Extension of linear operators and polynomial approximation, with applications to Markov moment problem and Mazur-Orlicz theorem

Octav Olteanu

Department of Mathematics-Informatics, Politehnica University of Bucharest, Splaiul Independenței 313, 060042 Bucharest, Romania

olteanuoctav@yahoo.ie

Abstract

One recalls the relationship between the Markov moment problem and extension of linear functionals (or operators), with two constraints. One states necessary and sufficient conditions for the existence of solutions of some abstract vector-valued Markov moment problems, by means of a general Hahn-Banach principle. The classical moment problem is discussed as a particular important case. This is the first aim of this review article (see sections 1 and 2). Secondly short subsection (namely subsection 3.1) is devoted to applications of polynomial approximation in studying the existence and uniqueness of the solutions for two types of Markov moment problems. We use these general type results in studying related problems which involve concrete spaces of functions and self-adjoint operators (subsection 3.2). This is the third purpose of the paper. Sometimes, the uniqueness of the solution follows too. Most of our solutions are operator-valued or function-valued. The methods follow from the corresponding proofs or via references citations. All the results have been previously published (see the references mentioned in the beginning of each subsection or before the statements of the theorems).

Keywords: extension of linear operators, Markov moment problem, Mazur-Orlicz theorem, polynomial approximation on unbounded subsets, concrete spaces

Mathematics Subject Classification (2010): 47A57; 41A10; 47A63

1 Introduction

We recall the classical formulation of the moment problem, under the terms of T. Stieltjes, given in 1894-1895 (see the basic book of N.I. Akhiezer [1] for details): find the repartition of the positive mass on the nonnegative semi-axis, if the moments of arbitrary orders k ($k = 0,1,2,\ldots$) are given. Precisely, in the Stieltjes moment problem, a sequence of real numbers $(s_k)_{k\geq 0}$ is given and one looks for a nondecreasing real function $\sigma(t)$ $(t \geq 0)$, which verifies the moment conditions:

$$\int_0^\infty t^k d\sigma = s_k \qquad (k = 0,1,2,\ldots)$$

This is a one dimensional moment problem, on an unbounded interval. Namely, is an interpolation problem with the constraint on the positivity of the measure dσ. The numbers $s_k, k \in \mathbb{N}$ are called the moments of the measure $d\sigma$. Existence, uniqueness and construction of the solution σ are studied. The present work concerns firstly the existence problem. The connection with the positive polynomials and extensions of linear positive functional and operators is quite clear. Namely, if one denotes by $\varphi_j, \varphi_j(t) := t^j, j \in \mathbb{N}, t \in [0,\infty), P$ the vector space of polynomials with real coefficients and $F_0: P \to \mathbb{R}, F_0\left(\sum_{j\in J_0} \alpha_j \varphi_j\right) := \sum_{j\in J_0} \alpha_j s_j$, where $J_0 \subset \mathbb{N}$ is a finite subset, then the moment conditions $F_0(\varphi_j) = s_j, j \in \mathbb{N}$ are obviously verified. It remains to check whether the linear form F_0 has nonnegative values at nonnegative polynomials. If the latter condition is also accomplished, one looks for the existence of a linear positive extension F of F_0 to a larger ordered function space X which contains both P and the space of continuous compactly supported functions, then representing F by means of a positive regular Borel measure μ on $[0,\infty)$, via Riesz representation theorem [2]. To see applications and proofs of such an extension result for linear functionals or operators, we refer to [1], [3]-[6]. Alternatively one can apply directly Haviland theorem [7] (see the next section). For more general extension type results for linear operators, giving necessary and sufficient conditions, see [8], [9], [10]. To obtain the function σ by means of the measure μ mentioned above one can define $\sigma(t) \coloneqq \mu([0,t]), t \in [0,\infty)$. If an interval (for example $[a,b]$, $\mathbb{R}$, or $[0,\infty)$) is replaced by a closed subset of $\mathbb{R}^n, n \geq 2$, we have a multidimensional moment problem. The case of multidimensional moment problem on compact semi-algebraic subsets in $\mathbb{R}^n$ was intensively studied. Observe that any compact is contained in a semi-algebraic compact in $\mathbb{R}^n$. The analytic form of positive polynomials on special closed finite dimensional subsets is crucial in solving classical moment problems on such subsets (see subsection 3.1). In case of Markov moment problem, approximation of nonnegative compactly supported continuous functions (with their support contained in a closed subset) by special nonnegative polynomials on that subset, having known analytic form is very important. Details and other aspects of the moment problem can be found in [11]-[32]. Connections of the moment problem with operator theory appear in [4], [16], [18], [19], [20]. Uniqueness of the solution is discussed in [30], [31], [32]. The rest of this work is organized as follows. Section 2 is devoted to general extension Hahn Banach type results for linear operators acting between abstract spaces. Necessary and sufficient or only sufficient conditions for the existence of a solution of some moment problems are recalled. Section 3 contains various applications to spaces of functions or/and operators. In some cases, the uniqueness of the solution follows from the proof of its existence. In subsection 3.1 polynomial approximations on unbounded subsets are applied, completing the review paper [6]. Some of the results in this subsection are new (such as Theorem 3.1.7). Section 4 concludes the paper.

2 Extension of linear operators, the abstract Markov moment problem and Mazur-Orlicz theorem (general-type results)

The main problem was to find necessary and sufficient conditions for the existence of a solution of the interpolation problem, preserving sandwich conditions. In this general case, the operators involved in the (convex and respectively concave) constraints are defined on arbitrary convex subsets. Throughout this first part of this section, X will be a real vector space, Y an order-complete vector lattice, $A, B \subset X$ convex subsets, $W: A \to Y$ a concave operator, $T: B \to Y$ a convex operator, $S \subset X$ a vector subspace, $f: S \to Y$a linear operator. All vector spaces and linear operators are considered over the real field.

Theorem 2.1 (see [8], [9]). *Assume that*

$$f|_{S\cap A} \geq W|_{S\cap A}, \quad f|_{S\cap B} \leq T|_{S\cap B}.$$

The following statements are equivalent:

(a) *there exists a linear extension* $F: X \to Y$ *of the operator f such that* $F|_A \geq W, F|_B \leq T$;

(b) *there exists* $T_1 : A \to Y$ *convex and* $W_1 : B \to Y$ *concave operator such that for all*

$$(\rho, t, \lambda', a_1, a', b_1, b', v) \in [0,1]^2 \times (0,\infty) \times A^2 \times B^2 \times S,$$

one has

$$(1-t)a_1 - tb_1 = v + \lambda'[(1-\rho)a' - \rho b'] \Rightarrow$$
$$(1-t)T_1(a_1) - tW_1(b_1) \geq f(v) + \lambda'[(1-\rho)W(a') - \rho T(b')].$$

Thus in the last relation, we have a convex operator in the left hand side, and a concave operator in the right hand side. The following result related to the theorem of H. Bauer follows.

Theorem 2.2 (see [8], [9]). *Let X be a preordered vector space with its positive cone* X_+, *Y an order complete vector lattice,* $T: X \to Y$ *a convex operator,* $S \subset X$ *a vector subspace,* $f: S \to Y$ *a linear positive operator. The following assertions are equivalent*

(a) *there exists a linear positive extension* $F: X \to Y$ *of f such that* $F(x) \le T(x)$, $\forall x \in X$;

(b) $f(s) \le T(x)$ *for all* $(s, x) \in S \times X$ *such that* $s \le x$.

Now we can deduce the main results on the abstract moment problem.

Theorem 2.3 (see [10]). *Let* $X, Y, T: X \to Y$ *be as in Theorem 3.1.2,* $\{x_j\}_{j \in J} \subset X$, $\{y_j\}_{j \in J} \subset Y$ *given families. The following assertions are equivalent*

(a) *there exists a linear positive operator* $F: X \to Y$ *such that*

$$F(x_j) = y_j \ \forall j \in J, \quad F(x) \le T(x) \ \forall x \in X;$$

(b) *for any finite subset* $J_0 \subset J$ *and any* $\{\lambda_j\}_{j \in J_0} \subset R$, *we have*

$$\sum_{j \in J_0} \lambda_j x_j \le x \Rightarrow \sum_{j \in J_0} \lambda_j y_j \le T(x)$$

In the classical real moment problem, X is a space of functions containing the polynomials and the compactly supported continuous functions, defined on a closed subset A in $\mathbb{R}^n$, while $x_j(t) = t_1^{j_1} \cdots t_n^{j_n}, t = (t_1, \dots, t_n) \in A, j = (j_1, \dots, j_n) \in \mathbb{N}^n, n \in \mathbb{N}, n \ge 1, Y = \mathbb{R}$. A clearer sandwich-moment problem variant is the following one.

Theorem 2.4 (see [10]). Let X *be an ordered vector space,* Y *an order complete vector lattice,* $\{x_j\}_{j \in J} \subset X$, $\{y_j\}_{j \in J} \subset Y$ *given families and* $F_1, F_2 \in L(X, Y)$ *two linear operators. The following statements are equivalent*

(a)*there is a linear operator* $F \in L(X, Y)$ *such that*

$$F_1(x) \le F(x) \le F_2(x) \ \forall x \in X_+, \ F(x_j) = y_j \ \forall j \in J;$$

(b)*for any finite subset* $J_0 \subset J$ *and any* $\{\lambda_j\}_{j \in J_0} \subset R$, *we have*

$$\left(\sum_{j \in J_0} \lambda_j x_j = \psi_2 - \psi_1, \psi_1, \psi_2 \in X_+ \right) \Rightarrow \sum_{j \in J_0} \lambda_j y_j \le F_2(\psi_2) - F_1(\psi_1).$$

The next result of this subsection is an earlier extension result, sometimes called Lemma of the majorizing subspace, for positive linear operators on subspaces in ordered vector spaces (X, X_+), for which the positive cone X_+ is generating $(X = X_+ - X_+)$. Recall that in such an ordered vector space X, a vector subspace S is called a majorizing subspace if for any $x \in X$, there exists $s \in S$ such that $x \le s$.

Theorem 2.5. *Let* X *be an ordered vector space whose positive cone is generating,* $S \subset X$ *a majorizing vector subspace,* Y *an order complete vector lattice,* $F_0: S \to Y$ *a linear positive operator. Then* F_0 *has a linear positive extension* $F: X \to Y$ *at least.*

Theorem 2.5 was proved or/and applied in [1], [3], [4], [5], [6], [25].

Theorem 2.6. (E.K. Haviland; see [7]). *Let $A \subset \mathbb{R}^n$ and $L: P := \mathbb{R}[t = (t_1, \dots, t_n)] \to \mathbb{R}$ be a linear form. Then L is given by a positive Borel measure μ on A (i.e. $L(p) = \int_A p d\mu$ for all $p \in P$) if and only if $L(p) \geq 0$ for all nonnegative p on A : $(p(t) \geq 0, \forall t \in A \Rightarrow L(p) \geq 0$).*

The next result is a variant of Mazur-Orlicz theorem.

Theorem 2.7. (see [10]). *Let $\boldsymbol{X}$ be an ordered vector space, $\boldsymbol{Y}$ an order complete vector lattice, $\{\boldsymbol{x}_j\}_{j \in J}, \{\boldsymbol{y}_j\}_{j \in J}$ arbitrary families in $\boldsymbol{X}$, respectively in $\boldsymbol{Y}$ and $\boldsymbol{T}: \boldsymbol{X} \to \boldsymbol{Y}$ a sublinear operator. The following statements are equivalent*

(a) $\exists F \in L(X, Y)$ *such that* $F(x_j) \geq y_j, \forall j \in J, F(x) \geq 0, \forall x \in X_+, \ F(x) \leq T(x), \forall x \in X$;

(b) *for any finite subset $J_0 \subset J$ and any $\{\lambda_j\}_{j \in J_0} \subset \mathbb{R}, \lambda_j \geq 0, \forall j \in J_0$, we have*

$$\sum_{j \in J_0} \lambda_j x_j \leq x \in X \Rightarrow \sum_{j \in J_0} \lambda_j y_j \leq T(x)$$

The last result of this section states a sufficient condition for the existence of some constrained extensions. It has an interesting geometric meaning.

Theorem 2.8 (see [9]). *Let X be a locally convex space, Y an order complete vector lattice with strong order unit u_0 and $S \subset X$ a vector subspace. Let $A \subset X$ be a convex subset with the following properties*

(a) *there exists a neighborhood V of the origin such that $(S + V) \cap A = \Phi$*

(A and S are distanced);

(b) *A is bounded.*

Then for any equicontinuous family of linear operators $\{f_j\}_{j \in J} \subset L(S, Y)$ and for any $\tilde{y} \in Y_+ \setminus \{0\}$, there exists an equicontinuous family $\{F_j\}_{j \in J} \subset L(X, Y)$ such that

$$F_j(s) = f_j(s), s \in S, \text{ and } F_j(\psi) \geq \tilde{y}, \psi \in A, j \in J.$$

Moreover, if V is a neighborhood of the origin such that

$$f_j(V \cap S) \subset [-u_0, u_0], (S + V) \cap A = \Phi$$

and if $\alpha > 0$ is such that $p_V(a) \leq \alpha \ \forall a \in A$, while *$\alpha_1 > 0$ is large enough such that $\tilde{y} \leq \alpha_1 u_0$, then the following relations hold*

$$F_j(x) \leq (1 + \alpha + \alpha_1) p_V(x) \cdot u_0, \quad x \in X, \ j \in J.$$

3 Markov moment problem and Mazur-Orlicz theorem on concrete spaces

3.1 Approximation and Markov moment problem (results and methods of proving them)

Next we complete some results from [6] (see also the references therein). The present section is mainly based on the articles [6], [22], [23], [25], [26], [27].

Lemma 3.1.1. *Let* $A \subset R^n$ *be a closed unbounded subset and* ν *a positive regular* $M-$ *determinate Borel measure on* A, *with finite moments of all orders. Then for any* $\psi \in (C_0(A))_+$, *there is a sequence* $(p_m)_m$ *of polynomials on* A, $p_m \geq \psi$, $p_m \to \psi$ *in* $L^1_\nu(A)$. *We have*

$$\lim \int_A p_m d\nu = \int_A \psi \, d\nu,$$

the cone P_+ *of positive polynomials is dense in* $\left(L^1_\nu(A)\right)_+$ *and* P *is dense in* $L^1_\nu(A)$.

Recall that a determinate (M−determinate) measure is uniquely determinate by its moments, or, equivalently, by its values on polynomials. The following statement holds for any closed unbounded subset $A \subset \mathbb{R}^n$, hence does not depend on the form of positive polynomials on A. One denotes $\varphi_j(t) := t_1^{j_1} \cdots t_n^{j_n}$, $j = (j_1, \dots, j_n) \in \mathbb{N}^n, t = (t_1, \dots, t_n) \in A$.

Theorem 3.1.1. *Let* A *be a closed unbounded subset of* $\mathbb{R}^n$, Y *an order complete Banach lattice,* $(y_j)_{j\in\mathbb{N}^n}$ *a given sequence in* Y, ν *a positive regular* M *−determinate Borel measure on* A, *with finite moments of all orders. Let* $F_2 \in B(L^1_\nu(A), Y)$ *be a linear positive bounded operator from* $L^1_\nu(A)$ *to* Y. *The following statements are equivalent:*

(a)*there exists a unique linear operator* $F \in B(L^1_\nu(A), Y)$ *such that* $F(\varphi_j) = y_j, j \in \mathbb{N}^n$, F *isbetween* 0 *and* F_2 *on the positive cone of* $L^1_\nu(A)$, *and* $\|F\| \leq \|F_2\|$;

(b) *for any finite subset* $J_0 \subset \mathbb{N}^n$, *and any* $\{a_j\}_{j\in J_0} \subset \mathbb{R}$, *we have*

$$\sum_{j\in J_0} a_j \varphi_j \geq 0 \text{ on } A \Rightarrow 0 \leq \sum_{j\in J_0} a_j y_j \leq \sum_{j\in J_0} a_j F_2(\varphi_j).$$

Proof. Let F_0 be the linear operator defined on the subspace of polynomials, such that the moment conditions to be accomplished.Then condition (b) says that for any polynomial p which is nonnegative on A, we have

$$0 \leq F_0(p) \leq F_2(p) \quad (1)$$

Hence the implication (a)⇨(b) is obvious. In order to prove the converse, one applies lemma 3.1.1. Let ψ be a continuous nonnegative compactly supported function on A, and $(p_m)_m$ a sequence of polynomials given by Lemma 3.1.1. Then all polynomials $p_m, m \in \mathbb{N}$ are nonnegative on A, so that they verify (1). On the other hand, there exists a linear positive extension $F$of F_0 to the space of all functions in $L^1_\nu(A)$ whose absolute values are dominated by a polynomial. This space contains the subspace of continuous compactly supported functions and the polynomials. Observe also that for any linear positive continuous functional y^* on $Y, y^* \circ F$ can be represented by means of a positive Borel measure on A, which Fatou lemma works for. Using (1), one deduces

$$y^*(F(\psi)) \leq liminf\, y^*(F(p_m)) \leq lim\, y^*(F_2(p_m)) = y^*(F_2(\psi)) \quad (2)$$

Assume that $F_2(\psi) - F(\psi) \notin Y_+$. Using a separation argument, it should exist a linear positive continuous functional y^* on Y, such that $y^*(F_2(\psi) - F(\psi)) < 0$, that is

$$y^*(F_2(\psi)) < y^*(F(\psi)).$$

This contradicts (2), so that we must have

$$0 \leq F(\psi) \leq F_2(\psi), \forall \psi \in (C_c(A))_+.$$

If φ is an arbitrary continuous compactly supported function, then

$$|F(\varphi)| \leq F(|\varphi|) \leq F_2(|\varphi|)$$

Since the norm on Y is solid, the last inequalities imply

$$\|F(\varphi)\| \leq \|F_2(|\varphi|)\| \leq \|F_2\| \cdot \|\varphi\|_1, \forall \varphi \in C_c(A)$$

Hence $\|F\|$ is dominated by $\|F_2\|$ on a dense subspace of $L^1_\nu(A)$. Now the bounded positive linear operator F, having the norm-property mentioned above, has a unique extension to the whole space $L^1_\nu(A)$, preserving its properties. This concludes the proof.

Next, we give the "scalar version" of Theorem 3.1.1.

Theorem 3.1.2. *Let A, ν be as in Theorem 3.1.1, and $(y_j)_{j\in\mathbb{N}^n}$ a multi-sequence of real numbers.*

The following statements are equivalent:

(a)*there exists a unique $h \in L^1_\nu(A), 0 \leq h \leq 1$ a..e. on A, such that $\int_A \varphi_j \cdot h d\nu = y_j$, $j \in \mathbb{N}^n$;*

(b)*for any finite subset $J_0 \subset \mathbb{N}^n$, and any $\{a_j\}_{j\in J_0} \subset \mathbb{R}$, we have*

$$\sum_{j\in J_0} a_j \varphi_j \geq 0 \text{ on } A \Rightarrow 0 \leq \sum_{j\in J_0} a_j y_j \leq \sum_{j\in J_0} a_j \int_A \varphi_j d\nu.$$

Note that for particular sets A, for which the form of positive polynomials on A in terms of sums of squares is known, one can give a characterization in terms of "computable' quadratic forms or mappings. For example, if $n = 1, A = [0,\infty)$, using the form of positive polynomials on $[0,\infty)$ [1]:

$$p(t) \geq 0, \forall t \in [0,\infty) \Leftrightarrow p(t) = p_1^2(t) + t p_2^2(t), \forall t \in [0,\infty), \text{ where } p_k \in \mathbb{R}[t], k = 1,2$$

one obtains:

Theorem 3.1.3. *Let ν be as in Theorem 3.1.1 on $A := [0,\infty), Y, (y_j)_{j\in\mathbb{N}}$, F_2 be as in Theorem 3.1.1. The following statements are equivalent*

(a) *there exists a unique linear operator $F \in B(L^1_\nu([0,\infty)), Y)$ such that $F(\varphi_j) = y_j, j \in \mathbb{N}, F$ is between 0 and F_2 on the positive cone of $L^1_\nu(([0,\infty))$, and $\|F\| \leq \|F_2\|$, where $\varphi_j(t) = t^j$, $j \in \mathbb{N}, t \geq 0$*

(b) *for any finite subset $J_0 \subset \mathbb{N}$, and any $\{\lambda_j\}_{j\in J_0} \subset \mathbb{R}$, we have*

$$0 \leq \sum_{i,j\in J_0} \lambda_i \lambda_j y_{i+j+k} \leq \sum_{i,j\in J_0} \lambda_j \lambda_j F_2(\varphi_{i+j+k}), k \in \{0,1\}$$

Next one applies a quite similar result, but for a concrete operator valued moment problem, replacing $L^1_\nu([0,\infty))$ by $X = C_\mathbb{R}(\sigma(A))$, where $\sigma(A) \subset [0,\infty)$ is the spectrum of a fixed positive self-adjoint operator A acting on a complex (or real) Hilbert space H. So, X is the space of all real continuous functions on $\sigma(A)$. Let $\mathcal{A}$ be the real vector space of self-adjoint operators from H to itself. Then $\mathcal{A}$ is an ordered vector space, endowed with the order relation defined by

$$U \leq V \Leftrightarrow < U(h), h > \leq < V(h), h >, \forall h \in H, U, V \in \mathcal{A}$$

Unfortunately, for arbitrary $U, V \in \mathcal{A}$, the supremum $sup\{U,V\} = U \vee V$ or/and the infimum $inf\{U,V\} = U \wedge V$ might not exist in $\mathcal{A}$. To avoid the fact that $\mathcal{A}$ is not a vector lattice, as well as the non commutativeness of multiplication of elements from $\mathcal{A}$, for any $A \in \mathcal{A}$ one uses the construction of the following space $Y = Y(A)$.

Theorem 3.1.4. (see [5]). *Let $A \in \mathcal{A}, Y_1 := \{U \in \mathcal{A}; AU = UA\}, Y = Y(A) := \{V \in Y_1; VU = UV, \forall U \in Y_1\}$. Then Y is a commutative (real) Banach algebra and an order-complete Banach lattice, where*

$$|V| := sup\{V, -V\} = \sqrt{V^2}, \forall V \in Y$$

($|V|$ is equal to the positive square root of the positive self-adjoint operator V^2).

Theorem 3.1.4 allows applying Hahn Banach extension type results of section 2 for linear operators taking values in $Y = Y(A)$. Let $\varphi \in X = C_{\mathbb{R}}(\sigma(A))$; one denotes $\|\varphi\|_\infty$ the sup-norm of φ in the space X, while $\|\cdot\|$ will be the operatorial norm on Y. As before, one denotes $\varphi_j(t) = t^j, j \in \mathbb{N}, t \in [0,\infty)$.

Lemma 3.1.2. *Let $\psi:[0,\infty) \to R_+$ be a continuous function, such that $\lim_{t\to\infty} \psi(t) \in R_+$ exists. Then there is a decreasing sequence $(h_l)_l$ in the linear hull of the functions*

$$\varphi_k(t) = \exp(-kt), \quad k \in \mathrm{N}, \quad t \ge 0,$$

such that $h_l(t) > \psi(t)$, $t \ge 0$, $l \in \mathrm{N}$, $\lim h_l = \psi$ uniformly on $[0,\infty)$. There exists a sequence of polynomial functions $(\tilde{p}_l)_{l\in\mathbb{N}}, \tilde{p}_l \ge h_l > \psi, \lim \tilde{p}_l = \psi$, uniformly on compact subsets of $[0,\infty)$.

The idea of the proof is to add the ∞ point and to apply the Stone-Weierstrass Theorem to the subalgebra generated by the functions $\exp(-mt)$, $m \in Z_+$. Then one uses for each such exp – function suitable majorizing or minorizing polynomials, as well as the elementary equality

$$\exp(s) - \left(1 + \frac{s}{1!} + \frac{s^2}{2!} + \Lambda + \frac{s^m}{m!}\right) = \frac{\exp(s)}{m!}\int_0^s \exp(-t)\cdot t^m dt, \quad m \in N, s \in R$$

Using the notations preceding Lemma 3.1.2, we prove the following theorem.

Theorem 3.1.5. *Let $A, X, Y = Y(A)$ be as above, $(U_n)_{n\ge 0}$ be a sequence of operators in Y. The following statements are equivalent*

(a) *there exists a unique linear bounded operator $F: X \to Y$ such that the moment interpolation conditions $F(\varphi_n) = U_n, n \in \mathbb{N}$ are verified and $0 \le F(\psi) \le \psi(A), \forall \psi \in X_+, \|F\| \le 1$;*

(b) *for any finite subset $J_0 \subset \mathbb{N}$ and any $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$, the following implication holds true*

$$\sum_{j\in J_0} \lambda_j t^j \ge 0, \forall t \in \sigma(A) \Rightarrow 0 \le \sum_{j\in J_0} \lambda_j U_j \le \sum_{j\in J_0} \lambda_j A^j;$$

(c) *for any finite subset $J_0 \subset \mathbb{N}$ and any $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$, the following relations hold*

$$0 \le \sum_{i,j\in J_0} \lambda_i \lambda_j U_{i+j+k} \le \sum_{i,j\in J_0} \lambda_i \lambda_j A^{i+j+k}, k \in \{0,1\}$$

Proof. Observe that the implications $(a) \Rightarrow (b), (a) \Rightarrow (c)$ are obvious, due to the properties of F. The next idea is to extend relations from (b) and (c) on nonnegative polynomials to arbitrary nonnegative functions from X, by means of a passing to the limit process. Namely, to prove the converses (b)⇒(a), (c)⇒(a), denote by P the vector space of polynomials with real coefficients, P_+ the convex cone of all polynomials which are nonnegative on $\sigma(A)$ and by P_{++} the convex cone of polynomials which are nonnegative on the whole interval $[0,\infty)$. Define the linear operator $F_0: P \to Y$ by

$$F_0\left(\sum_{j\in J_0} \lambda_j \varphi_j\right) := \sum_{j\in J_0} \lambda_j U_j$$

where $J_0 \subset \mathbb{N}$ is an arbitrary finite subset. Than (b) can be written as

$$0 \le F_0(p) \le p(A), p \in P_+ \tag{3}$$

so that, in particular, F_0 is a linear positive operator from P to Y. It has a linear positive extension $F: X \to Y$, since each element from X is bounded above by a constant, so that one can apply Theorem 2.5 from above. Obviously, F verifies the interpolation moment conditions, because of

$$F(\varphi_j) = F_0(\varphi_j) := U_j, j \in \mathbb{N}$$

It remains to prove that $F(\psi) \le \psi(A), \forall\psi \in X_+, \|F\| \le 1$. Let $\psi \in X_+$. Then there exists a sequence $(p_m)_{m\ge0}$ of polynomials such that $\lim_m p_m = \psi$ in X, i. e.

$$\|p_m - \psi\|_\infty \to 0$$

Since the convergence is uniform on $\sigma(A)$, one can assume that $p_m \ge \psi(\ge 0)$ in X for all m, so that $p_m \in P_+, m \in \mathbb{N}$. These comments imply

$$\|p_m(A) - \psi(A)\| = \sup\sigma\left((p_m - \psi)(A)\right) = sup(p_m - \psi)(\sigma(A)) = \|p_m - \psi\|_\infty \to 0$$

Consequently, $\lim_m p_m(A) = \psi(A)$. As in the proof of Theorem 3.1.1, from (3) written for all $p_m, m \in \mathbb{N}$, it results (passing to the limit): $F(\psi) \le \psi(A), \forall\psi \in X_+$. This further yield

$$|F(\varphi)| \le |\varphi|(A), \varphi \in X$$

Since the norm on Y is solid (Y is a Banach lattice), it results

$$\|F(\varphi)\| \le \||\varphi|(A)\| = \||\varphi|\|_\infty = \|\varphi\|_\infty, \forall\varphi \in X \Rightarrow \|F\| \le 1$$

In particular, $\|U_0\| = \|F(\varphi_0)\| \le \|\varphi_0\|_\infty = 1$. The proof of $(b) \Rightarrow (a)$ is complete. To prove $(c) \Rightarrow (a)$, recall that $p \in P_{++}$ if and only if there exist polynomials p_1, p_2 with real coefficients such that

$$p(t) = p_1^2(t) + tp_2^2(t) = \sum_{i,j=0}^{n} \lambda_i\lambda_j t^{i+j} + \sum_{k,l=0}^{p} \alpha_k\alpha_l\, t^{k+l+1}, t \in \mathbb{R}_+$$

where $p_1(t) = \sum_{j=0}^{n} \lambda_j\, t^j, p_2(t) = \sum_{k=0}^{p} \alpha_k t^k$. It follows that the relations of point (c) can be written as those from (3), but only for $p \in P_{++}$. Let $\psi \in X_+$. Then there exists a nonnegative compactly supported extension $\tilde{\psi}$ of ψ, such that $\tilde{\psi}$ is continuous on $[0,\infty)$, $suppp(\tilde{\psi}) \subset [0,\infty)$.

Applying Lemma 3.1.2, there exists a sequence $(p_m)_{m\ge0}$ of polynomial functions, with

$$p_m > \tilde{\psi}, m \ge 0, \lim_m p_m = \tilde{\psi} \ge 0,$$

the convergence being uniform on compact subsets in $[0,\infty)$. In particular, $p_m \in P_{++}$ for all $m \in \mathbb{N}$, so that for F as above it results

$$0 \leq F(p_m) \leq p_m(A), \forall m \in \mathbb{N}$$

by hypothesis (c), also using the form of nonnegative polynomials on $[0, \infty)$ in terms of a sum of square of a polynomial and another square of polynomial multiplied by $\varphi_1(t) = t$. Using the uniform convergence $\lim_m p_m = \tilde{\psi}$ on the compact $\sigma(A)$ and the fact that $\tilde{\psi}(t) = \psi(t), \forall t \in \sigma(A)$, we derive

$$\|(p_m - \psi)(A)\| = \|p_m - \psi\|_\infty = \|p_m - \tilde{\psi}\|_\infty \to 0 \Rightarrow \lim_m p_m(A) = \psi(A)$$

Repeating the passing to the limit process discussed in the proof of Theorem 3.1.1, one obtains

$$F(\psi) \leq \psi(A), \psi \in X_+$$

Now the last assertion $\|F\| \leq 1$ is a consequence of the preceding one, as discussed at (b)⇒(a). This concludes the proof. □

As we have observed above, the analytic form of positive polynomial on an arbitrary closed subset $A \subset \mathbb{R}^n$ is not known. If A is a particular subset for which the form of positive polynomials over A is known in terms of sums of squares, then Markov moment problem on A can be solved in terms of quadratic mappings (or products of quadratic mappings). This is the case of a strip $A \subset \mathbb{R}^2$, as claimed in the next theorem.

Theorem 3.1.6. (M. Marshall [12]). *Suppose that $p(t_1, t_2) \in \mathbb{R}[t_1, t_2]$ is non – negative on the strip $A = [0,1] \times \mathbb{R}$. Then $p(t_1, t_2)$ is expressible as*

$$p(t_1, t_2) = \sigma(t_1, t_2) + \tau(t_1, t_2) t_1 (1 - t_1),$$

where $\sigma(t_1, t_2), \tau(t_1, t_2)$ are sums of squares in $\mathbb{R}[t_1, t_2]$.

Let $A = [0,1] \times \mathbb{R}$, ν a positive $M -$ determinate regular Borel measure on A, with finite moments of all orders, $X := L^1_\nu(A), \varphi_j(t_1, t_2) := t_1^{j_1} t_2^{j_2}, j = (j_1, j_2) \in \mathbb{N}^2, (t_1, t_2) \in A$. Let Y be on order complete Banach lattice, $(y_j)_{j \in \mathbb{N}^2}$ a sequence of given elements in Y. The following result seems to be new. It is a consequence of Lemma 3.1.1 and Theorems 3.1.1, 3.1.6.

Theorem 3.1.7. *Let $F_2 \in B_+(X, Y)$ be a linear bounded positive operator from X to Y. The following statements are equivalent:*

(a) *there exists a unique bounded linear operator $F : X \to Y$, such that*

$$F(\varphi_j) = y_j, \forall j \in \mathrm{N}^2,$$

F is between zero and F_2 on the positive cone of X, $\|F\| \leq \|F_2\|$;

(b) *for any finite subset $J_0 \subset \mathbb{N}^2$, and any $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$, we have*

$$0 \leq \sum_{i,j \in J_0} \lambda_i \lambda_j \, y_{i+j} \leq \sum_{i,j \in J_0} \lambda_i \lambda_j \, F_2(\varphi_{i+j});$$

$$0 \leq \sum_{i,j \in J_0} \lambda_i \lambda_j \left(y_{i_1+j_1+1, i_2+j_2} - y_{i_1+j_1+2, i_2+j_2} \right) \leq$$

$$\sum_{i,j \in J_0} \lambda_i \lambda_j \left(F_2\left(\varphi_{i_1+j_1+1, i_2+j_2} - \varphi_{i_1+j_1+2, i_2+j_2} \right) \right), i = (i_1, i_2), j = (j_1, j_2) \in J_0$$

Proof. Theorem 3.1.6 shows that the condition at point (b) of the present theorem is equivalent to (1), written for $P_+ = P_+(A)$, where F_0 is the linear operator defined on the space P of all polynomial functions, such that the moment conditions $F_0(\varphi_j) = y_j, j \in \mathbb{N}^2$ be accomplished. The conclusion follows via Theorem 3.1.1.

3.2 Applications of extension theorems for linear operator to concrete spaces (results and methods of proving them)

The purpose of this section is to show how the results of section 2 can be applied to concrete function and/or operator spaces. We follow the results from [28], [29]. The next proposition is an application of Theorem 2.7 to the space X of power series in the disc $|z| < r$, continuous up to the boundary, with real coefficients. The order relation is given by the coefficients: we write

$$\sum_{n\in \mathrm{N}} \lambda_n z^n \ \pi \ \sum_{n\in \mathrm{N}} \gamma_n z^n \Leftrightarrow (\lambda_n \le \gamma_n, \forall n \in \mathrm{N}).$$

Denote $\varphi_n(z) = z^n, n \in \mathrm{N}, |z| \le r$. Let Y be the space defined in Theorem 3.1.4, $(B_n)_{n\in \mathrm{N}}$ a sequence in Y, and $U \in Y$ such that $\| U \| < r$.

Proposition 3.2.1. *Consider the following statements*

(a) *there exists a linear positive bounded operator* $F \in L_+(X,Y)$, *such that*

$$F(\varphi_n) \ge B_n, n \in \mathrm{N}, |F(\psi)| \le \|\psi\|_\infty r(rI - U)^{-1}, \forall \psi \in X,$$

$$\|F\| \le \frac{r}{r - \|U\|};$$

(b) *the following relations hold*

$$0 \le B_n \le U^n, \quad n \in \mathrm{N};$$

(c) *the following inequalities hold*

$$B_n \le r^{n+1}(rI - U)^{-1}, \quad n \in \mathrm{N}.$$

Then $(b) \Rightarrow (a) \Rightarrow (c)$.

Proof. $(b) \Rightarrow (a)$. One applies theorem 2.7, (b) implies (a), to $x_j = \varphi_j, j \in \mathrm{N}$. If

$$\sum_{j\in J_0} \lambda_j \varphi_j \le \psi = \sum_{n\in \mathrm{N}} \alpha_n \varphi_n, \lambda_j \in R_+,$$

then the hypothesis, Cauchy inequalities and the above relation yield

$$0 \le \lambda_j B_j \le \lambda_j U^j \le |\alpha_j| U^j, \ j \in \mathrm{N} \Rightarrow$$

$$\sum_{j\in J_0} \lambda_j B_j \le \sum_{j\in J_0} |\alpha_j| U^j \le \sum_{n\in N} \frac{\|\psi\|_\infty}{r^n} U^n =$$

$$\|\psi\|_\infty \left(I - \frac{U}{r}\right)^{-1} = r(rI - U)^{-1} \|\psi\|_\infty =: T(\psi) = T(-\psi)$$

Hence, the implication of (b), Theorem 2.7 is accomplished and an application of the latter theorem leads to the existence of a linear positive operator F applying X into Y, with the properties stated at point (a):

$$F(\varphi_n) \geq B_n, n \in \mathbb{N}, |F(\varphi)| \leq r(rI - U)^{-1}\|\varphi\|_\infty, \varphi \in X.$$

Since the norm on Y is solid, we infer that

$$\|F(\varphi)\| \leq \|(I - U/r)^{-1}\| \cdot \|\varphi\|_\infty, \varphi \in X$$

In particular, the following evaluation for the norm of F holds

$$\|F\| \leq \left\|\sum_{n=0}^{\infty} \frac{U^n}{r^n}\right\| \leq \sum_{n=0}^{\infty} \frac{\|U\|^n}{r^n} = \frac{r}{r - \|U\|}$$

On the other hand, $(a) \Rightarrow (c)$ is obvious, because of:

$$B_n = F(\varphi_n) \leq \|\varphi_n\|_\infty r(rI - U)^{-1} = r^{n+1}(rI - U)^{-1},$$

also using $\varphi_n \in X_+$ for all $n \in \mathrm{N}$. The conclusion follows. □

Theorem 3.2.1. *Let* $X = L_\nu^1(M), \nu \geq 0$ *and* $(\varphi_n)_{n\in \mathrm{N}}$ *a sequence of positive functions in* X, *such that* $\int_M \varphi_n d\nu = 1, \forall n \in \mathrm{N}.$ *Let* $Y = L_\mu^\infty(\Omega), \mu \geq 0, (y_n)_{n\in \mathrm{N}}$ *a sequence of positive functions in* Y. *Then* $\sup_{n\in \mathrm{N}}\|y_n\| = b < \infty$ *if and only if there is a linear positive operator* $F \in L(X,Y)$ *such that*

$$F(\varphi_n) \geq |y_n|, n \in \mathrm{N}, \quad |F(\psi)| \leq b \cdot \left(\int_M |\psi| d\nu\right) \cdot \chi_\Omega, \forall \psi \in X.$$

Proof. For the "only if" part, let $J_0 \subset \mathrm{N}$ be a finite subset, $\{\lambda_j\}_{j\in J_0} \subset R_+$ be such that $\sum_{j\in J_0} \lambda_j \varphi_j \leq \psi$ in X. Hypothesis on the functions $\varphi_n, n \in \mathrm{N}$ and integration in the relation $\sum_{j\in J_0} \lambda_j \varphi_j \leq \psi$ yield

$$\sum_{j\in J_0} \lambda_j = \sum_{j\in J_0} \lambda_j \int_M \varphi_j d\nu \leq \int_M \psi \, d\nu \Rightarrow$$
$$\sum_{j\in J_0} \lambda_j |y_j| \leq \left\|\sum_{j\in J_0} \lambda_j |y_j|\right\| \cdot \chi_\Omega \leq \left(\int_M \psi \, d\nu\right) \cdot b \cdot \chi_\Omega \leq$$
$$\left(\int_M |\psi| \, d\nu\right) \cdot b \cdot \chi_\Omega =: T(\psi) = T(-\psi), \psi \in X$$

Application of theorem 2.7 leads to the existence of a linear positive operator $F \in L(X,Y)$ with the following properties

$$F(\varphi_n) \geq |y_n|, n \in \mathrm{N}, \quad |F(\psi)| \leq b \cdot \left(\int_M |\psi| d\nu\right) \cdot \chi_\Omega, \forall \psi \in X.$$

In particular, one has $\|F\| \leq b$. Next, we prove the "if" part. Assume that $|y_n| \leq F(\varphi_n), n \in N$ and F has the qualities in the statement, then, because the norm on Y is solid, we derive

$$\|y_n\| \leq \|F(\varphi_n)\| \leq b\left(\int_M \varphi_n d\nu\right) = b, \quad n \in \mathrm{N}.$$

This concludes the proof. □

Theorem 3.2. 2.. *Let* $X, Y, (\varphi_n)_n, (y_n)_n$ *be as in Theorem 3.2.1, and* $0 < b < \infty$.; *consider the following statements*

(a) *there exists a linear positive operator* $F \in L(X, Y)$ *such that*

$$F(\varphi_n) = y_n, n \in \mathrm{N}, |F(\varphi)| \leq b \cdot \int_M |\varphi| d\nu \cdot \chi_\Omega, \varphi \in X, \|F\| \leq b;$$

(b) *for any finite subset* $J_0 \subset \mathrm{N}$ *and any* $\{\lambda_j\}_{j \in J_0} \subset R$, *the following relation holds*

$$\sum_{j \in J_0} |\lambda_j| \cdot \|y_j\| \leq b\left|\sum_{j \in J_0} \lambda_j\right|.$$

Then $(b) \Rightarrow (a)$.

Proof. We apply Theorem 2.4, (b) implies (a). If $\sum_{j \in J_0} \lambda_j \varphi_j = \psi_2 - \psi_1$, where $\psi_1, \psi_2 \in X_+$, then the following implications hold

$$-\int_M \psi_1 d\nu \leq \sum_{j \in J_0} \lambda_j \int_M \varphi_j d\nu = \sum_{j \in J_0} \lambda_j \leq \int_M \psi_2 d\nu \Rightarrow$$
$$\left|\sum_{j \in J_0} \lambda_j\right| \leq \int_M \psi_2 + \int_M \psi_1 = \int_M \psi_2 d\nu - \left(-\int_M \psi_1 d\nu\right).$$

Now the hypothesis (b) yields

$$\left|\sum_{j \in J_0} \lambda_j y_j\right| \leq \left\|\sum_{j \in J_0} \lambda_j y_j\right\|_\infty \cdot \chi_\Omega \leq \left(\sum_{j \in J_0} |\lambda_j| \cdot \|y_j\|\right) \cdot \chi_\Omega \leq$$
$$\leq b \cdot \left|\sum_{j \in J_0} \lambda_j\right| \chi_\Omega \leq b\left(\int_M \psi_2 d\nu \cdot \chi_\Omega - \left(-\int_M \psi_1 d\nu \cdot \chi_\Omega\right)\right) =$$
$$F_2(\psi_2) - F_1(\psi_1), F_2(\psi) := b\int_M \psi \, d\nu \cdot \chi_\Omega, \; F_1 := -F_2$$

Application of theorem 2.4 leads to the existence of a linear operator $F \in L(X, Y)$ such that

$$F(\varphi_n)=y_n, n\in \mathrm{N}, \quad F_1(\psi)=-b\int_M \psi\, d\nu\cdot \chi_\Omega \le F(\psi)\le F_2(\psi)=$$
$$b\int_M \psi\, d\nu\cdot \chi_\Omega, \psi\in X_+ \Leftrightarrow |F(\psi)|\le b\int_M \psi\, d\nu\cdot \chi_\Omega, \psi\in X_+ \Rightarrow$$
$$|F(\varphi)|\le \left|F(\varphi^+)\right|+\left|F(\varphi^-)\right|\le b\cdot \int_M (\varphi^+ +\varphi^-)d\nu\, \chi_\Omega = b\int_M |\varphi|\, d\nu\cdot \chi_\Omega, \varphi\in X$$

This concludes the proof. □

Theorems 3.2.1 and 3.2.2 show how different might be the moment problem and Mazur-Orlicz theorem, even for similar statements. For the next result, let X be the space of all absolutely convergent power series in the closed polydisc

$$\overline{D}_R = \left\{z = (z_1, \dots, z_n) : |z_p| \le R_p, p \in \{1, \dots, n\}\right\}, R := (R_1, \dots, R_n),$$

with real coefficients. The positive cone of X consists in all power series in X, having all the coefficients nonnegative numbers. The space Y is the same as in Theorem 3.1.4. Denote

$$\|h\|_\infty := \sup_{z\in \overline{D}_R} |h(z)|, h \in X.$$

Theorem 3.2.3. *Let* $0 < r_p < R_p, p = 1, \dots, n, h_k(z) = z_1^{k_1} \cdots z_n^{k_n}, k \in \mathbb{N}^n, z \in \overline{D}_R, \alpha > 0$. *Let* $(B_k)_{k\in\mathbb{N}^n}$ *be a multi indexed sequence of positive operators in* Y. *Consider the following statements*

(a) *there exists a linear positive bounded operator* F *applying* X *to* Y *such that*

$$F(h_k) \ge B_k, \forall k \in \mathbb{N}^n, |F(\varphi)| \le \alpha \prod_{p=1}^{n} \frac{R_p}{R_p - r_p} \|\varphi\|_\infty I,$$

$$\|F\| \le \alpha \prod_{p=1}^{n} \frac{R_p}{R_p - r_p};$$

(b) $B_k \le \alpha r_1^{k_1} \cdots r_n^{k_n} I, \forall k \in \mathbb{N}^n$, *where* I *is the identity operator.*

Then (b) *implies* (a).

Proof. Let $J_0 \subset \mathbb{N}^n$ be a finite subset and $(\lambda_j)_{j\in J_0}$ be a set of nonnegative real scalars, such that $\sum_{j\in J_0} \lambda_j h_j \le \varphi = \sum_{k\in\mathbb{N}^n} \gamma_k h_k \Rightarrow \lambda_j \le \gamma_j, j \in J_0, \gamma_k \ge 0, \forall k \in \mathbb{N}^n$. Let ε be an arbitrary number such that $0 < \varepsilon < \min_{1\le p\le n}\{R_p - r_p\}$. The Cauchy's inequalities for the analytic function φ lead to

$$\gamma_k = |\gamma_k| \le \frac{\|\varphi\|_\infty}{(R_1 - \varepsilon)^{k_1} \cdots (R_n - \varepsilon)^{k_n}}, k \in \mathbb{N}^n.$$

Using these relations and the preceding ones, as well as the hypothesis on $B_k, k \in \mathbb{N}^n$, we infer that

$$\sum_{j\in J_0} \lambda_j B_j \le \sum_{j\in J_0} \gamma_j B_j \le \alpha \|\varphi\|_\infty \sum_{k\in\mathbb{N}^n} \left(\frac{r_1}{R_1 - \varepsilon}\right)^{k_1} \cdots \left(\frac{r_n}{R_n - \varepsilon}\right)^{k_n} I =$$

$$= \alpha\|\varphi\|_\infty \prod_{p=1}^{n}\left(\sum_{k_p\in\mathbb{N}}\left(\frac{r_p}{R_p-\varepsilon}\right)^{k_p}\right) I = \alpha\|\varphi\|_\infty \prod_{p=1}^{n} \frac{R_p-\varepsilon}{R_p-\varepsilon-r_p} I ,$$

$$\forall\varepsilon \in (0, R_p - r_p), p = 1, \dots, n.$$

Passing through the limit with $\varepsilon \downarrow 0$, the following basic relation follows

$$\sum_{j\in J_0} \lambda_j B_j \le \alpha\|\varphi\|_\infty \prod_{p=1}^{n} \frac{R_p}{R_p - r_p} I =: T(\varphi) = T(-\varphi).$$

Application of Theorem 2.7 leads to the existence of a linear positive operator F,

$$F(h_k) \ge B_k, \forall k \in \mathbb{N}^n, |F(\varphi)| \le \alpha \prod_{p=1}^{n} \frac{R_p}{R_p - r_p} \|\varphi\|_\infty I, \forall\varphi \in X.$$

Since the norm on Y is solid, we derive

$$\|F(\varphi)\| \le \alpha \prod_{p=1}^{n} \frac{R_p}{R_p - r_p} \|\varphi\|_\infty, \forall\varphi \in X$$

This concludes the proof. □

The next two theorems are applications of the last theorem of section 2 (Theorem 2.8). The first one refers to a space of analytic functions, while the second one involves a space of continuous functions. Both these problems are multidimensional-type Markov moment problems. Let $n \neq 0$ be a natural number and X be the space of absolutely convergent power series in the unit closed poly - disc $\bar{D}_1 = \{z = (z_1, \dots, z_n): |z_p| \le 1, p \in \langle 1, \dots, n\rangle\}$, with real coefficients. The norm on X is defined by

$$\|\varphi\|_\infty = sup\{|\varphi(z)|: z \in \bar{D}_1\}.$$

Denote

$$h_k(z) = z_1^{k_1} \cdots z_n^{k_n}, k = (k_1, \dots, k_n) \in \mathbb{N}^n, z \in \bar{D}_1,$$

$|k| \coloneqq k_1 + \cdots + k_n$.. On the other hand, let H be a complex Hilbert space, $\mathcal{A}$ the real vector space of all self adjoint operators acting on $H, A \in \mathcal{A}$. Define the space $Y = Y(A)$ as in Theorem 3.1.4. Let $(B_k)_{k\in\mathbb{N}^n}$ be a multi - indexed sequence of operators in Y, *and* $\tilde{B} \in Y_+\backslash\{0\}$.

Theorem 3.2.4. *Assume that* $A_1, \dots, A_n$ *are elements of* Y *such that there exists a real number* $M > 0$, *so that*

$$|B_k| \le M \frac{A_1^{2k_1}}{k_1!} \cdots \frac{A_n^{2k_n}}{k_n!}, \forall k \in \mathbb{N}^n, \sum_{p=1}^{n} A_p^2 \le I,$$

where I *is the identity operator. Let* $\{\varphi_k\}_{k\in\mathbb{N}^n} \subset X$ *be such that* $1 = \|\varphi_k\| = \varphi_k(0), \forall k \in \mathbb{N}^n$. *Then there exists a linear bounded operator* $F \in B(X, Y)$ *such that*

$$F(h_k) = B_k, |k| \ge 1, F(\varphi_k) \ge \tilde{B}, \forall k \in \mathbb{N}^n,$$

$$F(h) \le \left(2 + \|\tilde{B}\| M^{-1} e^{-1}\right)\|h\|_\infty u_0, \forall h \in X, u_0 \coloneqq MeI.$$

In particular, the following evaluation holds: $\|F\| \leq 2Me + \|\tilde{B}\|$.

Proof. One applies Theorem 2.8. The subspace generated by $\{h_k : |k| \geq 1\}$ stands for S of Theorem 2.1, and the convex hull of the set of the functions $\varphi_k, k \in \mathbb{N}^n$, stands for the set A. The following remark is essential:

$$\|s-\varphi\|_\infty \geq |s(0)-\varphi(0)| = |0-1| = 1, \forall s \in S, \forall \varphi \in A.$$

This proves that $(S + B(0,1)) \cap A = \emptyset$, so that $B(0,1)$ stands for V and $\|\cdot\|_\infty$ stands for p_V from Theorem 2.1. The operator $\tilde{B}$ will stand for $\tilde{y}$. Now let

$$\varphi = \sum_{j \in J_0} \beta_j h_j \in S \cap B(0,1),$$

where J_0 is a finite subset of $\mathbb{N}^n$. The following relations hold

$$\left|\sum_{j \in J_0} \beta_j B_j\right| \leq \sum_{j \in J_0} |\beta_j||B_j| \leq \|\varphi\|_\infty \sum_{j \in J_0} \frac{1}{r_1^{j_1} \dots r_n^{j_n}} |B_j|,$$

for any $0 < r_p < 1, p \in \{1, \dots, n\}$, thanks to Cauchy inequalities. Passing to the limit with $r_p \uparrow 1. p \in \{1, \dots, n\}$ and using the fact that $\varphi \in B(0,1)$, as well as the hypothesis in the statement, the preceding relation further yields

$$\left|\sum_{j \in J_0} \beta_j B_j\right| \leq \sum_{j \in J_0} |B_j| \leq M \sum_{j \in J_0} \frac{A_1^{2j_1}}{j_1!} \cdots \frac{A_n^{2j_n}}{j_n!} \leq M \left(\sum_{k_1 \in \mathbb{N}} \frac{A_1^{2k_1}}{k_1!}\right) \cdots \left(\sum_{k_n \in \mathbb{N}} \frac{A_n^{2k_n}}{k_n!}\right) =$$

$$= Mexp\left(\sum_{p=1}^{n} A_p^2\right) \leq Mexp(I) = MeI = u_0.$$

The conclusion is that denoting by $f: S \to Y$ the linear operator which satisfies the moment conditions $f(h_k) = B_k, k \in \mathbb{N}^n, |k| > 1$, we have

$$-MeI \leq f(s) \leq MeI = u_0, \forall s \in S \cap B(0,1).$$

On the other hand, the following relations hold

$$\tilde{B} \leq \|\tilde{B}\| I = \|\tilde{B}\| M^{-1} e^{-1} u_0 = \alpha_1 u_0,$$

where $\alpha_1 := \|\tilde{B}\| M^{-1} e^{-1}$. The conditions on the norms of the functions $\varphi_k, k \in \mathbb{N}^n$ lead to

$$\|\varphi\| \leq 1, \forall \varphi \in A.$$

So, the constant 1 stands for α from Theorem 2.8. Now all the conditions from the statement of theorem 2.8 are accomplished. Application of the latter theorem, leads to the existence of a linear mapping $F: X \to Y$, such that

$$F(h_k) = f(h_k) = B_k, k \in \mathbb{N}^n, |k| > 1, F(\varphi_k) \geq \tilde{B}, \forall k \in \mathbb{N}^n,$$

$$F(h) \leq (2 + \|\tilde{B}\| M^{-1} e^{-1}) \|h\|_\infty MeI, \forall h \in X.$$

From the last inequality, we derive

$$|F(h)| \leq (2Me + \|\tilde{B}\|) \|h\|_\infty I, \quad \forall h \in X.$$

Since the norm on Y is solid, we infer that

$$\|F(h)\| \leq \left(2Me + \|\tilde{B}\|\right)\|h\|_{\infty}, \forall h \in X \Rightarrow \|F\| \leq 2Me + \|\tilde{B}\|.$$

This concludes the proof.

Let H be a complex Hilbert space, $A_k, k = 1,\ldots,n,$ linear positive commuting self -adjoint operators on H.

$$Y_1 = \{U \in \mathcal{A}(H);\ A_k U = U A_k,\ k = 1,\ldots,n\},\ Y = \{V \in Y_1; VU = UV, \forall U \in Y\},$$
$$Y_+ = \{V \in Y; \langle V(h), h\rangle \geq 0, \forall h \in H\}$$

X will be the space $C([0,b_1] \times \cdots \times [0,b_n]),\ Y = Y(A_1,\ldots,A_n)$ is the space just defined above. It seems that repeating the arguments in [5], one can prove that Y is an order complete Banach lattice (and a commutative real Banach algebra). Assume additionally that $\sigma(A_k) = [0, b_k],\ k = 1,\ldots,n.$

Let

$$\varphi_j \in X,\ \varphi_j(t_1,\ldots,t_n) = t_1^{j_1} \cdots t_n^{j_n},\ (t_1,\ldots,t_n) \in [0,b_1] \times \cdots \times [0,b_n],$$
$$j = (j_1,\ldots,j_n) \in \mathbb{N}^n,\ |j| := \sum_{k=1}^{n} j_k \geq 1.$$

Theorem 3.2.5. *Let* $(\psi_j)_{j \in \mathbb{N}^n}$ *be a sequence in* X *such that* $\psi_j(0,\ldots,0) = 1$, $\psi_{(0,\ldots,0)} \equiv 1$, $\|\psi_j\|_{\infty} \leq 1$ *for all* $j \in \mathbb{N}^n$, *and let* $B \in Y, B \geq I$. *Then there exists a linear bounded positive operator* $F \in B(X,Y)$, *which is multiplicative on the subspace of continuous functions vanishing at the origin, such that*

$$F(\varphi_j) = A_1^{j_1} \cdots A_n^{j_n},\ \ j \in \mathbb{N}^n,\ |j| \geq 1,$$
$$F(\psi_j) \geq B,\ \forall j \in \mathbb{N}^n, |F(\varphi)| \leq (2 + \|B\|) \cdot \|\varphi\| \cdot I,\ \forall \varphi \in X.$$

Proof. Denote $A = conv\{\psi_j; j \in \mathbb{N}^n\},\ S = Span(\{\varphi_j; |j| \geq 1\})$. Then we get

$$\|s - a\|_{\infty} \geq |s(0) - a(0)| = 1, \forall s \in S, \forall a \in A \Rightarrow$$

$$(S + B(0,1)) \cap A = \Phi,\ V := B(0,1),\ \|\psi\|_{\infty} \leq 1 =: \alpha,\ \forall \psi \in A$$

Thus, the unit ball $B(0,1)$ of the space X stands for V of Theorem 2.8, $\|\ \|$ stands for p_V, and A is the convex hull of the collection of functions $\psi_j,\ j \in N^n$. Define

$$f : S \to Y,\ f(s) = f\left(\sum_{j \in J_0} a_j \varphi_j\right) = \sum_{j \in J_0} a_j A_1^{j_1} \cdots A_n^{j_n},$$

where $J_0 \subset \{j \in N^n; |j| \geq 1\} \subset N^n$ is a finite subset. If $s \in S \cap B(0,1)$, then

$$-1 \le s = \sum_{j \in J_0} a_j t_1^{j_1} \Lambda\ t_n^{j_n} \le 1 \quad \forall (t_1,\ldots,t_n) \in [0,b_1] \times \Lambda\ [0,b_n] \Rightarrow$$
$$-I \le f(s) = \sum_{j \in J_0} a_j A_1^{j_1} \Lambda\ A_n^{j_n} \le I,$$

because of the positivity of the spectral measures associated to the $n-$ tuple $(A_1,\ldots,A_n)$. On the other hand $B \le \|B\| \cdot I$, so that all conditions of theorem 2.8 are verified for

$$\alpha_1 = \|B\|, \alpha = 1, u_0 = I$$

Application of theorem 2.8 leads to the existence of a linear extension F of f, such that

$$F(\varphi) \le (2 + \|B\|) \|\varphi\|_\infty I, \quad \forall \varphi \in X \Rightarrow |F(\varphi)| \le (2 + \|B\|) \cdot \|\varphi\|_\infty I \Rightarrow \|F\| \le 2 + \|B\|,$$
$$F(\psi_j) \ge B := \tilde{y}, \forall j \in \mathrm{N}^n \Rightarrow F(1) \ge B$$

In particular, F is continuous. Now we prove that F is also positive. Let p be a polynomial

$$p(t_1,\ldots,t_n) = \sum_{j \in J_1} a_j t_1^{j_1} \Lambda\ t_n^{j_n} \ge 0 \quad \forall (t_1,\ldots,t_n) \in [0,b_1] \times \Lambda \times [0,b_n],$$

where $J_1 \subset N^n$ is a finite subset. Then using the positivity of the spectral measures attached to $n-$ tuple of

operators $(A_1, \ldots, A_n)$, as well as the relations

$$F(1) = F(\psi_{(0,\ldots,0)}) \ge B \ge I, a_{(0,\ldots,0)} = p(0,\ldots,0) \ge 0,$$

we derive the following implications

$$s := \sum_{j \in J_1, |j| \ge 1} a_j t_1^{j_1} \Lambda\ t_n^{j_n} \ge -a_{(0,\ldots,0)} \Rightarrow F(s) = \sum_{j \in J_1, |j| \ge 1} a_j A_1^{j_1} \Lambda\ A_n^{j_n} \ge -a_{(0,\ldots,0)} \cdot I \Rightarrow$$
$$F(p) = a_{(0,\ldots,0)} F(1) + F(s) \ge a_{(0,\ldots,0)} (F(1) - I) \ge a_{(0,\ldots,0)} (B - I) \ge 0$$

Application of Weierstrass approximation theorem and the continuity of F lead to the positivity of F on X.

Hypothesis on the fact that $A_1,\ldots,A_n$ are permutable and a straightforward computation shows that

$$f(p_1 p_2) = f(p_1) f(p_2)$$

for all polynomials of n variables, vanishing at the origin. Since F is a continuous linear extension of f and the product operation on the Banach algebra Y is continuous, we infer that F is multiplicative on the subspace of continuous functions vanishing at the origin (use Bernstein approximating polynomials of n variables: if a continuous function vanishes at the origin, then all the corresponding Bernstein polynomials do the same). This concludes the proof.

4 Conclusions

The present review article is essentially based on results in constrained extension for linear operators, polynomial approximation on unbounded subsets and the analytic form of positive polynomials on certain closed subsets.

These are earlier results, which are recalled without their proofs. Using this background and general results in classical and functional analysis, one obtains solutions for Markov moment problems and Mazur-Orlicz type problems on concrete spaces. All our solutions involving concrete spaces are operator-valued or function-valued. These are relative new published results and are accompanied by their proofs. The methods can be seen following these proofs. Between the main background-domains mentioned above there is a strong relationship. For example, in the proof of the approximation result given by Lemma 3.1.1, Theorem 2.5 and Haviland theorem are applied, as discussed in [6], [25]. On the other hand**,** approximation and extension of linear operators solve Markov moment problems. To conclude, in particular, the relationship between the subjects in the title is illustrated. The results are stated and the corresponding methods follow from their proofs or from appropriate reference citations. The main purpose of this review article is to prove theorems of subsections 3.1 and 3.2. This aim is attained by means of methods recalled in the other two sections (no. 1 and no. 2), also using general type results in measure theory and functional analysis, referred during the proofs.

Conflicts of interest

The author declares that there is no conflict of interest in publishing this article.

References

1. N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*, Oliver and Boyd, Edinburgh-London, 1965.
2. W. Rudin, *Real and Complex Analysis*, Third Edition, Theta, Bucharest, 1999 (Romanian).
3. G. Choquet, *Le problème des moments*, Séminaire d'Initiation à l'Analise, Institut H. Poincaré, Paris, 1962.
4. K. Schmüdgen, *The Moment Problem,* Graduate Texts in Mathematics, Springer, 2017.
5. R. Cristescu, *Ordered Vector Spaces and Linear Operators*, Academiei, Bucharest, Romania, and Abacus Press, Tunbridge Wells, Kent, England, 1976.
6. O. Olteanu, J. M. Mihăilă*, Polynomial approximation on unbounded subsets and some applications,* Asian Journal of Science and Technology, 9, 10 (2018), 8875-8890.
7. E.K. Haviland, *On the moment problem for distribution functions in more than one dimension II,* Amer. J. math. **58** (1936), 164-168.
8. O. Olteanu, *Convexité et prolongement d'opérateurs linéaires*, C. R. Acad. Sci. Paris, **286**, Série A, (1978), 511-514.
9. O. Olteanu, *Théorèmes de prolongement d'opérateurs linéaires*, Rev. Roumaine Math. Pures Appl., **28**, 10 (1983), 953-983.
10. O. Olteanu*, Application de théorèmes de prolongement d'opérateurs linéaires au problème des moments et à une généralisation d'un théorème de Mazur-Orlicz*, C. R. Acad. Sci Paris **313**, Série I, (1991), 739-742.
11. C. Berg, J.P.R. Christensen, and C.U. Jensen, *A remark on the multidimensional moment problem*, Mathematische Annalen, 243(1979), 163-169.
12. M. Marshall, *Polynomials non–negative on a strip*, Proceedings of the American Mathematical Society, **138** (2010), 1559 - 1567.
13. M. Marshall, *Positive Polynomials and Sums of Squares*, American Mathematical Society, 2008.
14. G. Cassier, *Problèmes des moments sur un compact de $\mathbb{R}^n$ et décomposition des polynômes à plusieurs variables*, Journal of Functional Analysis, 58 (1984), 254-266.
15. K. Schmüdgen, *The $K-$ moment problem for compact semi-algebraic sets,* Mathematische Annalen, 289 (1991), 203-206.
16. B. Fuglede, *The multidimensional moment problem*, Expo. Math. I (1983), 47-65.

17. M.G. Krein and A.A. Nudelman, *Markov Moment Problem and Extremal Problems*, American Mathematical Society, Providence, RI, 1977.

18. M. Putinar, *Positive polynomials on compact semi-algebraic sets*, Indiana University Mathematical Journal, **42**, (3) (1993), 969-984.

19. F.H. Vasilescu, *Spectral measures and moment problems*. In "Spectral Analysis and its Applications. Ion Colojoară Anniversary Volume", Theta, Bucharest, 2003, pp. 173-215

20. L. Lemnete-Ninulescu, O.C. Olteanu, *Applications of classical and functional analysis*, LAMBERT Academic Publishing, Saarbrűcken, 2017.

21. C. Niculescu and N. Popa, *Elements of Theory of Banach Spaces*, Academiei, Bucharest, 1981 (Romanian).

22. J.M. Mihăilă J.M., O. Olteanu, C. Udrişte, *Markov – type moment problems for a rbitrary compact and some noncompact Borel subsets of* $\mathbb{R}^n$, Rev. Roum Math Pures Appl., 52, (5-6) (2007), 655-664.

23. A. Olteanu, O. Olteanu, *Some unexpected problems of the Moment Problem*. Proc. of the 6-th Congress of Romanian Mathematicians, Bucharest, 2007, Vol I, Scientific contributions, Romanian Academy Publishing House, Bucharest, 2009, pp. 347-355.

24. O. Olteanu, *Applications of a general sandwich theorem for operators to the moment problem*, Rev. Roum. Math. Pures Appl., **41,** (7-8) (1996), 513-521.

25. O. Olteanu, *Approximation and Markov moment problem on concrete spaces*. Rendiconti del Circolo Matematico di Palermo, 63 (2014), 161-172. DOI: 10.1007/s12215-014-0149-7

26. O. Olteanu, *Markov moment problems and related approximation*, Mathematical Reports, **17(67)**, (1) (2015), 107-117.

27. O. Olteanu, *Applications of Hahn-Banach principle to the moment problem,* Poincaré Journal of Analysis & Applications, 1, (2015), 1-28.

28. O. Olteanu, J. M. Mihăilă, *On Markov moment problem and Mazur – Orlicz theorem,* Open Access Library Journal, **4** (2017), 1-10.

29. O. Olteanu, J.M. Mihăilă, *Operator – valued Mazur – Orlicz and moment problems in spaces of analytic function,* U. P. B. Sci. Bull., Series A, **79**, 1 (2017), 175-184.

30. C. Kleiber and J. Stoyanov, *Multivariate distributions and the moment problem,* Journal of Multivariate Analysis, **113** (2013), 7-18.

31. G.D. Lin, J. Stoyanov, *Moment Determinacy of Powers and Products of Nonnegative Random Variables*, J. Theor Probab (2015) 28: 1337-1353.

32. J Stoyanov and C. Vignat, *Non-conventional limits of random sequences related to partitions of integers*, asXiv: 1901 .04029v 1 [math.PR] 13 Jan 2019 (submitted to the Annals of Applied Probability).