

Garays Condition of Deformed Cylindrical and Translation Surfaces in E^3

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Abstract

The motivation of the present work is to develop the finiteness property in our work [1, 2, 3, 4] by using Garay's condition [5]. The mean curvature flow and the finiteness property of the cylindrical surfaces in E^3 are investigated. Additionally, the linear deformation of such surfaces is studied. Finally, the translation surfaces are discussed.

Keywords: Cylindrical Surfaces, Translation Surfaces, Variation, Garay's condition.

Introduction

The study of submanifolds of finite type began in the late 1970s through the author's attempts to find the best possible estimate of the total mean curvature of a compact submanifold of a Euclidean space and to find a notion of "degree" for submanifolds of a Euclidean space. The family of submanifolds of finite type is large, which contains many important families of submanifolds; including minimal submanifolds of Euclidean space, minimal submanifolds of hyperspheres, parallel submanifolds as well as all equivariantly immersed compact homogeneous submanifolds.

On one hand, the notion of finite type submanifolds provides a very natural way to apply spectral geometry to study submanifolds where spectral geometry is a field in mathematics which concerns relationships between geometric structures of manifolds and spectra of canonically defined differential operators. On the other hand, one can also apply the theory of finite type submanifolds to investigate the spectral geometry of submanifolds. The first results on submanifolds of finite type were collected in [6, 7]. A list of twelve open problems and three conjectures on submanifolds of finite type was published in [8]. Furthermore, a detailed report of the progress on this theory was presented in [9]. Also, the study of finite type submanifolds have received a growing attention with many progresses since the beginning of this century. In [10], is provided a detailed account of recent development on the problems and conjectures listed in [8].

One of the most interesting and profound aspects of classical differential geometry is its interplay with the calculus of variations. The calculus of variations have their roots in the very origins of subject, such as, for instance, in the theory of minimal surfaces. More recently, the variational principles which give rise to the field equations of the general theory of relativity have suggested the systematic investigation of a seemingly new type of variational problem. In the case of the earlier applications one is, at least implicitly, concerned with a multiple integral in the calculus of variations. In addition, the normal variational problem on general surfaces and hyperruled surfaces were studied by some geometers, specifically one may cite [11]-[17].

The mean curvature flow has many physical problems in the nature, starting from the well-known Poisson-Laplace theorem which relates, the pressure and the mean curvature flow of a surface immersed in a liquid until the capillary theory [18].

In this paper, first, we study Garay's condition on the cylindrical surfaces and their deformations in E^3 . Secondly, we deal with Garay's condition on the translation surfaces in E^3 before and after their deformations. Finally, we give the necessary conditions to satisfy Garay's condition of deformed translation surfaces.

1 Basic concepts

Here, we introduce some basic definitions and relations. Let a surface $M : \mathbf{X} = \mathbf{X}(s, v)$ in an Euclidean 3-space E^3 . The map $\mathbf{G} : M \rightarrow S^2(1) \subset E^3$ which sends each point of M to the unit normal vector to M at the point is called the Gauss map of a surface M ; where $S^2(1)$ denotes the unit sphere of E^3 . The standard unit normal vector field \mathbf{G} on the surface M can be defined by:

$$\mathbf{G} = \frac{\mathbf{X}_s \times \mathbf{X}_v}{|\mathbf{X}_s \times \mathbf{X}_v|}, \quad (1)$$

where \mathbf{X}_s and \mathbf{X}_v are the first partial derivatives with respect to the parameters of \mathbf{X} . [19, 20] Let M be an n -dimensional surface. Then the Laplacian Δ operator (or Laplacian-Beitrami operator) associated with the induced metric on M is a mapping which sends any differentiable function f to the function Δf of the form

$$\Delta = -\frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x_i} (\sqrt{g} g^{ij} \frac{\partial}{\partial x_j}), \tag{2}$$

where $\{x_i, x_j\}$ are the local coordinates on M , (g_{ij}) is the matrix of the Riemannian metric on M where $(g^{ij}) = (g_{ij})^{-1}$ and $g = \det(g_{ij})$.

We recall theorem of T.Takahashi [21] and [9] which states that a submanifold M of a Euclidean space is of 1-type, i.e. the position vector field of the submanifold in the Euclidean space satisfies the differential equation

$$\Delta \mathbf{X} = \lambda \mathbf{X}, \tag{3}$$

for some real number λ , if and only if either the submanifold is a minimal submanifold of the Euclidean space ($\lambda = 0$) or it is a minimal submanifold of a hypersphere of the Euclidean space centered at the origin ($\lambda \neq 0$).

As a generalization of Takahashi's condition 2eq5, Garay [22] studied hypersurfaces in E^m whose coordinate functions are eigenfunctions of the Laplacian operator of the hypersurface, but not necessarily associated to the same eigenvalue. Specifically, he considered hypersurfaces in E^m satisfying the differential equation

$$\Delta \mathbf{X} = A \mathbf{X}, \tag{4}$$

where $A \in \text{Diag}(m; E)$ is an $m \times m$ -diagonal matrix, and proved that such hypersurfaces are minimal in E^m and open pieces of either round hyperspheres or generalized right spherical cylinders. Garay called such submanifolds coordinate finite type. Related to this, Dillen, Pas and Verstraelen [23] observed that Garay's condition 6eq4 is not coordinate invariant and they proposed the study of submanifolds of E^m satisfying the following equation:

$$\Delta \mathbf{X} = A \mathbf{X} + B, \tag{5}$$

where $A \in \text{Mat}(m; E)$ is a $m \times m$ matrix and $B \in E^m$. On the other hand, the class of submanifolds satisfying 6eq4 and the class of submanifolds satisfying 6eq5 are the same if the submanifolds are hypersurfaces of Euclidean space [24]. Also, the above mentioned study can be extended the notion of an immersion of submanifolds into pseudo-Euclidean space [25]. Recently, many geometers are studying an extension of Takahashi theorem for the linearized operators of the higher order mean curvatures of hypersurfaces [26]-[29].

Let M be a connected (not necessary compact) surface in E^3 . Then the position vector \mathbf{X} and the mean curvature vector \mathbf{H} of M in E^3 satisfy

$$\Delta \mathbf{X} = -2 \mathbf{H}, \tag{6}$$

where $\mathbf{H} = H \mathbf{G}$ and H is the mean curvature of the surface which defined by

$$H = \frac{1}{2} \sum_{i,j=1}^2 g^{ij} L_{ij}, \tag{7}$$

where L_{ij} are the coefficients of the second fundamental form. Form Eq. eq10 yields the following well-known result: A surface M in E^3 is minimal if and only if all coordinate functions of E^3 , restricted to M , are harmonic functions, that is,

$$\Delta \mathbf{X} = 0. \tag{8}$$

[11, 15] Let $X : U \rightarrow E^3$ be a parameterized 2-surface in E^3 . A variation of X is a smooth map $\bar{\mathbf{X}} : U \times [0, 1] \rightarrow E^3$ with the property that $\bar{\mathbf{X}}(u^i, 0) = \mathbf{X}(u^i)$ for all $u^i \in U$. Thus a variation surrounds the 2-surface \mathbf{X} with a family of singular 2-surface $\mathbf{X}_t : U \rightarrow E^3$ defined by

$$\bar{\mathbf{M}} : \bar{\mathbf{X}}_t(u^i) = \bar{\mathbf{X}}(u^i, t) = \mathbf{X}(u^i) + t \phi(u^i) \mathbf{G}(u^i), \quad i = 1, 2, \quad u^i = (u, v), \tag{9}$$

where ϕ is a smooth function along \mathbf{X} and \mathbf{G} is the Gauss map of \mathbf{X} , is called a normal variation of \mathbf{X} , where t is a parameter and $t \in [0, 1]$. And the family of surfaces represented by $\bar{\mathbf{X}}(u^i, t)$ is called a deformable surfaces resulting

from $\mathbf{X}(u^i)$ by the normal variation. Now, we define a ruled surface M in E^3 . Let I be an open interval containing 0 in R . The ruled surface M is parametrized by

$$M : \mathbf{X}(u, v) = \alpha(u) + v \beta(u), \quad u \in I, \quad v \in R, \quad (10)$$

where $\alpha = \alpha(u)$ and $\beta = \beta(u)$ are smooth mappings from I into E^3 . The map α is called a base curve and β is called a director curve. M is said to be cylindrical if $\beta(u)$ is parallel to fixed direction in E^3 . It is called non-cylindrical otherwise see [30, 32].

Also, we define the translation surfaces M in E^3 . Let $\mathbf{X} : M \rightarrow E^3$ be a translation surface in E^3 with planar generating curves lying in orthogonal planes. Then, it can be parameterized, locally, as

$$\mathbf{X}(u, v) = (u, v, f(u) + h(v)), \quad (11)$$

where $f = f(u)$ and $h = h(v)$ are smooth functions on M [33, 34, 35].

2 Cylindrical surfaces in E^3

Let M be a cylinder over a plane curve $\alpha(s) = \{\alpha_1(s), \alpha_2(s), 0\}$. We suppose that $\alpha(s)$ is parameterized by its arc length s . Then, M is written as:

$$M : \mathbf{X}(s, v) = \alpha(s) + v \beta, \quad (12)$$

where β is a constant unit vector, namely $\beta = \{0, 0, 1\}$. Then the unit normal vector of M is given by

$$\mathbf{G} = (\alpha'_2, -\alpha'_1, 0), \quad (13)$$

where $\alpha_1 = \alpha_1(s)$, $\alpha_2 = \alpha_2(s)$. Using the coefficients of the first fundamental form we find that

$$(g_{ij}) = \text{diag}(1, 1), \quad g = 1, \quad (14)$$

where $\text{diag}(1, 1)$ is 2×2 -diagonal matrix. As well known the formula of Laplacian is taking the following form:

$$\Delta = -\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial v^2}. \quad (15)$$

Therefore, the Laplacian operator Δ of \mathbf{X} is given by

$$\Delta \mathbf{X} = (-\alpha'_2 \psi, \alpha'_1 \psi, 0); \quad \psi = \alpha''_1 \alpha'_2 - \alpha'_1 \alpha''_2. \quad (16)$$

Suppose \mathbf{X} satisfies Garay's Condition (coordinate finite type) Eq. 6eq4 where A is given by

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}. \quad (17)$$

Then, Eq. 6eq4 turn to

$$\alpha'_2 \psi + \lambda_1 \alpha_1 = 0, \quad \alpha'_1 \psi - \lambda_2 \alpha_2 = 0, \quad \lambda_3 v = 0. \quad (18)$$

Since α is parameterized by the arc length, i.e., $(\alpha'_1)^2 + (\alpha'_2)^2 = 1$. Then, we may put [30]

$$\alpha'_1 = \cos \theta(s), \quad \alpha'_2 = \sin \theta(s), \quad (19)$$

for some function $\theta(s)$. Using R5eq2 into 0eq9, we get

$$\theta' \sin \theta - \lambda_1 \alpha_1 = 0, \quad \theta' \cos \theta + \lambda_2 \alpha_2 = 0, \quad \lambda_3 v = 0, \quad (20)$$

where $\theta = \theta(s)$. From the third equation, we get $\lambda_3 = 0$. Differentiate the first and seconde equations in R5eq1 w.r.t. s , it yields

$$\begin{aligned} (\theta'^2 - \lambda_1) \cos \theta + \theta'' \sin \theta &= 0, \\ (\theta'^2 - \lambda_2) \sin \theta - \theta'' \cos \theta &= 0. \end{aligned} \quad (21)$$

Since, $\cos \theta$ and $\sin \theta$ are linearly independent functions. Consequently, $\theta'' = 0$ and $\lambda_1 = \lambda_2 = \theta'^2$. Hence, $\theta' = \text{constant}$. If $\theta' = 0$, then $\lambda_1 = \lambda_2 = 0$. That's means M is a plane or minimal surface. But if $\theta' \neq 0$, then

$$\begin{aligned} \theta &= as + c; a = \sqrt{\lambda_1} \Rightarrow \alpha'_1 = \cos(as + c), \alpha'_2 = \sin(as + c) \Rightarrow \\ \alpha_1 &= \frac{1}{a} \sin(as + c), \quad \alpha_2 = -\frac{1}{a} \cos(as + c). \end{aligned}$$

That means α is a circular curve and \mathbf{X} is a circular cylinder.

As known, the plane and circular cylinder are the only circular ruled surfaces which are finite type in E^3 .

Supposes \mathbf{G} satisfies Garay's Condition, i.e.,

$$\Delta \mathbf{G} = A \mathbf{G}. \tag{22}$$

Then, we get

$$\{ \sin \theta (\theta'^2 - \lambda_1) - \theta'' \cos \theta, \cos \theta (\lambda_2 - \theta'^2) - \theta'' \sin \theta, 0 \} = (0, 0, 0) = \mathbf{0}, \tag{23}$$

where $\mathbf{0}$ zero vector. Using the previous technique, we obtain the same result as in the case of the surface \mathbf{X} see [31].

2.1 Mean curvature flow of cylindrical surfaces

Now, we research the effect of the deformation of the circular ruled surfaces in direction of mean curvature flow. Putting $u = s$ in Eq. var17 then, we get

$$\bar{\mathbf{X}}(s, v) = \mathbf{X}(s, v) + t H \mathbf{G}, \tag{24}$$

where $H = \frac{\psi(s)}{2}$ and $\psi(s)$ is given from 0eq10. Then

$$\bar{\mathbf{X}}(s, v) = \{ \alpha_1 + \frac{1}{2}t \psi \alpha'_2, \alpha_2 - \frac{1}{2}t \psi \alpha'_1, v \}. \tag{25}$$

Therefore, one can get the coefficients of the first fundamental form as the following:

$$(\bar{g}_{ij}) = \begin{pmatrix} 1 - t \psi^2 & 0 \\ 0 & 1 \end{pmatrix}. \tag{26}$$

Consequently, one can find the Laplacian $\bar{\Delta}$ of \bar{M} which is given by

$$\bar{\Delta} = \frac{1}{(t \psi^2 - 1)^2} \left((2t \psi^2 - 1) \frac{\partial^2}{\partial v^2} + (t \psi^2 - 1) \frac{\partial^2}{\partial s^2} - t (\alpha_1^{(3)} \alpha'_2 - \alpha'_1 \alpha_2^{(3)}) \psi \frac{\partial}{\partial s} \right), \tag{27}$$

where $t \psi^2 - 1 \neq 0$. Therefore $\bar{\Delta} \bar{\mathbf{X}} = \frac{1}{2(t \psi^2 - 1)^2} \left(2t (\alpha'_2)^2 (\alpha''_1)^3 - 4t \alpha'_1 \alpha'_2 \alpha''_2 (\alpha''_1)^2 + 2\alpha''_1 \left(t \alpha'_1 (\alpha'_1 (\alpha_2^{(3)} \alpha'_2 + (\alpha''_2)^2) - \alpha_1^{(3)} (\alpha'_2)^2) - 1 \right) + t \left(\alpha'_2 (\alpha_1^{(3)} (2(\alpha'_1)^2 - 3)\alpha''_2 + \alpha_2^{(4)} \alpha'_1) + \alpha_2^{(3)} \alpha'_1 (3 - 2(\alpha'_1)^2)\alpha''_2 - \alpha_1^{(4)} (\alpha_2^{(2)})^2 \right), 2t (\alpha'_1)^2 (\alpha''_2)^3 - 4t \alpha'_1 \alpha'_2 \alpha''_1 (\alpha''_2)^2 + 2\alpha''_2 \left(t \alpha'_2 (\alpha'_2 (\alpha_1^{(3)} \alpha'_1 + (\alpha''_1)^2) - \alpha_2^{(3)} (\alpha'_1)^2) - 1 \right) + t \left(-2\alpha_1^{(3)} (\alpha_2^{(2)})^3 \alpha''_1 + 2\alpha_2^{(3)} \alpha'_1 (\alpha_2^{(2)})^2 \alpha''_1 + \alpha_2^{(4)} \alpha'_1 + 3\alpha_1^{(3)} \alpha''_1 \right) - \alpha'_1 (\alpha_2^{(4)} \alpha'_1 + 3\alpha_2^{(3)} \alpha''_1) \right), 0 \right)$. Suppose $\bar{\mathbf{X}}$ satisfy Garay's condition, i.e.,

$$\bar{\Delta} \bar{\mathbf{X}} = \bar{A} \bar{\mathbf{X}}; \quad \bar{A} = \begin{pmatrix} \bar{\lambda}_1 & 0 & 0 \\ 0 & \bar{\lambda}_2 & 0 \\ 0 & 0 & \bar{\lambda}_3 \end{pmatrix}. \tag{28}$$

From the foregoing results, the condition 0eq11 splitted into three equations as the following:

$$\begin{aligned} (t \theta^{(3)} + (t \bar{\lambda}_1 + 2)\theta' - 3t \theta'^3) \sin \theta + t \theta' \theta'' \cos \theta + 2\bar{\lambda}_1 \alpha_1 (2t \theta'^2 - 1) &= 0, \\ (3t \theta'^3 - t \theta^{(3)} - (t \bar{\lambda}_2 + 2) \theta') \cos \theta + t \theta' \theta'' \sin \theta + 2\bar{\lambda}_2 \alpha_2 (2t \theta'^2 - 1) &= 0, \\ -v \bar{\lambda}_3 &= 0. \end{aligned} \tag{29}$$

Since, $\cos \theta$ and $\sin \theta$ are linearly independent functions. Then, we get some possibilities:

- If $\theta' = 0$, then $\theta = c = \text{constant}$, and $\bar{\lambda}_1 = \bar{\lambda}_2 = \bar{\lambda}_3 = 0$. Then, $\alpha_1 = a_1 s + b_1$, $\alpha_2 = a_2 s + b_2$ where $a_1 = \cos c$, $a_2 = \sin c$. Therefore, $\bar{\mathbf{X}}$ is a plane.
- If $\theta'' = 0$, then Eqs. R5eq3 becomes

$$\begin{aligned} 2\bar{\lambda}_1 \alpha_1 (2t \theta'^2 - 1) + \sin \theta ((t \bar{\lambda}_1 + 2)\theta' - 3t \theta'^3) &= 0, \\ 2\bar{\lambda}_2 \alpha_2 (2t \theta'^2 - 1) + \cos \theta (3t \theta'^3 - (t \bar{\lambda}_2 + 2)\theta') &= 0, \\ -v \bar{\lambda}_3 &= 0. \end{aligned} \tag{30}$$

Also, we use the idea of linear independence for \sin and \cos , then

$$(t \bar{\lambda}_1 + 2)\theta' - 3t \theta'^3 = 0, \quad (t \bar{\lambda}_2 + 2)\theta' - 3t \theta'^3 = 0. \tag{31}$$

Solving above equations, we get

(1) $\theta = \text{constant}$. Then, $\bar{\mathbf{X}}$ is a plane.

(2)

$$\theta = c_1 \pm s \sqrt{\frac{\bar{\lambda}_1 t + 2}{3t}} = c_1 \pm s \sqrt{\frac{\bar{\lambda}_2 t + 2}{3t}} \tag{32}$$

One can conclude $\bar{\lambda}_1 = \bar{\lambda}_2$. Since $2t \theta'^2 - 1 \neq 0$ (the denominator not equal zero) then, $\bar{\lambda}_1 = \bar{\lambda}_2 = 0$. Hence, we conclude that \bar{M} is an open portion of the plane or circular cylinder.

The deformed cylindrical surface of the mean curvature flow in E^3 satisfies Garay's condition if and only if

- M is a plane where $A = \bar{A}$.
- M is a circular cylinder where $A \neq \bar{A} = 0$.

2.2 Linear deformation of cylindrical surfaces

Here, we study the deformation of the cylindrical surfaces in direction in its tangent plane. Then one can write the parametrization of this surface as

$$\tilde{\mathbf{X}} = \mathbf{X} + t H (a_1 \mathbf{X}_s + a_2 \mathbf{X}_v). \tag{33}$$

Hence

$$\tilde{\mathbf{X}} = \left(\frac{1}{2}t a_1 \alpha'_1 \psi + \alpha_1, \frac{1}{2}t a_1 \alpha'_2 \psi + \alpha_2, \frac{1}{2}t a_2 \psi + v \right). \tag{34}$$

The coefficients of the first fundamental form given by

$$\begin{aligned} (\tilde{g}_{ij}) &= \begin{pmatrix} 1 + t a_1 \psi' & \frac{1}{2}t a_2 \psi' \\ \frac{1}{2}t a_2 \psi' & 1 \end{pmatrix}, \quad \tilde{g} = 1 + t a_1 \psi' \neq 0, \\ (\tilde{g}^{ij}) &= \frac{1}{2(1 + t a_1 \psi')} \begin{pmatrix} 2 & -t a_2 \psi' \\ -t a_2 \psi' & 2(1 + t a_1 \psi') \end{pmatrix}. \end{aligned} \tag{35}$$

The unit normal vector of \tilde{M} is defined as

$$\begin{aligned} \tilde{\mathbf{G}} &= \frac{1}{2\sqrt{t a_1 \psi' + 1}} (t a_1 \alpha'_2 (\alpha''_1 \alpha''_2 - \alpha_2^{(3)} \alpha'_1) - t a_1 (\alpha'_1 (\alpha''_2)^2 - \alpha_1^{(3)} (\alpha'_2)^2) + 2\alpha'_2, \\ &\quad -t a_1 \alpha'_1 \psi' - t a_1 \psi \alpha''_1 - 2\alpha'_1, 0) \end{aligned} \tag{36}$$

Therefore, The Laplacian $\tilde{\Delta}$ of \tilde{M} is written as

$$\begin{aligned} \tilde{\Delta} &= \frac{1}{2(1 + t a_1 \psi')^2} \left(t \psi'' \left(a_1 \frac{\partial}{\partial s} + a_2 \frac{\partial}{\partial v} \right) - (2t a_1 \psi' + 2) \frac{\partial^2}{\partial s^2} + 2t a_2 \psi' \frac{\partial^2}{\partial s \partial v} \right. \\ &\quad \left. - (4t a_1 \psi' + 2) \frac{\partial^2}{\partial v^2} \right). \end{aligned} \tag{37}$$

Using Eq. R5eq4 to find $\tilde{\Delta} \tilde{\mathbf{X}}$. Then

$$\tilde{\Delta} \tilde{\mathbf{X}} = \frac{-1}{2(1+t a_1 \psi')^2} \left\{ t a_1 (4\alpha_1'' \psi' + \alpha_1^{(3)} \psi) + 2\alpha_1'', t a_1 (4\alpha_2'' \psi' + \alpha_2^{(3)} \psi) + 2\alpha_2'', 0 \right\}. \tag{38}$$

Let $\tilde{\mathbf{X}}$ be satisfy Garay's condition, i.e.,

$$\tilde{\Delta} \tilde{\mathbf{X}} = \tilde{A} \tilde{\mathbf{X}}; \quad \tilde{A} = \begin{pmatrix} \tilde{\lambda}_1 & 0 & 0 \\ 0 & \tilde{\lambda}_2 & 0 \\ 0 & 0 & \tilde{\lambda}_3 \end{pmatrix}. \tag{39}$$

Similar to the previous subsection, we get

$$\begin{aligned} & \frac{1}{2(t a_1 \theta'' - 1)^2} \left(2\tilde{\lambda}_1 \alpha_1(s) (2t a_1 \theta'' - 1) \right. \\ & \left. + t a_1 \theta' (\tilde{\lambda}_1 - \theta'^2) \cos \theta + \theta' (2 - 5t a_1 \theta'') \sin \theta \right) = 0; \\ & \frac{1}{2(t a_1 \theta'' - 1)^2} \left(2\tilde{\lambda}_2 \alpha_2(s) (2t a_1 \theta'' - 1) \right. \\ & \left. + t a_1 \theta' (\tilde{\lambda}_2 - \theta'^2) \sin \theta + \theta' (5t a_1 \theta'' - 2) \cos \theta \right) = 0; \\ & -\frac{1}{2} \tilde{\lambda}_3 (2v - t a_2 \theta') = 0, \end{aligned} \tag{40}$$

where $t a_1 \theta'' - 1 \neq 0$. From third equation, we get $\tilde{\lambda}_3 = 0$, and from first and seconde equation we get $\theta' = 0$. That means $\tilde{\mathbf{X}}$ ia a plane. The plane in E^3 is the only surface which satisfies Garay's condition after its deformation. Moreover, we deal with Gauss map $\tilde{\mathbf{G}}$ of $\tilde{\mathbf{X}}$. Using Eq. R5eq4 we have

$$\begin{aligned} \tilde{\Delta} \tilde{\mathbf{G}} = & \frac{1}{4(t a_1 \psi' + 1)^{7/2}} \left\{ -14t a_1 \alpha_2^{(3)} \psi' - 2t a_1 \alpha_2^{(4)} \psi - 4\alpha_2^{(3)}, 14t a_1 \alpha_1^{(3)} \psi' \right. \\ & \left. + 2t a_1 \alpha_1^{(4)} \psi + 4\alpha_1^{(3)}, 0 \right\} \end{aligned} \tag{41}$$

Suppose $\tilde{\mathbf{G}}$ satisfies Garay's condition, i.e., $\tilde{\Delta} \tilde{\mathbf{G}} = A \tilde{\mathbf{G}}$. Then, this condition is splitted into two equations as follows:

$$\begin{aligned} & \frac{1}{2(1-t a_1 \theta'')^{7/2}} \left(\sin \theta (t a_1 \theta'' (7\tilde{\lambda}_1 - 10 \theta'^2) - 2\tilde{\lambda}_1 + 2\theta'^2) + \right. \\ & \left. \cos \theta (t a_1 (7\theta''^2 + \theta' (\theta^{(3)} + \tilde{\lambda}_1 \theta' - \theta'^3)) - 2\theta'') \right) = 0 \\ & \frac{1}{2(1-t a_1 \theta'')^{7/2}} \left(\cos \theta (\tilde{\lambda}_2 (2 - 7t a_1 \theta'') + 2\theta'^2 (5t a_1 \theta'' - 1)) + \right. \\ & \left. \sin \theta (t a_1 (7\theta''^2 + \theta' (\theta^{(3)} + \tilde{\lambda}_2 \theta' - \theta'^3)) - 2\theta'') \right) = 0 \end{aligned} \tag{42}$$

Combining the previous two equations, we have the following:

$$\begin{aligned} & (7t a_1 (\tilde{\lambda}_1 - \tilde{\lambda}_2) \theta'' + 2(\tilde{\lambda}_2 - \tilde{\lambda}_1)) \sin 2\theta + t a_1 (\tilde{\lambda}_1 - \tilde{\lambda}_2) \theta'^2 \cos 2\theta \\ & + 14t a_1 \theta''^2 + t a_1 \tilde{\lambda}_1 \theta'^2 + t a_1 \tilde{\lambda}_2 \theta'^2 - 2t a_1 \theta'^4 + 2t a_1 \theta^{(3)} \theta' \\ & - 4\theta'' = 0. \end{aligned} \tag{43}$$

Based on the linearly independent of cos and sin one can get if $\theta'' = 0$, then

- $\theta' = \tilde{\lambda}_1 = \tilde{\lambda}_2 = 0$. Then $\tilde{\mathbf{X}}$ is plane.
- $\theta'^2 = \tilde{\lambda}_1 = \tilde{\lambda}_2 \neq 0$. Then $\tilde{\mathbf{X}}$ is circular cylinder.

The plane and circular cylinder are the only surfaces whose Gauss map satisfies Garay's condition before and after their deformations. Let $\alpha(s)$ be circle curve. Then, the circular cylinder given by

$$\mathbf{X}(s, v) = (\cos s, \sin s, v). \tag{44}$$

Therefore, by using Eq. R5eq6, we obtain

$$\Delta \mathbf{X} = (\cos s, \sin s, 0). \quad (45)$$

Applying Garay's condition, then

$$\Delta \mathbf{X} - A \mathbf{X} = ((1 - \lambda_1) \cos s, (1 - \lambda_2) \sin s, -\lambda_3 v) = 0. \quad (46)$$

From this we get $\lambda_1 = \lambda_2 = 1, \lambda_3 = 0$. From Eq.R5eq5 we get the deformed surface is given by

$$\tilde{\mathbf{X}} = \left(\cos s + \frac{t a_1}{2} \sin s, \sin s - \frac{t a_1}{2} \cos s, v - \frac{t a_2}{2} \right). \quad (47)$$

The coefficients of the first fundamental form are

$$(g_{ij}) = (\tilde{g}_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g = \tilde{g} = 1. \quad (48)$$

Therefore

$$\tilde{\Delta} \tilde{\mathbf{X}} = \frac{1}{2} \left((2 \cos s + t a_1 \sin s), (2 \sin s - t a_1 \cos s), 0 \right). \quad (49)$$

Consequently

$$\tilde{\Delta} \tilde{\mathbf{X}} - \tilde{A} \tilde{\mathbf{X}} = \left(-\frac{1}{2}(\tilde{\lambda}_1 - 1)(a_1 t \sin s + 2 \cos s), \frac{1}{2}(\tilde{\lambda}_2 - 1)(t a_1 \cos s - 2 \sin s), \tilde{\lambda}_3 \left(\frac{1}{2} t a_2 - v \right) \right). \quad (50)$$

Using the notion of linearly independent for \sin and \cos we get $\tilde{\lambda}_1 = \tilde{\lambda}_2 = 1$ and $\tilde{\lambda}_3 = 0$. This example confirms the theorem.

3 Translation surfaces in E^3

In this section, we shall study Garay's condition of the translation surfaces in E^3 where we study these surfaces before and after their deformations.

3.1 Garay's condition of translation surfaces

Let M be a surface has the position vector as Eq. 66eq. Then the unit normal vector field G on the surface M is given by

$$\mathbf{G} = \frac{1}{\sqrt{w}} (-\gamma, -\rho, 1), \quad (51)$$

where $w = \gamma^2 + \rho^2 + 1 \neq 0$, and $\gamma = f'(u), \rho = h'(v)$. The metric (g_{ij}) and the contravariant metric (g^{ij}) can be written as the following

$$(g_{ij}) = \begin{pmatrix} \gamma^2 + 1 & \gamma \rho \\ \gamma \rho & \rho^2 + 1 \end{pmatrix}, \quad (g^{ij}) = \frac{1}{w} \begin{pmatrix} \rho^2 + 1 & -\gamma \rho \\ -\gamma \rho & \gamma^2 + 1 \end{pmatrix}, \quad g = w. \quad (52)$$

Then, via a straightforward computation one can obtain the formula of Laplacian as the following:

$$\Delta = \frac{1}{w^2} \left(\left(\gamma \frac{\partial}{\partial u} + \rho \frac{\partial}{\partial v} \right) \delta - w(\rho^2 + 1) \frac{\partial^2}{\partial u^2} + 2w \gamma \rho \frac{\partial^2}{\partial u \partial v} - w(\gamma^2 + 1) \frac{\partial^2}{\partial v^2} \right). \quad (53)$$

where $\delta = \gamma'(1 + \rho^2) + \rho'(1 + \gamma^2)$. The coefficients of the second fundamental form are

$$(L_{ij}) = \frac{1}{\sqrt{w}} \begin{pmatrix} \gamma' & 0 \\ 0 & \rho' \end{pmatrix}. \quad (54)$$

Hence, the mean curvature of M take the following form

$$H = \frac{\delta}{2w^{\frac{3}{2}}}. \quad (55)$$

Therefore,

$$\Delta \mathbf{X} = \frac{\delta}{w^2} (\gamma, \rho, -1). \quad (56)$$

Let \mathbf{X} be satisfy Garay's condition Eq. 6eq4, then one can get

$$\frac{1}{w^2} (\delta \gamma - \lambda_1 u w^2, \delta \rho - \lambda_2 v w^2, -\delta - \lambda_3 (f + h) w^2) = \mathbf{0}. \quad (57)$$

One can rewrite the pervious equation as

$$\delta \gamma - \lambda_1 u w^2 = 0, \quad \delta \rho - \lambda_2 v w^2 = 0, \quad \lambda_3 (f + h) w^2 + \delta = 0. \quad (58)$$

Combining the first and seconde equation in Eq. 6eq1, we obtain

$$w^2 (\lambda_2 v \gamma - \lambda_1 u \rho) = 0. \quad (59)$$

Separating the variables u and v , then one can get

$$\frac{\lambda_1 u}{\gamma} = -\frac{\lambda_2 v}{\rho} = -\xi^2, \quad (60)$$

Since u and v are independent variables, each side of these equations must be constant i.e., ξ^2 is a constant. Hence, we conclude that

$$f = -\frac{\lambda_1 u^2}{2 \xi^2} + c_1, \quad h = -\frac{\lambda_2 v^2}{2 \xi^2} + c_2, \quad (61)$$

where c_1, c_2 are constants. Using Eq.6eq7 in the third equation of 6eq1 we get

$$\begin{aligned} & 2\xi^8 (-c_3 \lambda_3 \xi^2 + \lambda_1 + \lambda_2) - 2\lambda_1^3 \lambda_3 \xi^2 (c_3 \lambda_1 - \xi^2) u^4 + \lambda_1 \xi^4 (2\lambda_1 (\lambda_2 - \\ & 2c_3 \lambda_3 \xi^2) + \lambda_3 \xi^4) u^2 + 2\lambda_1 \lambda_2 \lambda_3 \xi^2 (\lambda_1 (\xi^2 - 2c_3 \lambda_2) + \lambda_2 \xi^2) u^2 v^2 - \\ & 2\lambda_2^3 \lambda_3 \xi^2 (c_3 \lambda_2 - \xi^2) v^4 + \lambda_2 \xi^4 (\lambda_3 \xi^2 (\xi^2 - 4c_3 \lambda_2) + 2\lambda_1 \lambda_2) v^2 + \\ & \lambda_1^5 \lambda_3 u^6 + \lambda_1^3 \lambda_2 \lambda_3 (\lambda_1 + 2\lambda_2) u^4 v^2 + \lambda_1 \lambda_2^3 \lambda_3 (2\lambda_1 + \lambda_2) u^2 v^4 + \lambda_2^5 \lambda_3 v^6 = 0. \end{aligned} \quad (62)$$

where $c_3 = c_1 + c_2$. In view of the independence of the set $\{u^p v^q\}$, all coefficients of different combinations of powers $u^p v^q$ in the previous polynomial should vanish. Then $\lambda_1 = \lambda_2 = \lambda_3 = 0$. That means, \mathbf{X} is a plane or minimal surface. There is no translation surface satisfying Garay's condition in E^3 except the plane and minimal surfaces.

3.2 Mean curvature flow of translation surfaces in E^3

Let \bar{M} be the deformed translation surface according to var17 where $\phi = H$. Then, it has the parametrization as

$$\bar{\mathbf{X}}(u, v) = \frac{1}{2w^2} (2u w^2 - t \gamma \delta, 2v w^2 - t \rho \delta, 2w^2 (f + h) + t \delta). \quad (63)$$

Then, one can get the coefficients of the first fundamental form as the following:

$$\begin{aligned} (\bar{g}_{ij}) &= \begin{pmatrix} \frac{(\gamma^2+1)w^2-t\delta\gamma'}{w^2} & \frac{\gamma\rho}{(\rho^2+1)\frac{w^2-t\delta\rho'}{w^2}} \\ \frac{\gamma\rho}{(\rho^2+1)\frac{w^2-t\delta\rho'}{w^2}} & \frac{(\rho^2+1)w^2-t\delta\rho'}{w^2} \end{pmatrix}, \quad \bar{g} = \frac{w^3-t\delta^2}{w^2}, \\ (\bar{g}^{ij}) &= \frac{1}{w^3-t\delta^2} \begin{pmatrix} (\rho^2+1)w^2-t\delta\rho' & -w^2\gamma\rho \\ -w^2\gamma\rho & (\gamma^2+1)w^2-t\delta\gamma' \end{pmatrix}, \end{aligned} \quad (64)$$

where $w^3 - t\delta^2 \neq 0$. Consequently, one can find the Laplacian $\bar{\Delta}$ of \bar{M} is given by

$$\begin{aligned} \bar{\Delta} &= \frac{1}{(w^3-t\delta^2)^2} \left(w \left(\gamma \gamma' ((w-\gamma^2)(2\delta^2 t + w^3) + t w \rho' (2w \rho' - 7\delta)) \right. \right. \\ &\quad \left. \left. + \rho' (\gamma w (w^3 - t\delta^2) - \gamma \rho^2 (2\delta^2 t + w^3) + t w^2 \gamma'' (w - \gamma^2)) \right) \right) \end{aligned}$$

$$\begin{aligned}
 & +t \delta w (\gamma \rho \rho'' (w - \rho^2) - \gamma'' (w - \gamma^2)^2) \frac{\partial}{\partial u} + 2 \gamma \rho w^2 (w^3 - \delta^2 t) \frac{\partial^2}{\partial u \partial v} \\
 & + w^2 ((\gamma^2 - w) (w^3 - t \delta^2) + t \delta w \rho') \frac{\partial^2}{\partial u^2} + w^2 ((w - \rho^2) (t \delta^2 - w^3) \\
 & + t \delta w \gamma') \frac{\partial^2}{\partial v^2} + w (\rho (\gamma' (-\delta^2 t (2\gamma^2 + w) - 7t \delta w \rho' \\
 & + w^3 (w - \gamma^2)) + \rho' (w - \rho^2) (2t \delta^2 + w^3) + 2t w^2 (\gamma')^2 \rho' \\
 & + t \gamma \delta w \gamma'' (w - \gamma^2)) + t w \rho'' (w - \rho^2) (w \gamma' + \delta (\rho^2 - w))) \frac{\partial}{\partial v}.
 \end{aligned} \tag{65}$$

Therefore

$$\begin{aligned}
 \overline{\Delta \overline{X}}_1 &= \frac{w}{2(w^3 - t \delta^2)^2} \left(\rho' (\gamma (-2\rho^2 (2t \delta^2 + w^3) + 2w (w^3 - t \delta^2) - 9t \rho w \rho'' \right. \\
 & (w - \rho^2)^2) + t w \gamma'' (\gamma^2 \rho^2 (5w - 9\gamma^2) + (\gamma^2 + 2) w (w - \gamma^2)) + t w (\delta \gamma'' (2\gamma^4 \\
 & - w^2 - \gamma^2 w) + \gamma (\gamma^{(3)} w (w - \gamma^2)^2 + 2\delta \rho \rho'' (\rho^2 - w) + \rho^{(3)} w (w - \rho^2)^2)) + \\
 & \gamma' (2\gamma (w - \gamma^2) (2t \delta^2 + w^3) + t (2\gamma w (\rho')^2 (w^2 - w (\gamma^2 + 10\rho^2 - 2) + 9\rho^2 (w \\
 & - 1)) + \gamma \delta \rho' (\rho^2 (17w - 56\gamma^2) + 8\gamma^2 w - 5w (w + 2)) + \gamma \rho w \rho'' (\rho^2 (w - 9\gamma^2) \\
 & - w (w - 5\gamma^2)) + w \gamma'' (2w - 9\gamma^2) (w - \gamma^2)^2) + t \gamma (\gamma')^2 (\delta (w - \gamma^2) (28\gamma^2 \\
 & - 13w) - 6w \rho' (3\gamma^2 - w)) - 4t \gamma \delta (\rho')^2 (w - 7\rho^2) (w - \rho^2) \Big), \\
 \overline{\Delta \overline{X}}_2 &= \frac{w}{2(w^3 - t \delta^2)^2} \left(t w (-\delta \rho'' (-2\rho^4 + w^2 + \rho^2 w) + 2\gamma \delta \rho \gamma'' (\gamma^2 - w) + \rho w \right. \\
 & (\gamma^{(3)} (w - \gamma^2)^2 + \rho^{(3)} (w - \rho^2)^2)) + 2t \rho (\gamma')^2 (w \rho' (w^2 - w (10\gamma^2 + \rho^2 - 2) \\
 & + 9\gamma^2 (w - 1)) - 2\delta (w - 7\gamma^2) (w - \gamma^2)) + t \delta \rho (\rho')^2 (-28\rho^4 - 13w^2 + 41\rho^2 w) \\
 & + \rho' ((w - \rho^2) (2\rho (2\delta^2 t + w^3) + t w \rho'' (2w - 9\rho^2) (w - \rho^2)) + t \gamma \rho w \gamma'' \\
 & (-9\gamma^2 \rho^2 - w^2 + w (\gamma^2 + 5\rho^2))) + \gamma' (-2\rho (\delta^2 t (2\gamma^2 + w) + w^3 (\gamma^2 - w)) \\
 & + t w \rho'' (2w^2 + \rho^4 (-9\gamma^2 + w)) + \rho^2 w (5\gamma^2 + w - 2)) + t \rho (\delta \rho' (8\rho^2 (w \\
 & - 7\gamma^2) + 17\gamma^2 w - 5w (w + 2)) - 9\gamma w \gamma'' (w - \gamma^2)^2 + 6w (\rho')^2 (w - 3\rho^2)) \Big), \\
 \overline{\Delta \overline{X}}_3 &= \frac{w}{2(w^3 - t \delta^2)^2} 2t (\gamma')^2 (w \rho' (9\gamma^4 + w^2 - 3\rho^2 (w - 3\gamma^2) - 10\gamma^2 w) - 2\delta (w \\
 & - 7\gamma^2) (w - \gamma^2)) + \gamma' (2(w^3 (w - \gamma^2) - \delta^2 t (2\gamma^2 + w)) + t (2\delta \rho' (-28\gamma^2 \rho^2 \\
 & + 2(w - 2) w^2 - (w - 1) (2w - 9) w) + 2w (\rho')^2 (w^2 - w (3\gamma^2 + 10\rho^2) + 9\rho^2 \\
 & (w - 1)) + \rho w \rho^2)) - 9\gamma w \gamma'' (w - \gamma^2)^2) + \rho' (t \gamma w \gamma'' (-9\gamma^2 \rho^2 - w^2 \\
 & + w (\gamma^2 + 5\rho^2)) + 2(-\rho^2 (2t \delta^2 + w^3) - \delta^2 t w + w^4) - 9t \rho w \rho'' (w - \rho^2)^2) \\
 & + t w (2\gamma \delta \gamma'' (\gamma^2 - w) + \gamma^{(3)} w (w - \gamma^2)^2 + 2\delta \rho \rho'' (\rho^2 - w) + \rho^{(3)} w (w \\
 & - \rho^2)^2) - 4t \delta (\rho')^2 (w - 7\rho^2) (w - \rho^2) \Big).
 \end{aligned} \tag{66}$$

Supposes $\overline{\mathbf{X}}$ satisfies Garay's condition, $\overline{\Delta \overline{\mathbf{X}}} = \overline{A \overline{\mathbf{X}}}$. By a manner similar to the pervious subsections we have three equation:

$$\frac{w}{2(w^3 - t \delta^2)^2} \left(\overline{\lambda}_1 w (4t \delta^2 u + t \gamma \delta w - 2u w^3) - t \delta \gamma'' (-2\gamma^4 + w^2 + \gamma^2 w) \right)$$

$$\begin{aligned}
 &+t \gamma \left(\gamma^{(3)} w (w - \gamma^2)^2 + 2\delta \rho \rho'' (\rho^- w) + \rho^{(3)} w (w - \rho^2)^2 \right) + \rho' \left(\gamma (-2\rho^2 (2t \delta^2 \right. \\
 &+ w^3) + 2w (w^3 - t \delta^2) - 9t \rho w \rho'' (w - \rho^2)^2) + t w \gamma'' (\gamma^2 \rho^2 (5w - 9\gamma^2) + (\gamma^2 + 2) \\
 &w (w - \gamma^2)) \left. \right) + t \gamma (\gamma')^2 (\delta (-28\gamma^4 - 13w^2 + 41\gamma^2 w) + 6w\rho' (3\gamma^2 - w) (\gamma^2 + \rho^2 - w)) \\
 &- 4\gamma \delta t (\rho')^2 (7\rho^4 + w^2 - 8\rho^2 w) + \gamma' \left(2\gamma (w - \gamma^2) (2t \delta^2 + w^3) + t \left(2\gamma w (\rho')^2 (9\rho^2 \right. \right. \\
 &(\gamma^2 + \rho^2) + w^2 - w (\gamma^2 + 10\rho^2 - 2)) + \gamma \delta \rho' (\rho^2 (17w - 56\gamma^2) + 8\gamma^2 w - 5w(w + 2)) \\
 &\left. \left. + \gamma \rho w \rho'' (\rho^2 (w - 9\gamma^2) - w (w - 5\gamma^2)) + w \gamma'' (2w - 9\gamma^2) (w - \gamma^2)^2 \right) \right) = 0,
 \end{aligned}$$

$$\begin{aligned}
 &\frac{w}{2(w^3 - t \delta^2)^2} \left(w \left(\lambda_2 w (4t \delta^2 v + t \delta \rho w - 2v w^3) + t (-\delta \rho'' (-2\rho^4 + w^2 + \rho^2 w) \right. \right. \\
 &+ 2\gamma \delta \rho \gamma'' (\gamma^2 - w) + \rho w (\gamma^{(3)} (w - \gamma^2)^2 + \rho^{(3)} (w - \rho^2)^2)) \left. \right) + 2t \rho (\gamma')^2 \left(w \rho' (9\gamma^2 \right. \\
 &(\gamma^2 + \rho^2) + w^2 - w (10\gamma^2 + \rho^2 - 2)) - 2\delta (7\gamma^4 + w^2 - 8\gamma^2 w) \left. \right) + t \delta \rho (\rho')^2 (-28\rho^4 \\
 &- 13w^2 + 41\rho^2 w) + \rho' \left((w - \rho^2) (2\rho (2\delta^2 t + w^3) + t w \rho'' (2w - 9\rho^2) (w - \rho^2)) \right. \\
 &\left. + t \gamma \rho w \gamma'' (-9\gamma^2 \rho^2 - w^2 + w (\gamma^2 + 5\rho^2)) \right) + \gamma' \left(-2\rho (\delta^2 t (2\gamma^2 + w) + w^3 \right. \\
 &(\gamma^2 - w)) + t w \rho'' (2w^2 - \rho^4 (9\gamma^2 + w)) + \rho^2 w (5\gamma^2 + w - 2) \left. \right) + t \rho \left(\delta \rho' (8\rho^2 \right. \\
 &(w - 7\gamma^2) + 17\gamma^2 w - 5w(w + 2)) + 6w (\rho')^2 (w - 3\rho^2) (-\gamma^2 - \rho^2 + w) - 9\gamma w \gamma'' \\
 &\left. \left. (w - \gamma^2)^2 \right) \right) = 0,
 \end{aligned}$$

$$\begin{aligned}
 &-\frac{w}{2(w^3 - t \delta^2)^2} \left(2\gamma' w^5 + 2\rho' w^5 - 4\gamma^2 \gamma' w^4 - 2\rho^2 \gamma' w^4 - 2\gamma^2 \rho' w^4 - 4\rho^2 \rho' w^4 \right. \\
 &+ t \gamma^{(3)} w^4 + 2t \gamma' (\rho')^2 w^3 + t \delta \bar{\lambda}_3 w^3 + 2\gamma^4 \gamma' w^3 + 2\gamma^2 \rho^2 \gamma' w^3 + 2\rho^4 \rho' w^3 \\
 &+ 2\gamma^2 \rho^2 \rho' w^3 + 2t (\gamma')^2 \rho' w^3 + 2t \gamma \delta \gamma'' w^3 - 9t \gamma \gamma' \gamma'' w^3 - t \gamma \rho' \gamma'' w^3 \\
 &+ 2t \delta \rho \rho'' w^3 - t \rho \gamma' \rho'' w^3 - 9t \rho \rho' \rho'' w^3 - 2t \gamma^2 \gamma^{(3)} w^3 - 4t \delta (\gamma')^2 w^2 \\
 &- 4t \delta (\rho')^2 w^2 - 6t \gamma^2 \gamma' (\rho')^2 w^2 - 20t \rho^2 \gamma' (\rho')^2 w^2 + 2(w^3 - 2t \delta^2) f \bar{\lambda}_3 w^2 \\
 &+ 2(w^3 - 2t \delta^2) h \bar{\lambda}_3 w^2 - 2t \delta^2 \gamma' w^2 - 2t \delta^2 \rho' w^2 - 20t \gamma^2 (\gamma')^2 \rho' w^2 \\
 &- 6t \rho^2 (\gamma')^2 \rho' w^2 + 4t \delta \rho^2 \gamma' \rho' w^2 + 4t \gamma^2 \delta \gamma' \rho' w^2 - 4t \delta \gamma' \rho' w^2 \\
 &- 2t \gamma \delta \rho^2 \gamma'' w^2 - 4t \gamma^3 \delta \gamma'' w^2 - 4t \gamma \delta \gamma'' w^2 + 18t \gamma^3 \gamma' \gamma'' w^2 + t \gamma^3 \rho' \gamma'' w^2 \\
 &+ 5t \gamma \rho^2 \rho' \gamma'' w^2 - 4t \delta \rho^3 \rho'' w^2 - 2t \gamma^2 \delta \rho \rho'' w^2 - 4t \delta \rho \rho'' w^2 + t \rho^3 \gamma' \rho'' w^2 \\
 &+ 5t \gamma^2 \rho \gamma' \rho'' w^2 + 18t \rho^3 \rho' \rho'' w + t \gamma^4 \gamma^{(3)} w^2 + t (w - \rho^2)^2 \rho^{(3)} w^2 + 32t \gamma^2 \delta (\gamma')^2 w \\
 &+ 32t \delta \rho^2 (\rho')^2 w + 18t \rho^4 \gamma' (\rho')^2 w + 18t \gamma^2 \rho^2 \gamma' (\rho')^2 w - 2t \gamma^2 \delta^2 \gamma' w \\
 &+ 2t \delta^2 \rho^2 \gamma' w + 2t \gamma^2 \delta^2 \rho' w - 2t \delta^2 \rho^2 \rho' w + 18t \gamma^4 (\gamma')^2 \rho' w + 18t \gamma^2 \rho^2 (\gamma')^2 \rho' w \\
 &- 4t \delta \rho^4 \gamma' \rho' w - 8t \gamma^2 \delta \rho^2 \gamma' \rho' w + 14t \delta \rho^2 \gamma' \rho' w - 4t \gamma^4 \delta \gamma' \rho' w \\
 &+ 14t \gamma^2 \delta \gamma' \rho' w + 2t \gamma^3 \delta \rho^2 \gamma'' w + 2t \gamma^5 \delta \gamma'' w + 4t \gamma^3 \delta \gamma'' w - 9t \gamma^5 \gamma' \gamma'' w \\
 &- 9t \gamma^3 \rho^2 \rho' \gamma'' w + 2t \delta \rho^5 \rho'' w + 2t \gamma^2 \delta \rho^3 \rho'' w + 4t \delta \rho^3 \rho'' w - 9t \gamma^2 \rho^3 \gamma' \rho'' w \\
 &- 9t \rho^5 \rho' \rho'' w - 28t \gamma^4 \delta (\gamma')^2 - 28t \delta \rho^4 (\rho')^2 + 4t \gamma^4 \delta^2 \gamma' + 4t \gamma^2 \delta^2 \rho^2 \gamma' \\
 &\left. + 4t \delta^2 \rho^4 \rho' + 4t \gamma^2 \delta^2 \rho^2 \rho' - 56t \gamma^2 \delta \rho^2 \gamma' \rho' \right) = 0. \tag{67}
 \end{aligned}$$

The translation surfaces satisfy Garay’s condition if and only if it satisfy equations system in Eq. 6eq6.

4 Conclusion

The effect of the deformed cylindrical and translation surfaces in deferent directions of finiteness property (Garay’s condition) is summarized as the following:

- (1) The deformed cylindrical surfaces in E^3 satisfy Garay's condition if and only if they are planes. Reality, the circular cylinder doesn't satisfy Garay's condition if $\bar{A} \neq 0$.
- (2) A plane is the only surface which satisfies Garay's condition after its deformation in the direction of the tangent plane of \mathbf{X} .
- (3) The deformed plane and circular cylinder are the only surfaces whose Gauss map satisfies Garay's condition in direction of the tangent plane of \mathbf{X} .
- (4) There are no translation surfaces satisfying Garay's condition in E^3 except the plane and minimal surfaces.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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