Extending the applicability of an efficient fifth order method under weak conditions in Banach space

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Abstract

We extend the applicability of an efficient fifth order method for solving Banach space valued equations. To achieve this we use weaker Lipschitz-type conditions in combination with our idea of the restricted convergence region. Numerical examples are used to compare our results favorably to the ones in earlier works.

Keywords: Banach space; three step method; local-semi-local convergence; weak conditions.

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Introduction

Fifth convergence order of a three step method for approximating a solution of the equation

$$Q(x) = 0, \tag{0.1}$$

where $Q : D \subset E_1 \longrightarrow E_2$ is a continuously differentiable operator in the sense of Fréchet was proved in [22]. Here E_1, E_2 are Banach spaces. The fifth order method in [22] was defined for $n = 0, 1, 2, \ldots,$ by

$$\begin{aligned} y_n &= Q'(x_n)^{-1}Q(x_n) \\ z_n &= y_n - Q'(x_n)^{-1}Q(y_n) \\ x_{n+1} &= z_n - Q'(y_n)^{-1}Q(z_n), \end{aligned} \tag{0.2}$$

where x_0 is an initial point.

The semi-local convergence analysis of method (0.2) was based on hypotheses reaching the second derivatives limit the applicability of method (0.2) to functions Q that are at least twice differentiable on some domain containing the solution x_* of the equation $Q(x) = 0$.

The study of convergence of iterative algorithms is usually centered into two categories: semi-local and local convergence analysis. The semi-local convergence is based on the information around an initial point, to obtain conditions ensuring the convergence of these algorithms, while the local convergence is based on the information around a solution to find estimates of the computed radii of the convergence balls. Local results are important since they provide the degree of difficulty in choosing initial points.

Our local convergence analysis of method (0.2) is based only on the first derivative. This way we expand the applicability of method (0.2). Furthermore, in [22] the initial guess x_0 should be close to x_* for the convergence of method (0.2). But, how close the initial guess should be for the convergence of method (0.2), for this we need computable error bounds on the distances $\|x_n - x_0\|$. These concerns are addressed in what follows. Notice that the convergence order of method (0.2) can be obtained by computing the computational order of convergence (COC) [1] given by

$$\xi = \frac{\ln(\frac{\|x_{n+2}-x_*\|}{\|x_{n+1}-x_*\|})}{\ln(\frac{\|x_{n+1}-x_*\|}{\|x_n-x_*\|})} \tag{0.3}$$

or the approximate computational order of convergence (ACOC) [1] given by

$$\xi^* = \frac{\ln(\frac{\|x_{n+2}-x_{n+1}\|}{\|x_{n+1}-x_n\|})}{\ln(\frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|})} \tag{0.4}$$

that do not require high order derivatives. Notice also that ξ^* does not even require the knowledge of the exact root x_*. Moreover, we present our semi-local convergence analysis that also compares favorably to the one given in [22].

The rest of the article is organized as follows: In Section 2, we present the local convergence analysis of method (0.2). The semi-local convergence follows in Section 3. The numerical examples appear in Section 4.

1 Local convergence analysis

It is convenient for us to introduce some parameters and functions. Set $I = [0, \infty)$. Let $\mu_0 : I \longrightarrow I$ be a continuous and increasing function with $\mu_0(0) = 0$. Assume equation

$$\mu_0(t) = 1 \tag{1.1}$$

has at least one positive solution. We denote by ρ_0 the smallest such solution. Set $I_1 = [0, \rho_0)$. Let $\mu : I_1 \longrightarrow I$ and $\mu_1 : I_1 \longrightarrow I$ be continuous and increasing functions with $\mu(0) = 0$. Define functions a and $\bar{a}$ on the interval I_1 bt

$$a(t) = \frac{\int_0^1 \mu((1-\theta)t)d\theta}{1-\mu_0(t)}$$

and

$$\bar{a}(t) = a(t) - 1.$$

We have $\bar{a}(0) = -1$ and $\bar{a}(t) \longrightarrow \infty$ as $t \longrightarrow \rho_0^-$. By the Intermediate value theorem equation $\bar{\alpha}(t) = 0$ has at least one solution in $(0, \rho_0)$. Denote by r_1 the smallest such solution. Assume equation

$$\mu_0(a(t)t) = 1 \tag{1.2}$$

has at least one positive solution. Denote by ρ_1 the smallest such solution. Set $\rho_2 = \min\{\rho_0, \rho_1\}$ and $I_2 = [0, \rho_2)$. Define functions b and $\bar{b}$ on the interval I_2 by

$$\begin{aligned} b(t) \quad = \quad & \left[\frac{\int_0^1 \mu((1-\theta)a(t)t)d\theta}{1-\mu_0(a(t)t)} \right. \\ & \left. + \frac{(\mu_0(a(t)t) + \mu_0(t)) \int_0^1 \mu_1(\theta a(t)t)d\theta}{(1-\mu_0(a(t)t))(1-\mu_0(t))} \right] a(t) \end{aligned}$$

and

$$\bar{b}(t) = b(t) - 1.$$

We get $\bar{b}(0) = -1$ and $\bar{b}(t) \longrightarrow \infty$ as $t \longrightarrow \rho_2^-$. Denote by r_2 the smallest solution of equation $\bar{b}(t) = 0$ in $(0, \rho_2)$. Assume equation

$$\mu_0(b(t)t) = 1 \tag{1.3}$$

has at least one positive solution. Denote by ρ_3 the smallest such solution. Set $\rho = \min\{\rho_2, \rho_3\}$ and $I_3 = [0, \rho)$. Define functions c and $\bar{c}$ on the interval I_3 by

$$\begin{aligned} c(t) \quad = \quad & \left[\frac{\int_0^1 \mu((1-\theta)b(t)t)d\theta}{1-\mu_0(b(t)t)} \right. \\ & \left. + \frac{(\mu_0(b(t)t) + \mu_0(t)) \int_0^1 \mu_1(\theta b(t)t)d\theta}{(1-\mu_0(b(t)t))(1-\mu_0(a(t)t))} \right] b(t) \end{aligned}$$

and

$$\bar{c}(t) = c(t) - 1.$$

We get $\bar{c}(0) = -1$ and $\bar{c}(t) \longrightarrow \infty$ as $t \longrightarrow \rho^-$. Denote by r_3 the smallest solution of equation $\bar{c}(t) = 0$ in $(0, \rho)$.

Define a radius of convergence r by

$$r = \min\{r_j\},\ j = 1, 2, 3. \tag{1.4}$$

Then, we have for each $t \in [0, r)$

$$0 \leq \mu_0(t) < 1 \tag{1.5}$$
$$0 \leq \mu_0(a(t)t) < 1 \tag{1.6}$$
$$0 \leq \mu_0(b(t)t) < 1 \tag{1.7}$$
$$0 \leq a(t) < 1 \tag{1.8}$$
$$0 \leq b(t) < 1 \tag{1.9}$$

and

$$0 \leq c(t) < 1. \tag{1.10}$$

By $S(x, d)$ we denote a ball in E_1 with center $x \in E_1$ and of radius $d > 0$. Then, $\bar{S}(x, d)$ denotes its closure. We shall rely on the conditions (A):

(a1) $F : D \longrightarrow E_2$ is a continuously differentiable operator in the sense of Fréchet and there exists $x_* \in D$ such that $F(x_*) = 0$ and $F'(x_*)^{-1} \in \mathcal{L}(E_2, E_1)$ (the set of all bounded linear operators from E_2 into E_1).

(a2) There exists function $\mu_0 : I \longrightarrow I$ continuous and increasing with $\mu_0(0) = 0$ such that for each $x \in D$

$$\|F'(x_*)^{-1}(F'(x) - F'(x_*))\| \leq \mu_0(\|x - x_*\|)$$

Set $D_0 = D \cap S(x_*, \rho_0)$, where ρ_0 is given in (1.1).

(a3) There exist functions $\mu : I_1 \longrightarrow I$ and $\mu_1 : I_1 \longrightarrow I$ continuous and increasing with $\mu(0) = 0$ such that for each $x, y \in D_0$

$$\|F'(x_*)^{-1}(F'(y) - F'(x))\| \leq \mu(\|y - x\|)$$

and

$$\|F'(x_*)^{-1}F'(x)\| \leq \mu_1(\|x - x_*\|).$$

(a4) $\bar{S}(x_*, r) \subseteq D, \rho_0, \rho_1, \rho_3$ given by (1.1), (1.2), (1.3) respectively exist, and r is defined in (1.4).

(a5) There exists $\bar{r} \geq r$ such that

$$\int_0^1 \mu_0(\theta\bar{r})d\theta < 1.$$

Set $D_1 = D \cap \bar{S}(x_*, \bar{r})$.

Next, we provide the local convergence analysis of method (0.2) using the preceding notation, and the conditions (A).

THEOREM 1.1 *Under the conditions (A), choose* $x_0 \in S(x_*, r) - \{x_*\}$. *Then,* $y_n, z_n, x_{n+1} \in S(x_*, r)$, $\lim_{n\longrightarrow\infty} x_n = x_*$,

$$\|y_n - x_*\| \leq a(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\| < r, \tag{1.11}$$
$$\|z_n - x_*\| \leq b(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\| \tag{1.12}$$

and

$$\|x_{n+1} - x_*\| \leq c(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\|, \tag{1.13}$$

where functions a, b, c *are given previously and* r *is defined in (1.4). Moreover, the limit point* x_* *is the unique solution of equation* $F(x) = 0$ *in the set* D_1 *defined in (a5).*

Proof. Let $x \in S(x_*, r) - \{x_*\}$. By (a1), (a2), (1.4) and (1.5), we obtain

$$\|F'(x_*)^{-1}(F'(x) - F'(x_*))\| \leq \mu_0(\|x - x_*\|) \leq \mu_0(r) < 1, \tag{1.14}$$

so by the Banach perturbation Lemma [1–5], we get $F'(x)^{-1} \in \mathcal{L}(E_2, E_1)$,

$$\|F'(x)^{-1}F'(x_*)\| \leq \frac{1}{1 - \mu_0(\|x - x_*\|)} \tag{1.15}$$

and y_0, z_0 are well defined. By method (0.2), substep one, we can write

$$y_0 - x_* = x_0 - x_* - F'(x_0)^{-1}F(x_0). \tag{1.16}$$

Using (1.4), (1.8), (a3), (1.15) and (1.16), we get in turn that

$$\begin{aligned} \|y_0 - x_*\| &\leq \|F'(x_0)^{-1}F'(x_*)\| \\ &\quad \|\int_0^1 F'(x_*)^{-1}(F'(x_* + \theta(x_0 - x_*) - F'(x_0)d\theta(x_0 - x_*)\| \\ &\leq \frac{\int_0^1 \mu((1-\theta)\|x_0 - x_*\|)d\theta}{1 - \mu_0(\|x_0 - x_*\|}\|x_0 - x_*\| \\ &= a(\|x_0 - x_*\|)\|x_0 - x_*\| \\ &\leq \|x_0 - x_*\| < r, \end{aligned} \tag{1.17}$$

so (1.11) holds for $n = 0$ and $y_0 \in S(x_*, r)$. We can write by (a1) that

$$F(x) = F(x) - F(x_*) = \int_0^1 F'(x_* + \theta(x - x_*))d\theta(x - x_*). \tag{1.18}$$

Then, by (a3) and (1.18), we get that

$$\|F'(x_*)^{-1}F(x)\| \leq \int_0^1 \mu_1(\theta\|x - x_*\|)d\theta. \tag{1.19}$$

We can write by the second substep of method (0.2) that

$$\begin{aligned} z_0 - x_* &= (y_0 - x_* - F'(y_0)^{-1}F(y_0)) \\ &\quad + F'(y_0)^{-1}(F'(x_0) - F'(y_0))F'(x_0)^{-1}F(y_0). \end{aligned} \tag{1.20}$$

In view of (1.4), (1.6), (1.9), (1.15) (for $x = y_0$), (1.17), (1.19) (for $x = y_0$) and (1.20), we have in turn that

$$\begin{aligned} \|z_0 - x_*\| &\leq \|y_0 - x_* - F'(y_0)^{-1}F(y_0)\| \\ &\quad + \|F'(y_0)^{-1}F'(y_*)\|[\|F'(x_*)^{-1}(F'(x_0) - F'(x_*))\| \\ &\quad + \|F'(x_*)^{-1}(F'(y_0) - F'(x_*))\| \\ &\quad \times \|F'(x_0)^{-1}F'(x_*)\|\|F'(x_*)^{-1}F(y_0)\| \\ &\quad \Bigg[\frac{\int_0^1 \mu((1-\theta)\|y_0 - x_*\|)d\theta}{1 - \mu_0(\|y_0 - x_*\|)} \\ &\quad + \frac{(\mu_0(\|y_0 - x_*\|) + \mu_0(\|x_0 - x_*\|)\int_0^1 \mu_1(\theta\|y_0 - x_*\|)d\theta}{(1 - \mu_0(\|y_0 - x_*\|))(1 - \mu_0(\|x_0 - x_*\|))}\Bigg]\|y_0 - x_*\| \\ &\leq b(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\| < r, \end{aligned} \tag{1.21}$$

so (1.12) holds for $n = 0$ and $z_0 \in S(x_*, r)$. The point x_1 is well defined by the last substep of method (0.2) for $n = 0$ and (1.15) (for $x = y_0$). Then, we can write

$$x_1 - x_* = z_0 - x_* - F'(z_0)^{-1}F(z_0) + F'(z_0)^{-1}(F'(y_0) - F'(z_0))F'(y_0)^{-1}F(z_0). \tag{1.22}$$

By, (1.4), (1.7), (1.10), (1.17), (1.21) and (1.22), we obtain in turn that

$$\begin{aligned}
\|x_1 - x_*\| &= \|(z_0 - x_* - F'(z_0)^{-1}F(z_0)) \\
&\quad + F'(z_0)^{-1}(F'(y_0) - F'(z_0))F'(y_0)^{-1}F(z_0)\| \\
&\le \|z_0 - x_* - F'(z_0)^{-1}F(z_0)\| \\
&\quad + \|F'(z_0)^{-1}F'(x_*)\|[\|F'(x_*)^{-1}(F'(y_0) - F'(x_*))\| \\
&\quad + \|F'(x_*)^{-1}(F'(z_0) - F'(x_*))\|]\|F'(y_0)^{-1}F'(x_*)\|\|F'(x_*)^{-1}F(z_0)\| \\
&\le \Bigg[\frac{\mu((1-\theta)\|z_0 - x_*\|)d\theta}{1 - \mu_0(\|z_0 - x_*\|)} \\
&\quad + \frac{(\mu_0(\|z_0 - x_*\|) + \mu_0(\|y_0 - x_*\|))\int_0^1 \mu_1(\theta\|z_0 - x_*\|)d\theta}{(1 - \mu_0(\|z_0 - x_*\|))(1 - \mu_0(\|y_0 - x_*\|))}\Bigg]\|z_0 - x_*\| \\
&\le c(\|x_0 - x_*\|)\|x_0 - x_*\| \le \|x_0 - x_*\| < r, \qquad (1.23)
\end{aligned}$$

so (1.13) holds for $n = 0$ and $x_1 \in D(x_*, r)$. Replace x_0, y_0, z_0, x_1 by x_k, y_k, z_k, x_{k+1} in the preceding estimates to complete the induction for (1.11)–(1.13). It then follows from the estimate

$$\|x_{k+1} - x_*\| \le \tau\|x_k - x_*\|,\ \tau = c(\|x_0 - x_*\|) \in [0, 1) \qquad (1.24)$$

that $\lim_{k\longrightarrow\infty} x_k = x_*$ and $x_{k+1} \in S(x_*, r)$. Let $G = \int_0^1 F'(x_* + \theta(x_{**} - x_*))d\theta$ for some $x_{**} \in D_1$ with $F(x_{**}) = 0$. Next, by (a2) and (a5), we have

$$\begin{aligned}
\|F'(x_*)^{-1}(G - F'(x_*))\| &\le \int_0^1 \mu_0(\theta\|x_{**} - x_*\|)d\theta \\
&\le \int_0^1 \mu_0(\theta\bar{r})d\theta < 1,
\end{aligned}$$

so $G^{-1} \in \mathcal{L}(E_1, E_1)$. Finally, from the estimate $0 = F(x_{**}) - F(x_*) = G(x_{**} - x_*)$, we get $x_* = x_{**}$.

□

REMARK 1.2 *(a) In view of (a2), we can write*

$$\begin{aligned}
\|F'(x_*)^{-1}F'(x)\| &= \|F'(x_*)^{-1}[(F'(x) - F'(x_*)) + F'(x_*)]\| \\
&\le 1 + \|F'(x_*)^{-1}(F'(x) - F'(x_*))\| \\
&\le 1 + \mu_0(\|x - x_*\|), \qquad (1.25)
\end{aligned}$$

so the second condition in (a3) can be dropped, and we choose $\mu_1(t) = 1 + \mu_0(t)$, *with* $\mu_0(t) = L_0 t$ *or* $\mu_1(t) = 2$, *since* $t \in [0, \frac{1}{L_0})$.

2 Semi-local convergence analysis

The authors in [22] used the following standard conditions (C):

(c1) $\|Q'(x_0)^{-1}\| \le \beta_0$

(c2) $\|Q'(x_0)^{-1}Q(x_0)\| \le \eta_0^1$

(c3) $\|Q''(x)\| \le M^1$

(c4) $\|Q''(x) - Q''(y)\| \le w^1(\|x - y\|),\ x, y \in \Omega$, where function $w^1(x), x > 0$, is continuous nondecreasing, with $w^1(0) \ge 0$ such that $w^1(sx) \le s^\lambda w^1(x)$ for $s \in [0, 1]$, $x \in (0, \infty)$ and $q \in [0, 1]$.

Let $r_0^1 = M^1\beta_0\eta_0^1$, $s_0^1 = \beta_0\eta_0^1 w(\eta_0^1)$ and define sequences $\{r_n^1\}, \{s_n^1\}$ and $\{\eta_n^1\}$ for $n = 0, 1, 2, \dots,$ by

$$\begin{aligned}
r_{n+1}^1 &= r_n^1\varphi^1(r_n^1)^2\psi^1(r_n^1, s_n^1), \qquad (2.1) \\
s_{n+1}^1 &= s_n^1\varphi^1(r_n^1)^{2+\lambda}\psi^1(r_n^1, s_n^1)^{1+\lambda}, \qquad (2.2) \\
\eta_{n+1}^1 &= \eta_n^1\varphi^1(r_n^1)\psi^1(r_n^1, s_n^1), \qquad (2.3)
\end{aligned}$$

where

$$\varphi^1(u) = \frac{1}{1-ug^1(u)}, \quad (2.4)$$

$$g^1(u) = \left(1+\frac{u}{2}+\frac{u^2}{2(1-u)}(1+\frac{u}{4})\right), \quad (2.5)$$

and

$$\psi^1(u,v) = \frac{u^2}{2(1-u)}(1+\frac{u}{4})\left[\frac{v}{1+\lambda}\left(\frac{u^{1+\lambda}}{2^{1+\lambda}}+\frac{1}{2+\lambda}\left(\frac{u^2}{2(1-u)}(1+\frac{u}{4})\right)^{1+\lambda}\right)\right.$$
$$\left.+\frac{u}{2}(u+\frac{u^2}{2(1-u)}(1+\frac{u}{4}))\right]. \quad (2.6)$$

Using the conditions (C), they showed the following semi-local result for method (0.2).

THEOREM 2.1 *Under the conditions (C) further suppose $r_0^1 < v^1$ and $\bar{S}(x_0, R^1\eta_0^1) \subseteq D$, where $R^1 = \frac{g^1(r_0^1)}{1-\gamma^1\delta^1}$, $\gamma^1 \in (0,1)$ and $\delta^1 = \frac{1}{\varphi^1(r_0^1)}$. Then, $y_n, z_n, x_{n+1} \in \bar{S}(x_0, R^1\eta_0^1)$; there exists $x_* \in \bar{S}(x_0, R^1\eta_0^1)$ which is the only solution of equation $F(x) = 0$ in $S^1 = S(x_0, \frac{2}{M^1\beta_0} - R^1\eta_0^1) \cap D$, and*

$$\|x_n - x_*\| \le g^1(r_0^1)(\delta^1)^k \frac{(\gamma^1)^{\frac{(4+\lambda)^n-1}{3+\lambda}}}{1-\delta^1(\gamma^1)^{(4+\lambda)^n}}\eta_0^1 \quad (2.7)$$

where v^1 is the smallest positive root of equation $h^1(t) = g^1(t)t - 1$.

□

Next, we present our improvements using conditions (H):

(h1) =(c1)

(h2) =(c2)

(h_2^0) $\|Q'(x) - Q'(x_0)\| \le L^0\|x - x_)\|$ for each $x \in D$. Set $D_2 = D \cap S(x_0, \frac{1}{\beta_0 L^0})$

(h3) $\|Q''(x)\| \le M^0$ for each $x \in D_2$.

(h4) $\|Q''(x) - Q''(y)\| \le w^0(\|x-y\|)$ for each $x, y \in D_2$, where function $w^0(x), x > 0$ is continuous, non-decreasing with $w^0(0) \ge 0$ such that $w^0(sx) \le s^\lambda w^0(x)$, for $s \in [0,1], x \in (0,\infty)$ and $\lambda \in [0,1]$.

Let $r_0^0 = M^0\beta_0\eta_0^1, s_0^0 = \beta_0\eta_0^1 w(\eta_0^1)$ and define sequences $\{r_n^0\}, \{s_n^0\}$ and $\{\eta_n^0\}$ for $n = 0, 1, 2, \ldots,$ by

$$r_{n+1}^0 = r_n^0\varphi^0(r_n^0)^2\psi^0(r_n^0, s_n^0), \quad (2.8)$$
$$s_{n+1}^0 = s_n^0\varphi^0(r_n^0)^{2+\lambda}\psi^0(r_n^0, s_n^0)^{1+\lambda}, \quad (2.9)$$
$$\eta_{n+1}^0 = \eta_n^0\varphi^0(r_n^0)\psi^0(r_n^0, s_n^0), \quad (2.10)$$

where

$$\varphi^0(u) = \frac{1}{1-ug^0(u)}, \quad (2.11)$$

$$g^0(u) = \left(1+\frac{u}{2}+\frac{u^2}{2(1-u)}(1+\frac{u}{4})\right), \quad (2.12)$$

and

$$\psi^0(u,v) = \frac{u^2}{2(1-u)}(1+\frac{u}{4})\left[\frac{v}{1+\lambda}\left(\frac{u^{1+\lambda}}{2^{1+\lambda}}+\frac{1}{2+\lambda}\left(\frac{u^2}{2(1-u)}(1+\frac{u}{4})\right)^{1+\lambda}\right)\right.$$
$$\left.+\frac{u}{2}(u+\frac{u^2}{2(1-u)}(1+\frac{u}{4}))\right]. \quad (2.13)$$

REMARK 2.2 *It follows from the conditions (C) and (H) that*

$$L^0 \le M^0 \le M^1, \tag{2.14}$$

and

$$w^0(t) \le w^1(t), \tag{2.15}$$

since $D_2 \subset D$. *Moreover, by (2.14) the estimate*

$$\|Q'(x)^{-1}\| \le \frac{\beta_0^1}{1-\beta_0^1 M^1\|x-x_0\|} \tag{2.16}$$

is used in [22] instead of the more precise

$$\|Q'(x)^{-1}\| \le \frac{\beta_0^1}{1-\beta_0^1 L_0\|x-x_0\|} \tag{2.17}$$

used by us. Notice that condition (h_2^0) *is used to help us define* D_2, *and consequently* M^0, w^0 *which are* $M^0 = M^0(L^0, D_2)$ *and* $w^0 = w^0(L^0, D_2)$. *By replacing* M^1, w^1 *by* M^0, w^0 *in the proof of Theorem 2.1, we obtain the improved version.*

THEOREM 2.3 *Under the conditions (H) further suppose* $r_0^0 < v^0$ *and* $\bar{S}(x_0, R^0\eta_0^1) \subseteq D$, *where* $R^0 = \frac{g^0(r_0^0)}{1-\gamma^0\delta^0}$, $\gamma^0 \in (0,1)$ *and* $\delta^0 = \frac{1}{\varphi^0(r_0^0)}$. *Then,* $y_n, z_n, x_{n+1} \in \bar{S}(x_0, R^0\eta_0^1)$; *there exists* $x_* \in \bar{S}(x_0, R^0\eta_0^1)$ *which is the only solution of equation* $F(x) = 0$ *in* $S^0 = S(x_0, \frac{2}{M^0\beta_0} - R^0\eta_0^1) \cap D$, *and*

$$\|x_n - x_*\| \le g^0(r_0^0)(\delta^0)^k \frac{(\gamma^0)^{\frac{(4+\lambda)^n-1}{3+\lambda}}}{1-\delta^0(\gamma^0)^{(4+\lambda)^n}}\eta_0^1 \tag{2.18}$$

where v^0 *is the smallest positive root of equation* $h^0(t) = g^0(t)t - 1$.

□

REMARK 2.4 *It follows from (2.14), (2.15) that the new functions and parameters are tighter than the functions and parameters in [22]. In particular, we have*

$$\begin{aligned}
r_0^0 &\le r_0^1, && (2.19)\\
s_0^0 &\le s_0^1, && (2.20)\\
r_n^0 &\le r_n^1, && (2.21)\\
s_n^0 &\le s_n^1, && (2.22)\\
\eta_n^0 &\le \eta_n^1, && (2.23)\\
\varphi^0(t) &\le \varphi^1(t), && (2.24)\\
g^0(t) &\le g^1(t), && (2.25)\\
\psi^0(t_1,t_2) &\le \psi^1(t_1,t_2), && (2.26)\\
h^0(t) &\le h^1(t), && (2.27)\\
v^1 &\le v^0 \ (by (2.27)), && (2.28)
\end{aligned}$$

and

$$S^1 \subseteq S^0. \tag{2.29}$$

The estimates (2.19)-(2.29) can be strict if (2.14) or (2.15) hold as strict inequalities. Notice also that the new constants and functions are special cases of the old ones. Therefore, the improved results are obtained without additional conditions. Hence, we have expanded the applicability of method (0.2) in the semi-local convergence case and also have justified the improvements as stated in the introduction of this article. Our technique can be used to expand the applicability of other iterative methods along the same lines [1–23].

3 Numerical examples

EXAMPLE 3.1 *Let $E_1 = E_2 = \mathbb{R}^3$, $D = U(0,1)$, $x^* = (0,0,0)^T$ and define Q on D by*

$$Q(x) = Q(x_1, x_2, x_3) = (e^{x_1} - 1, \frac{e-1}{2}{x_2}^2 + x_2, x_3)^T. \tag{3.1}$$

For the points $u = (u_1, u_2, u_3)^T$, the Fréchet derivative is given by

$$Q'(u) = \begin{pmatrix} e^{u_1} & 0 & 0 \\ 0 & (e-1)u_2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using the norm of the maximum of the rows and (c2)-(c4) and since $Q'(x^*) = diag(1,1,1)$, we can define functions for method (0.2) by $\mu_0(t) = (e-1)t, \mu(t) = e^{\frac{1}{e-1}}t, \mu_1(t) = e^{\frac{1}{e-1}}$. Then,

$$r_1 = 0.388692, r_2 = 0.254177, r_3 = 0.223778 = r.$$

EXAMPLE 3.2 *Let $E_1 = E_2 = C[0,1]$, the space of continuous functions defined on $[0,1]$ and be equipped with the max norm. Let $D = \overline{U}(0,1)$. Define function Q on D by*

$$Q(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{3.2}$$

We have that

$$Q'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in D.$$

Then, we get that $x^* = 0$, $\mu_0(t) = 7.5t, \mu(t) = 15t, \mu_1(t) = 1 + \mu_0(t)$ and $\delta(t) = 15$. Then

$$r_1 = 0.066667, r_2 = 0.0499893, r_3 = 0.04639 = r.$$

EXAMPLE 3.3 *Let $E_1 = E_2 = \mathbb{R}$, $D = [-\frac{5}{2}, \frac{3}{2}]$. Define Q on D by*

$$Q(x) = x^3 \log x^2 + x^5 - x^4$$

Then

$$Q'(x) = 3x^2 \log x^2 + 5x^4 - 4x^3 + 2x^2,$$

Then, we get that $\mu_0(t) = \mu(t) = 96.6629073t$ and $\mu_1(t) = 1 + \mu_0(t)$. then

$$r_1 = 0.00689682, r_2 = 0.00500794, r_3 = 0.00458943 = r.$$

EXAMPLE 3.4 *Let $E1 = E_2 = C[0,1], D = \bar{U}(x^*, 1)$ and consider the nonlinear integral equation of the mixed Hammerstein-type [1–3, 5, 11] defined by*

$$x(s) = \int_0^1 G(s,t)(x(t)^{3/2} + \frac{x(t)^2}{2})dt,$$

where the kernel G is the Green's function defined on the interval $[0,1] \times [0,1]$ by

$$G(s,t) = \begin{cases} (1-s)t, & t \leq s \\ s(1-t), & s \leq t. \end{cases}$$

The solution $x^*(s) = 0$ is the same as the solution of equation (0.1), where $Q : C[0,1] \longrightarrow C[0,1])$ is defined by

$$Q(x)(s) = x(s) - \int_0^1 G(s,t)(x(t)^{3/2} + \frac{x(t)^2}{2})dt.$$

Notice that

$$\| \int_0^1 G(s,t)dt \| \leq \frac{1}{8}.$$

Then, we have that

$$Q'(x)y(s) = y(s) - \int_0^1 G(s,t)(\frac{3}{2}x(t)^{1/2} + x(t))dt,$$

so since $F'(x^*(s)) = I$,

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq \frac{1}{8}(\frac{3}{2}\|x-y\|^{1/2} + \|x-y\|).$$

Then, we get that $\mu_0(t) = \mu(t) = \frac{1}{8}(\frac{3}{2}t^{1/2} + t)$ *and* $\mu(t) = 1 + \mu_0(t)$. *Then*

$$r_1 = 2.6303, r_2 = 1.6184, r_3 = 1,2024$$

so $r = 1$.

EXAMPLE 3.5 *Let* $E_1 = E_2 = \mathbb{R}$, $D = \bar{U}(x_0, 1-\xi)$, $x_0 = 1$ *and* $\xi \in [0, \frac{1}{2})$. *Define function* Q *on* D *by*

$$Q(x) = x^3 - \xi.$$

Then, we get $\beta_0 = \frac{1}{3}, \eta_0^1 = \frac{1}{3}(1-\xi), M^1 = 3$, $w^1(t) = 6$, $\lambda = 0$, $L^0 = 3(3-\xi), M^0 = \frac{6}{3-\xi} < M^1$, $w^0(t) = 6$, *and* D_2 *is a strict subset of* D.

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