\sigma}>\\ &> (\tau-\varepsilon)\left(\log_{q-1}\frac{1-\frac{1}{m}}{1-r}\right)^{\sigma}>\\ &> \beta\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}.\end{aligned}$$

Set $F=\bigcup_{m=m_2}^{\infty}I_m$ where $I_m=\left[r_m,1-\left(1-\frac{1}{m}\right)(1-r_m)\right]$. Then

$$m_l(F)=\sum_{m=m_2}^{\infty}\int_{I_m}\frac{dr}{1-r}=\sum_{m=m_2}^{\infty}\log\left(\frac{m}{m-1}\right)=\infty.$$

■

By the same method of the proof of Lemma 14, we can get the following three lemmas.

Lemma 15 *Let $f(z)$ be an analytic function in the unit disc D with $\sigma_{M,[p,q]}(f)=\sigma$, $0<\sigma<\infty$, then for any given $0<\beta<\sigma$, there exists a set $F\subset(0,1)$ that has infinite logarithmic measure such that for all $|z|=r\in F$ we have*

$$M(r,f)>\exp_p\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\beta}\right\}.$$

Lemma 16 *Let $f(z)$ be meromorphic function in the unit disc D with $\sigma_{[p,q]}(f)=\sigma$, $\tau_{[p,q]}(f)=\tau$, $0<\sigma<\infty$, $0<\tau<\infty$, then for any given $0<\beta<\tau$, there exists a set $F\subset(0,1)$ that has infinite logarithmic measure such that for all $|z|=r\in F$ we have*

$$T(r,f)>\exp_{p-1}\left\{\beta\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right\}.$$

Lemma 17 *Let $f(z)$ be meromorphic function in the unit disc D with $\sigma_{[p,q]}(f)=\sigma$, $0<\sigma<\infty$; then for any given $0<\beta<\sigma$, there exists a set $F\subset(0,1)$ that has infinite logarithmic measure such that for all $|z|=r\in F$ we have*

$$T(r,f)>\exp_{p-1}\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\beta}\right\}.$$

Lemma 18 *Let $p\geq q\geq 1$ be integers and let $A_j(z)$ $j=0,1,...,k-1$ be analytic functions in the unit disc D satisfying the condition (1) or (2) of Theorem 5. Then, every solution $f\not\equiv 0$ of (1) satisfies*

$$\sigma_{M,[p+1,q]}(f)=\sigma_{M,[p,q]}(A_0). \tag{12}$$

Proof. We will prove the case (2) and by the same method we can prove the case (1). Suppose that we have the condition (2) of Theorem 5. By Lemma 14, for any $\varepsilon>0$ there exists a set $F\subset[0,1)$ of infinite logarithmic measure such that for $|z|=r\in F$, we have

$$M(r,A_0)\geq\exp_p\left\{(\tau-\varepsilon)\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right\}. \tag{13}$$

From the condition (2), there exists a set $E\subset[0,1)$ of finite logarithmic measure such that for $|z|=r\in[0,1)-E$, we have

$$M\left(r,A_j\right)\leq\exp_p\left\{(\tau-2\varepsilon)\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right\},\ j\neq 0, \tag{14}$$

where $\varepsilon>0$ small enough. If $\sigma_{M,[p,q]}(f)<\infty$, then from Lemma 12, if $p>q$ we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right|\leq\exp_{p-1}\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{(\sigma+\varepsilon)}\right\}, \tag{15}$$

and if $p=q$ we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right|\leq\left(\frac{1}{1-|z|}\right)^{j(2+\varepsilon)}\left(\exp_{p-1}\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{(\sigma+\varepsilon)}\right\}\right)^{j},$$

for $|z|=r\notin E$. From (1) we can write

$$|A_0(z)|\leq\left|\frac{f^{(k)}(z)}{f(z)}\right|+|A_{k-1}(z)|\left|\frac{f^{(k-1)}(z)}{f(z)}\right|+...+|A_1(z)|\left|\frac{f'(z)}{f(z)}\right|. \tag{16}$$

Using (13)-(15) in (16) with $|z|=r\in F-E$, we get a contradiction; so $\sigma_{M,[p,q]}(f)=\infty$ and then for $p\leq q$, we have

$$\max\left\{\log\frac{1}{1-|z|},T(s(|z|),f)\right\}=T(s(|z|),f).$$

From Lemma 12, we can write

$$\left|\frac{f^{(j)}(z)}{f(z)}\right|\leq(T(s(r),f))^{j+\varepsilon}. \tag{17}$$

Now from (13)-(14) and (16)-(17), with $|z|=r\in F-E$, we get

$$\exp_p\left\{(\tau-\varepsilon)\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right\}\leq k(T(s(r),f))^{j+\varepsilon}\exp_p\left\{(\tau-2\varepsilon)\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right\}. \tag{18}$$

Set $s(r)=R$. We have $1-r=\frac{1}{d}(1-R)$ and for $R\in F$, (18) becomes

$$\exp_p\left\{(\tau-\varepsilon)\left(\log_{q-1}\frac{d}{1-R}\right)^{\sigma}\right\}\leq k(T(R,f))^{j+\varepsilon}\exp_p\left\{(\tau-2\varepsilon)\left(\log_{q-1}\frac{d}{1-R}\right)^{\sigma}\right\}. \tag{19}$$

From (19), we can easily obtain that $\sigma_{M,[p+1,q]}(f)=\sigma_{[p+1,q]}(f)\geq\sigma$. On the other hand, by lemma 13, we have $\sigma_{[p+1,q]}(f)\leq\max\left\{\sigma_{M,[p,q]}\left(A_j\right):j=0,1,...,k-1\right\}=\sigma_{M,[p,q]}(A_0)=\sigma$. So, we conclude that $\sigma_{M,[p+1,q]}(f)=\sigma_{M,[p,q]}(A_0)$. ■

Lemma 19 *Let $p > q \geq 1$ be integers and let $A_j(z)$ $j = 0, 1, \ldots, k-1$ be meromorphic functions in the unit disc D satisfying the condition (1) or (2) of Theorem 6 with $\delta(\infty, A_0) > 0$. Then, every solution $f \not\equiv 0$ of (1) satisfies*

$$\sigma_{[p+1,q]}(f) \geq \sigma_{[p,q]}(A_0).$$

Proof. For the condition (1), see [17, Thm 1.1]. Now, suppose that we have the condition (2). By the definition of $\tau_{[p,q]}(A_0)$ and Lemma 16, for any $\varepsilon > 0$ there exists a set $F \subset [0,1)$ of infinite logarithmic measure such that for $|z| = r \in F$, we have

$$T(r, A_0) > \exp_{p-1}\left\{(\tau - \varepsilon)\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right\}. \tag{20}$$

From the condition (2), there exists a set $E \subset [0,1)$ of finite logarithmic measure such that for $|z| = r \in [0,1) - E$, we have

$$T\left(r, A_j\right) \leq \exp_{p-1}\left\{(\tau - 2\varepsilon)\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right\}, \tag{21}$$

where $\varepsilon > 0$ small enough. Now, we follow the same stages (15)-(18) with

$$m(r, A_0) \leq \sum_{j=1}^{k-1} m\left(r, A_j\right) + \sum_{j=1}^{k} m\left(r, \frac{f^{(j)}}{f}\right) + O(1),$$

instead of (16) to obtain $\sigma_{[p+1,q]}(f) \geq \sigma$. ∎

Lemma 20 *Let $p \geq 1$ be an integer and let $A_j(z)$ $j = 0, 1, \ldots, k-1$ be analytic functions in the unit disc D satisfying the condition (1) or (2) of Theorem 7. Then, every solution $f \not\equiv 0$ of (1) satisfies*

$$\sigma_{[p+1,p]}(f) \geq \sigma_{[p,p]}(A_0).$$

Proof. For the condition (1), see [17, Thm 1.1]. The condition (2) implies that (20) and (21) hold for $p = q$, and so we can follow the same method of the proof of Lemma 18. ∎

Lemma 21 *Let $p \geq q \geq 1$ be integers and let $A_j(z)$ $j = 0, 1, \ldots, k-1$ be analytic functions in the unit disc D satisfying*

$$\max\left\{\sigma_{M,[p,q]}\left(A_j\right) : j = 1, 2, \ldots, k-1\right\} = \sigma_1 < \sigma_{M,[p,q]}(A_0) = \sigma < \infty \tag{22}$$

and U_j^i $(j = 0, 1, \ldots, k)$ $(i \in \mathbb{N})$ be stated as in (7). Then, for any given $\varepsilon > 0$ satisfying $\sigma - 2\varepsilon > \sigma_1$, there exists a set F of infinite logarithmic measure and a set E of finite logarithmic measure such that

$$\left|U_0^i\right| \geq \exp_p\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma-\varepsilon}\right\}, \ |z| = r \in F, \tag{23}$$

and

$$\left|U_j^i\right| \leq \exp_p\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma_1+\varepsilon}\right\}. \ j \neq 0, \ r \notin E. \tag{24}$$

Proof. The inductive method will be used. We start by (24) and $i = 1$. From (7), we have $U_j^1 = A_j + A_{j+1}\left(\frac{A'_{j+1}}{A_{j+1}} - \frac{A'_0}{A_0}\right)$; and by the triangular inequality, we get

$$\left|U_j^1\right| \leq \left|A_j\right| + \left|A_{j+1}\right|\left(\left|\frac{A'_{j+1}}{A_{j+1}}\right| + \left|\frac{A'_0}{A_0}\right|\right). \tag{25}$$

From the assumption (22), we have

$$\left|A_j\right| \le \exp_p\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma_1+\varepsilon/2}\right\} \quad (j \neq 0),\ r \notin E; \tag{26}$$

By Lemma 12, we get

$$\max\left\{\left|\frac{A'_{j+1}}{A_{j+1}}\right|, \left|\frac{A'_0}{A_0}\right|\right\} \le \exp_{p-1}\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{(\sigma_1+\varepsilon)}\right\},\ \varepsilon > 0. \tag{27}$$

From (25)-(27), for r near enough from 1^-, we obtain

$$\left|U_j^1\right| \le \exp_p\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma_1+\varepsilon}\right\}.$$

Now, we suppose that

$$\left|U_j^{i-1}\right| \le \exp_p\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma_1+\varepsilon/2}\right\},\ (j \neq 0). \tag{28}$$

From (7), we have

$$\left|U_j^i\right| \le \left|U_j^{i-1}\right| + \left|U_{j+1}^{i-1}\right|\left(\left|\frac{\left(U_{j+1}^{i-1}\right)'}{U_{j+1}^{i-1}}\right| + \left|\frac{\left(U_0^{i-1}\right)'}{U_0^{i-1}}\right|\right). \tag{29}$$

Using the properties of the order of growth of a meromorphic function and by induction on $i \in \mathbb{N}$, we can conclude that $\sigma_{[p,q]}\left(U_j^i\right) \le \max\left\{\sigma_{[p,q]}\left(A_j\right)\right\} \le \max\left\{\sigma_{M,[p,q]}\left(A_j\right)\right\}$ for every $i \in \mathbb{N}$ and $j = 0,1,\dots,k-1$; and by Lemma 12, we get

$$\left|\frac{\left(U_j^i\right)'}{U_j^i}\right| \le \exp_{p-1}\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{(\sigma_1+\varepsilon)}\right\},\ \varepsilon > 0. \tag{30}$$

From (28)-(30), we obtain

$$\left|U_j^i\right| \le \exp_p\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma_1+\varepsilon}\right\} \quad (j \neq 0). \tag{31}$$

Now we will prove (23) also by induction and we start by $i = 1$. Since $0 < \sigma_{M,[p,q]}(A_0) = \sigma < \infty$, then by Lemma 11, there exists a set F of infinite logarithmic measure such that for $|z| = r \in F$ we have

$$\lim_{r\to 1^-}\frac{\log^+_{p+1} M(r, A_0)}{\log_q\left(\frac{1}{1-r}\right)} = \sigma$$

and then, for every $\varepsilon > 0$ there exists $r_0 \in (0,1)$ such that for all $r \in F$ satisfying $r_0 < r < 1$ and $|A_0| = M(r, A_0)$ we have

$$|A_0| \ge \exp_p\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma-\varepsilon/2}\right\}. \tag{32}$$

Now we will prove that $\left|U_0^1\right| \ge \exp_p\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma-\varepsilon}\right\}$. From (7), we have $U_0^1 = A_0 + A_1\left(\frac{A'_1}{A_1} - \frac{A'_0}{A_0}\right)$; and so

$$\left|U_0^1\right| \ge |A_0| - |A_1|\left(\left|\frac{A'_1}{A_1}\right| + \left|\frac{A'_0}{A_0}\right|\right). \tag{33}$$

Using (26)-(27) and (32) in (33), we get

$$\left|U_0^1\right| \ge \exp_p\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma-\varepsilon/2}\right\} - \exp_p\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma_1+\varepsilon/2}\right\}, \tag{34}$$

which implies that

$$\left|U_0^1\right| \ge \exp_p\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma-\varepsilon}\right\}.$$

Suppose that

$$\left|U_0^{i-1}\right| \geq \exp_p\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma-\varepsilon/2}\right\}, \tag{35}$$

and we prove (22). From (7), we get

$$\left|U_0^i\right| \geq \left|U_0^{i-1}\right| - \left|U_1^{i-1}\right|\left(\left|\frac{\left(U_1^{i-1}\right)'}{U_1^{i-1}}\right| + \left|\frac{\left(U_0^{i-1}\right)'}{U_0^{i-1}}\right|\right). \tag{36}$$

Combining (30), (31) and (35) with (36), we get

$$\left|U_0^i\right| \geq \exp_p\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma-\varepsilon}\right\}.$$

■

By the same method of the proof of Lemma 21, we can prove the following lemma.

Lemma 22 *Let $p \geq q \geq 1$ be integers and let $A_j(z)$ $j = 0,1,...,k-1$ be analytic functions in the unit disc D satisfying $\max\left\{\sigma_{M,[p,q]}\left(A_j\right) : j = 1,2,...,k-1\right\} \leq \sigma_{M,[p,q]}(A_0) = \sigma < \infty$ and $\max\left\{\tau_{M,[p,q]}\left(A_j\right) : \sigma_{M,[p,q]}\left(A_j\right) = \sigma_{M,[p,q]}(A_0)\right\} = \tau_1 < \tau_{M,[p,q]}(A_0) = \tau < \infty$. Let U_j^i $(j = 0,1,...,k)$ $(i \in \mathbb{N})$ be stated as in (7). Then, for any given $\varepsilon > 0$ satisfying $\tau - \tau_1 > 2\varepsilon$, there exists a set F of infinite logarithmic measure and a set E of finite logarithmic measure such that*

$$\left|U_0^i\right| \geq \exp_p\left\{(\tau-\varepsilon)\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right\}, \quad r \in F,$$

and

$$\left|U_j^i\right| \leq \exp_p\left\{(\tau_1+\varepsilon)\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right\}. \quad j \neq 0, \ r \notin E.$$

Lemma 23 *Let $p \geq q \geq 1$ be integers and let $A_j(z)$ $j = 0,1,...,k-1$ be meromorphic functions in the unit disc D satisfying $\max\left\{\sigma_{[p,q]}\left(A_j\right) : j = 1,2,...,k-1\right\} = \sigma_1 < \sigma_{[p,q]}(A_0) = \sigma < \infty$ with $\delta(\infty, A_0) > 0$ and let U_j^i $(j = 0,1,...,k)$ $(i \in \mathbb{N})$ be stated as in (7). Then, for any given $\varepsilon > 0$ satisfying $\sigma - 2\varepsilon > \sigma_1$, there exists a set F of infinite logarithmic measure and a set E of finite logarithmic measure such that*

$$m\left(r, U_0^i\right) \geq \exp_{p-1}\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma-\varepsilon}\right\}, \quad r \in F, \tag{37}$$

and

$$m\left(r, U_j^i\right) \leq \exp_{p-1}\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma_1+\varepsilon}\right\}, \quad j \neq 0, \ r \notin E. \tag{38}$$

Proof. The inductive method will be used. We start by (38) and $i = 1$. From (7), we have $U_j^1 = A_j + A_{j+1}\left(\frac{A'_{j+1}}{A_{j+1}} - \frac{A'_0}{A_0}\right)$, $j \neq 0$; and by the proximity function properities, we get

$$m\left(r, U_j^1\right) \leq m\left(r, A_j\right) + m\left(r, A_{j+1}\right) + m\left(r, \frac{A'_{j+1}}{A_{j+1}}\right) + m\left(r, \frac{A'_0}{A_0}\right) + O(1).$$

From the assumption, we have

$$m\left(r, A_j\right) \leq T\left(r, A_j\right) \leq \exp_{p-1}\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma_1+\varepsilon/2}\right\}, \quad j \neq 0, \ r \notin E,$$

and by making use the logarithmic derivative formula

$$m\left(r,\frac{A_j'}{A_j}\right)=O\left(\log^+T\left(r,A_j\right)+\log\frac{1}{1-r}\right),$$

we deduce that

$$m\left(r,U_j^1\right)\leq \exp_{p-1}\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma_1+\varepsilon}\right\},\ \ j\neq 0,\ r\notin E.$$

Suppose that

$$m\left(r,U_j^{i-1}\right)\leq \exp_{p-1}\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma_1+\varepsilon/2}\right\},\ \ j\neq 0,\ r\notin E.$$

From (7), we have

$$m\left(r,U_j^i\right)\leq m\left(r,U_j^{i-1}\right)+m\left(r,U_{j+1}^{i-1}\right)+m\left(r,\frac{\left(U_{j+1}^{i-1}\right)'}{U_{j+1}^{i-1}}\right)+m\left(r,\frac{\left(U_0^{i-1}\right)'}{U_0^{i-1}}\right).$$

By making use the logarithmic derivative formula and by taking account that $\sigma_{[p,q]}\left(U_j^i\right)\leq\max\left\{\sigma_{[p,q]}\left(A_j\right)\right\}\leq\max\left\{\sigma_{[p,q]}\left(A_j\right)\right\}=\sigma$ for every $i\in\mathbb{N}$ and $j=0,1,\ldots,k-1$, we obtain that

$$m\left(r,U_j^i\right)\leq \exp_{p-1}\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma_1+\varepsilon}\right\},\ \ j\neq 0,\ r\notin E.$$

Now, we prove (37) for $i=1$. By Lemma 17 and $\delta(\infty,A_0)>0$, for any $\varepsilon>0$ there exists a set $F\subset[0,1)$ of infinite logarithmic measure such that for $|z|=r\in F$, we have

$$m(r,A_0)>\exp_{p-1}\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma-\varepsilon}\right\}. \tag{39}$$

From (7), we have $U_0^1=A_0+A_1\left(\frac{A_1'}{A_1}-\frac{A_0'}{A_0}\right)$; and by the proximity function properities, we get

$$m\left(r,U_0^1\right)\leq m(r,A_0)+m(r,A_1)+m\left(r,\frac{A_1'}{A_1}\right)+m\left(r,\frac{A_0'}{A_0}\right)+O(1). \tag{40}$$

On the other hand, also from (7), we have $A_0=U_0^1-A_1\left(\frac{A_1'}{A_1}-\frac{A_0'}{A_0}\right)$; and so

$$m(r,A_0)\leq m\left(r,U_0^1\right)+m(r,A_1)+m\left(r,\frac{A_1'}{A_1}\right)+m\left(r,\frac{A_0'}{A_0}\right)+O(1). \tag{41}$$

From (40)-(41), we conclude that $m\left(r,U_0^1\right)\sim m(r,A_0)$ as $r\to 1^-$ and by (39), we get

$$m\left(r,U_0^1\right)>\exp_{p-1}\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma-\varepsilon}\right\},\ \ r\in F.$$

If we suppose that

$$m\left(r,U_0^{i-1}\right)\geq\exp_{p-1}\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma-\varepsilon}\right\},\ \ r\in F,$$

then we can prove that

$$m\left(r,U_0^{i}\right)\geq\exp_{p-1}\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma-\varepsilon}\right\},\ \ r\in F,$$

by making use (7) and the proximity function properities as in (40)-(41). ■

By the same method of the proof of Lemma 23, we can prove the two following lemmas.

Lemma 24 *Let $p \geq q \geq 1$ be integers and let $A_j(z)$ $j = 0,1,...,k-1$ be meromorphic functions in the unit disc D satisfying $\max\left\{\sigma_{[p,q]}\left(A_j\right) : j = 1,2,...,k-1\right\} \leq \sigma_{[p,q]}(A_0) = \sigma < \infty$ and $\max\left\{\tau_{[p,q]}\left(A_j\right) : \sigma_{[p,q]}\left(A_j\right) = \sigma_{[p,q]}(A_0)\right\} = \tau_1 < \tau_{[p,q]}(A_0) = \tau < \infty$ with $\delta(\infty, A_0) > 0$. Let U_j^i $(j = 0,1,...,k)$ $(i \in \mathbb{N})$ be stated as in (7). Then, for any given $\varepsilon > 0$ satisfying $\tau - \tau_1 > 2\varepsilon$, there exists a set F of infinite logarithmic measure and a set E of finite logarithmic measure such that*

$$m\left(r, U_0^i\right) \geq \exp_{p-1}\left\{(\tau - \varepsilon)\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right\}, \quad r \in F,$$

and

$$m\left(r, U_j^i\right) \leq \exp_{p-1}\left\{(\tau_1 + \varepsilon)\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right\}. \quad j \neq 0, \; r \notin E.$$

Lemma 25 *Let $p = q \geq 1$ be integers and let $A_j(z)$ $j = 0,1,...,k-1$ be analytic functions in the unit disc D satisfying $\sum_{j\in J} \tau_{[p,q]}\left(A_j\right) = \tau_1 < \tau_{[p,q]}(A_0) = \tau < \infty$; where $J = \left\{j : \sigma_{[p,q]}\left(A_j\right) = \sigma_{[p,q]}(A_0)\right\}$ and $\sigma_{[p,q]}\left(A_j\right) < \sigma_{[p,q]}(A_0)$ for $j \notin J$. Let U_j^i $(j = 0,1,...,k)$ $(i \in \mathbb{N})$ be stated as in (7). Then, for any given $\varepsilon > 0$ satisfying $\tau - \tau_1 > 2\varepsilon$, there exists a set F of infinite logarithmic measure and a set E of finite logarithmic measure such that*

$$m\left(r, U_0^i\right) \geq \exp_{p-1}\left\{(\tau - \varepsilon)\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right\}, \quad r \in F,$$

and

$$m\left(r, U_j^i\right) \leq \exp_{p-1}\left\{(\tau_1 + \varepsilon)\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right\}, \quad j \neq 0, \; r \notin E.$$

Lemma 26 *Let $p \geq q \geq 1$ be integers and let $H_j(z)$ $j = 0,1,...,k-1$ be meromorphic functions in the unit disc D satisfying*

$$\max\left\{\left|H_j(z)\right|, \; j = 1,...,k-1\right\} \leq \exp_p\left\{\beta\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right\}$$

and

$$|H_0(z)| \geq \exp_p\left\{\alpha\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right\}$$

for $|z| = r \in F \subset (0,1)$ of infinite logarithmic measure, where $\alpha > \beta > 0$, $\sigma > 0$. Then, every meromorphic solution f of the differential equation

$$f^{(k)} + H_{k-1}(z) f^{(k-1)} + ... + H_1(z) f' + H_0(z) f = 0 \tag{42}$$

satisfies $\sigma_{[p+1,q]}(f) \geq \sigma$.

Proof. Suppose that $f \not\equiv 0$ is a meromorphic solution of (42) with $\sigma_{[p,q]}(f) = \rho < \infty$. From (42), we get

$$|H_0(z)| \leq \left|\frac{f^{(k)}}{f}\right| + \sum_{j=1}^{k-1}\left|H_j(z)\right|\left|\frac{f^{(j)}}{f}\right|. \tag{43}$$

By Lemma 12, for a given $\varepsilon > 0$ there exists a set $E \subset [0,1)$ of finite logarithmic measure such that for all $z \in D$ satisfying $|z| = r \notin E$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \exp_{p-1}\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\rho+\varepsilon}\right\}. \tag{44}$$

From (43)-(44) and the assumptions of Lemma 26, we get

$$\exp_p\left\{\alpha\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right\} \leq c\exp_p\left\{\beta\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right\}\exp_{p-1}\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\rho+\varepsilon}\right\}, \tag{45}$$

where $c>0$ is a constant. Since $\beta<\alpha$, a contradiction follows from (45) as $r\to 1^-$. So, $\sigma_{[p,q]}(f)=\infty$. Now by Lemma 12, we get

$$\left|\frac{f^{(j)}(z)}{f(z)}\right|\leq\frac{1}{(1-r)^{j(2+\varepsilon)}}\left(T\left(s(r),f\right)\right)^j,\ r\notin E. \tag{46}$$

From (43), (46) and the assumptions of this lemma, we get

$$\exp_p\left\{\alpha\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right\}\leq\frac{c}{(1-r)^{k(2+\varepsilon)}}\left(T\left(s(r),f\right)\right)^k\exp_p\left\{\beta\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right\}. \tag{47}$$

Set $s(r)=R$. We have $1-r=\frac{1}{d}(1-R)$ and for $R\in F$, (47) becomes

$$\exp_p\left\{\alpha\left(\log_{q-1}\frac{d}{1-R}\right)^{\sigma}\right\}\leq c\left(\frac{d}{1-R}\right)^{k(2+\varepsilon)}\left(T(R,f)\right)^k\exp_p\left\{\beta\left(\log_{q-1}\frac{d}{1-R}\right)^{\sigma}\right\}. \tag{48}$$

From (48), we conclude that

$$\sigma_{[p+1,q]}(f)\geq\sigma.$$

■

By using the same method of the proof of Lemma 26, we get the following lemma.

Lemma 27 *Let $p\geq q\geq 1$ be integers and let $H_j(z)$ $j=0,1,\dots,k-1$ be meromorphic functions in the unit disc D satisfying*

$$\max\left\{\left|H_j(z)\right|,\ j=1,\dots,k-1\right\}\leq\exp_p\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\beta}\right\}$$

and

$$|H_0(z)|\geq\exp_p\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right\}$$

for $|z|=r\in F\subset(0,1)$ of infinite logarithmic measure, where $\sigma>\beta>0$. Then, every meromorphic solution f of (42) satisfies $\sigma_{[p+1,q]}(f)\geq\sigma$.

Lemma 28 *Let $p\geq q\geq 1$ be integers and let $H_j(z)$ $j=0,1,\dots,k-1$ be meromorphic functions in the unit disc D satisfying*

$$m\left(r,H_j\right)\leq\exp_{p-1}\left\{\beta\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right\},\ j\neq 0, \tag{49}$$

and

$$m(r,H_0)\geq\exp_{p-1}\left\{\alpha\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right\}, \tag{50}$$

where $|z|=r\in F$ and $\alpha>\beta>0$. Then, every meromorphic solution f of (42) satisfies $\sigma_{[p+1,q]}(f)\geq\sigma$.

Proof. From (42), we have

$$m(r,H_0)\leq\sum_{j=1}^{k-1}m\left(r,H_j\right)+\sum_{j=1}^{k}m\left(r,\frac{f^{(j)}}{f}\right)+O(1).$$

By making use the logarithmic derivative formula

$$m\left(r,\frac{f^{(j)}}{f}\right)\leq c\left(\log^+T(r,f)+\log\frac{1}{1-r}\right)^j,$$

and by taking account the assumptions (49) and (50), we get

$$\exp_{p-1}\left\{\alpha\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right\} \leq (k-1)\exp_{p-1}\left\{\beta\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right\}+ c\left(\log^{+}T(r,f)+\log\frac{1}{1-r}\right)^{j}. \quad (51)$$

From (51), it is easy to obtain that $\sigma_{[p+1,q]}(f) \geq \sigma$. ■

By using the same method of the proof of Lemma 28, we get the following lemma.

Lemma 29 *Let $p \geq q \geq 1$ be integers and let $H_j(z)$ $j = 0,1,...,k-1$ be meromorphic functions in the unit disc D satisfying*

$$m(r,H_0) \geq \exp_{p-1}\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\sigma}\right\},$$

and

$$m\left(r,H_j\right) \leq \exp_{p-1}\left\{\left(\log_{q-1}\frac{1}{1-r}\right)^{\beta}\right\},\ j \neq 0,$$

where $|z| = r \in F$ and $0 < \beta < \sigma$. Then, every meromorphic solution f of (42) satisfies $\sigma_{[p+1,q]}(f) \geq \sigma$.

Lemma 30 *[17] Let $G(z) \not\equiv 0, H_j(z)$ $j = 0,1,...,k-1$ be meromorphic functions in the unit disc D. If f is a meromorphic solution of the differential equation*

$$f^{(k)} + H_{k-1}(z)f^{(k-1)} + ... + H_1(z)f' + H_0(z)f = G(z), \quad (52)$$

satisfying $\max\left\{\sigma_{[p,q]}(G), \sigma_{[p,q]}\left(H_j\right);\ j = 0,1,...,k-1\right\} < \sigma_{[p,q]}(f)$, *then* $\overline{\lambda}_{[p,q]}(f) = \lambda_{[p,q]}(f) = \sigma_{[p,q]}(f),\ (p \geq q \geq 1)$.

4 Proof of theorems

We signal here that the major part of the proof is made in the section Preliminaries lemmas.

Proof of Theorem 5.

Assume that $f \not\equiv 0$ is a solution of (1) and $\varphi(z) \not\equiv 0$ is an analytic function in the unit disc D satisfying $\sigma_{[p+1,q]}(\varphi) < \sigma_{[p,q]}(A_0)$. We start to prove (2) for $i = 0$, i.e. $\overline{\lambda}_{[p+1,q]}(f-\varphi) = \lambda_{[p+1,q]}(f-\varphi) = \sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$. By Lemma 18, we have $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$. Set $g = f - \varphi$. Since $\sigma_{[p+1,q]}(\varphi) < \sigma_{[p,q]}(A_0)$ then $\sigma_{[p+1,q]}(g) = \sigma_{[p+1,q]}(f)$. By Lemma 9, g satisfies (5). Set $G(z) = \varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + ... + A_0\varphi$. If $G \equiv 0$, then by Lemma 18 we have $\sigma_{[p+1,q]}(\varphi) = \sigma_{[p,q]}(A_0)$, a contradiction; thus $G \not\equiv 0$. Now, since $\sigma_{[p+1,q]}(g) = \sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0) > \max\left\{\sigma_{[p+1,q]}(G), \sigma_{[p+1,q]}\left(A_j\right)\right\}$, then the assumption of Lemma 30 holds, and then we have $\overline{\lambda}_{[p+1,q]}(g) = \lambda_{[p+1,q]}(g) = \sigma_{[p+1,q]}(g)$. Then, we conclude that $\overline{\lambda}_{[p+1,q]}(f-\varphi) = \lambda_{[p+1,q]}(f-\varphi) = \sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$. Now we prove the claim for $i \geq 1$. Set $g_i = f^{(i)} - \varphi$. Since $\sigma_{[p+1,q]}\left(f^{(i)}\right) = \sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$ and $\sigma_{[p+1,q]}(\varphi) < \sigma_{[p,q]}(A_0)$, then we have $\sigma_{[p+1,q]}(g_i) = \sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$. By Lemma 10, g_i satisfies (6). Set $G_i = \varphi^{(k)} + U_{k-1}^{i}\varphi^{(k-1)} + ... + U_0^{i}\varphi$. If $G_i \equiv 0$, by Lemma 21, Lemma 22, Lemma 26 and Lemma 27, we get $\sigma_{[p+1,q]}(\varphi) \geq \sigma_{[p,q]}(A_0)$, a contradiction; so $G_i \not\equiv 0$. Now, by Lemma 30, we obtain $\overline{\lambda}_{[p+1,q]}(g_i) = \lambda_{[p+1,q]}(g_i) = \sigma_{[p+1,q]}(g_i)$ i.e. $\overline{\lambda}_{[p+1,q]}\left(f^{(i)}-\varphi\right) = \lambda_{[p+1,q]}\left(f^{(i)}-\varphi\right) = \sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$. ■

Proof of Theorem 6.

Assume that $f \not\equiv 0$ is a meromorphic solution of (1) and $\varphi(z) \not\equiv 0$ is a meromorphic function in the unit disc D

satisfying $\sigma_{[p+1,q]}(\varphi) < \sigma_{[p,q]}(A_0)$. By Lemma 19, we have $\sigma_{[p+1,q]}(f) \geq \sigma_{[p,q]}(A_0)$. Set $g = f - \varphi$. Since $\sigma_{[p+1,q]}(\varphi) < \sigma_{[p,q]}(A_0)$ then $\sigma_{[p+1,q]}(g) = \sigma_{[p+1,q]}(f)$. By Lemma 9, g satisfies (5). Set $G(z) = \varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + ... + A_0\varphi$. If $G \equiv 0$, then by Lemma 19 we have $\sigma_{[p+1,q]}(\varphi) \geq \sigma_{[p,q]}(A_0)$, a contradiction; thus $G \not\equiv 0$; and by Lemma 30 we get the result for $i = 0$. Now for $i \geq 1$, using the same notation as in proof of Theorem 5: $g_i = f^{(i)} - \varphi$ and $G_i = \varphi^{(k)} + U^i_{k-1}\varphi^{(k-1)} + ... + U^i_0\varphi$; if $G_i \equiv 0$, then by Lemma 23, Lemma 24 ,Lemma 28 and Lemma 29, we have $\sigma_{[p+1,q]}(\varphi) \geq \sigma_{[p,q]}(A_0)$, a contradiction; thus $G_i \not\equiv 0$; and by Lemma 30, we get $\overline{\lambda}_{[p+1,q]}\left(f^{(i)} - \varphi\right) = \lambda_{[p+1,q]}\left(f^{(i)} - \varphi\right) = \sigma_{[p+1,q]}(f) \geq \sigma_{[p,q]}(A_0)$. ■

Proof of Theorem 7.

By Lemma 13 and Lemma 20, we get that every solution $f \not\equiv 0$ of (1) satisfies $\sigma_{[p,q]}(A_0) \leq \sigma_{[p+1,q]}(f) \leq \alpha_M$. We use the same method and notations of the proof of Theorem 5. If $G \equiv 0$, by Lemma 20, we get $\sigma_{[p+1,q]}(\varphi) \geq \sigma_{[p,q]}(A_0)$, a contradiction; so $G \not\equiv 0$; and by Lemma 30, we get $\overline{\lambda}_{[p+1,q]}(f - \varphi) = \lambda_{[p+1,q]}(f - \varphi) = \sigma_{[p+1,q]}(f)$. If $G_i \equiv 0$, by Lemma 23, Lemma 25 ,Lemma 28 and Lemma 29, we get $\sigma_{[p+1,q]}(\varphi) \geq \sigma_{[p,q]}(A_0)$, a contradiction; so $G_i \not\equiv 0$; and by Lemma 30, we get $\overline{\lambda}_{[p+1,q]}\left(f^{(i)} - \varphi\right) = \lambda_{[p+1,q]}\left(f^{(i)} - \varphi\right) = \sigma_{[p+1,q]}(f)$. ■

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