

## Implicit Iteration Process for Lipschitzian $\alpha$ -Hemicontraction Semigroups

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### Abstract:

In this paper, the concepts of  $\alpha$ -demicontractive semigroup and  $\alpha$ -hemicontractive semigroup are considered. We study the strong and weak convergence of an implicit iterative scheme to the common fixed points of Lipschitzian  $\alpha$ -hemicontractive semigroup. The result presented in this paper extend, generalize, improve and unify the results of several well known authors in the literature.

**keyword:** Banach space, Fixed point, Normalized duality mapping, Implicit iterative scheme, Strong convergence, weak convergence,  $\alpha$ -hemicontractive semigroup.

## 1 Introduction

Throughout this paper, we assume that  $E$  is a real Banach space with norm  $\|\cdot\|$ ,  $E^*$  the dual space of  $E$ ,  $\langle \cdot, \cdot \rangle$  the duality between  $E$  and  $E^*$  and  $C$  a nonempty closed convex subset of  $E$ .  $\mathfrak{R}^+$  denotes the set of nonnegative real numbers and  $\mathbb{N}$  the natural number set. The mapping  $J : E \rightarrow 2^{E^*}$  with

$$J(\psi) = \{f^* \in E^* : \langle \psi, f^* \rangle = \|\psi\|^2, \|f^*\| = \|\psi\|\}, \forall \psi \in E, \quad (1)$$

is called the normalized duality mapping. In the sequel, we shall use  $j$  to denote the single-valued duality mapping. Let  $S : C \rightarrow C$  be a nonlinear mapping.  $F(S)$  denotes the set of fixed point of  $S$ , i.e,  $F(S) = \{\psi \in C : S\psi = \psi\}$ . We use “ $\rightarrow$ ” to stand for strong convergence and “ $\rightharpoonup$ ” for weak convergence. For any given sequence  $\{\psi_n\} \subset C$ , let  $\omega_w(\psi_n)$  denotes the weak  $\omega$ -limit set. Firstly, we recall the following definitions:

**Definition 1.1.** A mapping  $S : C \rightarrow C$  is said to be

- *nonexpansive* if

$$\|S\psi - S\zeta\| \leq \|\psi - \zeta\|, \forall \psi, \zeta \in C; \quad (2)$$

- *strictly pseudocontractive* if there exists a constant  $\lambda \in (0, 1)$  and  $j(\psi - p) \in J(\psi - p)$  such that

$$\langle S\psi - S\zeta, j(\psi - \zeta) \rangle \leq \|\psi - \zeta\|^2 - \lambda\|(I - S)\psi - (I - S)\zeta\|^2; \quad (3)$$

$\forall \psi, \zeta \in C$ ;

- *pseudocontractive* if there exists  $j(\psi - \zeta) \in J(\psi - \zeta)$  such that

$$\langle S\psi - S\zeta, j(\psi - \zeta) \rangle \leq \|\psi - \zeta\|^2, \quad (4)$$

$\forall \psi, \zeta \in C$ ;



- *demiccontractive* if  $F(S) \neq \emptyset$ , there exists a constant  $\lambda \in (0, 1)$  and  $j(\psi - p) \in J(\psi - p)$  such that

$$\langle S\psi - p, j(\psi - p) \rangle \leq \|\psi - p\|^2 - \lambda\|(I - S)\psi\|^2, \tag{5}$$

$\forall \psi \in C$  and  $p \in F(S)$ ;

- *hemicontractive* if  $F(S) \neq \emptyset$  and there exists  $j(\psi - p) \in J(\psi - p)$  such that

$$\langle S\psi - p, j(\psi - p) \rangle \leq \|\psi - p\|^2, \tag{6}$$

$\forall \psi \in C$  and  $p \in F(S)$ ;

- $\alpha$ -*demiccontractive* if  $F(S) \neq \emptyset$ , there exists a constant  $\lambda \in (0, 1)$  and  $j(\psi - \alpha p) \in J(\psi - \alpha p)$  such that

$$\langle S\psi - \alpha p, j(\psi - \alpha p) \rangle \leq \|\psi - \alpha p\|^2 - \lambda\|(I - S)\psi\|^2, \tag{7}$$

for some  $\alpha \geq 1$ ,  $\forall \psi \in C$  and  $p \in F(S)$ . This class of mapping was introduced by Maruster and Maruster [6] in 2011. In [6], Maruster and Maruster proved that the class of  $\alpha$ -demiccontractive mapping is more general than the class of demicontractive mapping with an example, i.e., an example of an  $\alpha$ -demiccontractive mapping with  $\alpha > 1$  which is not a demicontractive mapping was given. Clearly, every demicontractive mapping is an  $\alpha$ -demiccontractive with  $\alpha = 1$ .

- $\alpha$ -*hemicontractive* if  $F(S) \neq \emptyset$ , there exists a constant  $\lambda \in (0, 1)$  and  $j(\psi - \alpha p) \in J(\psi - \alpha p)$  such that

$$\langle S\psi - \alpha p, j(\psi - \alpha p) \rangle \leq \|\psi - \alpha p\|^2, \tag{8}$$

for some  $\alpha \geq 1$ ,  $\forall \psi \in C$  and  $p \in F(S)$ . This class of mapping was introduced by Osilike and Onah [7] in 2015. In [7], Osilike and Onah [7] gave an example of an  $\alpha$ -hemicontractive mapping with  $\alpha > 1$  which is not demicontractive mapping,  $\alpha$ -demiccontractive mapping and hemicontractive mapping; which implies that the class of  $\alpha$ -hemicontractive mapping is more general than all of the classes of pseudocontractive mapping, demicontractive mapping,  $\alpha$ -hemicontractive mapping and hemicontractive mapping. Clearly, every hemicontractive mapping is an  $\alpha$ -hemicontractive with  $\alpha = 1$ , the converse case fails as shown in Osilike and Onah [7].

*Remark 1.2* (see [7]). Obviously, if  $S$  is an  $\alpha$ -hemicontractive mapping then  $\alpha p$  is a fixed point of  $S$ , i.e.,  $\alpha p \in F(S)$ ,  $\forall p \in F(S)$  and for some  $\alpha \geq 1$  such that  $\alpha p$  remains in the domain  $D(S)$  of  $S$ .

For more about the properties of “ $\alpha$ ”, the readers may consult Maruster and Maruster [6], Osilike and Onah [7].

**Definition 1.3.** A one parameter family  $\mathfrak{S} = \{S(s) : s \geq 0\}$  of self mappings of  $C$  is said to be *nonexpansive semigroup*; if the following conditions are satisfied:

- (i)  $S(s_1 + s_2)\psi = S(s_1)S(s_2)\psi$ , for any  $s_1, s_2 \in \mathbb{R}^+$  and  $\psi \in C$ ;
- (ii)  $S(0)\psi = \psi$ , for each  $\psi \in C$ ;
- (iii) for each  $\psi \in C$ ,  $s \mapsto S(s)\psi$  is continuous;
- (iv) for any  $s \geq 0$  and  $\psi, \zeta \in C$ ,

$$\|S(s)\psi - S(s)\zeta\| \leq \|\psi - \zeta\|. \tag{9}$$

If the family  $\mathfrak{S} = \{S(s) : s \geq 0\}$  satisfies conditions (i)–(iii), then it is said to be:

- (a) *Lipschitzian semigroup*, if there exists a bounded measurable function  $L : [0, \infty) \rightarrow [0, \infty)$  such that, for any  $\psi, \zeta \in C$  and  $s \geq 0$ ,

$$\|S(s)\psi - S(s)\zeta\| \leq L(s)\|\psi - \zeta\|; \quad (10)$$

- (b) *strictly pseudocontractive semigroup*, if there exists a bounded function  $\lambda : [0, \infty) \rightarrow [0, \infty)$  and for any given  $\psi, \zeta \in C$ , there exists  $j(\psi - \zeta) \in J(\psi - \zeta)$  such that

$$\langle S(s)\psi - S(s)\zeta, j(\psi - \zeta) \rangle \leq \|\psi - \zeta\|^2 - \lambda(s)\|(I - S(s))\psi - (I - S(s))\zeta\|^2, \quad (11)$$

for any  $s \geq 0$ ;

- (b) *pseudocontractive semigroup*, if for any given  $\psi, \zeta \in C$ , there exists  $j(\psi - \zeta) \in J(\psi - \zeta)$  such that

$$\langle S(s)\psi - S(s)\zeta, j(\psi - \zeta) \rangle \leq \|\psi - \zeta\|^2, \quad (12)$$

for any  $s \geq 0$ ;

- (c) *demiccontractive semigroup*, if  $\bigcap_{s \geq 0} F(S(s)) \neq \emptyset$  for all  $s \geq 0$ , there exists a bounded function  $\lambda : [0, \infty) \rightarrow [0, \infty)$  and for any given  $\psi \in C$  and  $p \in \bigcap_{s \geq 0} F(S(s))$ , there exists  $j(\psi - p) \in J(\psi - p)$  such that

$$\langle S(s)\psi - p, j(\psi - p) \rangle \leq \|\psi - p\|^2 - \lambda(s)\|(I - S(s))\psi\|^2; \quad (13)$$

- (c) *hemiccontractive semigroup*, if  $\bigcap_{s \geq 0} F(S(s)) \neq \emptyset$  for all  $s \geq 0$ , there exists a bounded function  $\lambda : [0, \infty) \rightarrow [0, \infty)$  and for any given  $\psi \in C$  and  $p \in \bigcap_{s \geq 0} F(S(s))$ , there exists  $j(\psi - p) \in J(\psi - p)$  such that

$$\langle S(s)\psi - p, j(\psi - p) \rangle \leq \|\psi - p\|^2; \quad (14)$$

In this paper, we introduce the following semigroups.

**Definition 1.4.** A one parameter family  $\mathfrak{S} = \{S(s) : s \geq 0\}$  of self mapping of  $C$  satisfying condition (i)–(iii) is said to be:

- (a)  $\alpha$ -*demiccontractive semigroup*, if  $\bigcap_{s \geq 0} F(S(s)) \neq \emptyset$  for all  $s \geq 0$ , there exists a bounded function  $\lambda : [0, \infty) \rightarrow [0, \infty)$ , for some  $\alpha \geq 1$ , for any  $\psi \in C$  and  $p \in \bigcap_{s \geq 0} F(S(s))$ , there exists  $j(\psi - \alpha p) \in J(\psi - \alpha p)$  such that

$$\langle S(s)\psi - \alpha p, j(\psi - \alpha p) \rangle \leq \|\psi - \alpha p\|^2 - \lambda(s)\|(I - S(s))\psi\|^2. \quad (15)$$

- (b)  $\alpha$ -*hemiccontractive semigroup* if  $\bigcap_{s \geq 0} F(S(s)) \neq \emptyset$  for all  $s \geq 0$ , there exists a bounded function  $\lambda : [0, \infty) \rightarrow [0, \infty)$ , for some  $\alpha \geq 1$ , for any  $\psi \in C$  and  $p \in \bigcap_{s \geq 0} F(S(s))$ , there exists  $j(\psi - \alpha p) \in J(\psi - \alpha p)$  such that

$$\langle S(s)\psi - \alpha p, j(\psi - \alpha p) \rangle \leq \|\psi - \alpha p\|^2 - \lambda(s)\|(I - S(s))\psi\|^2. \quad (16)$$

*Remark 1.5.* Clearly, the class of  $\alpha$ -hemiccontractive semigroup reduces to the class of hemiccontractive semigroup with  $\alpha = 1$ . Every pseudocontractive semigroup with a nonempty fixed point set is a hemiccontractive semigroup. The class of  $\alpha$ -demiccontractive semigroup is a proper subclass of the class of  $\alpha$ -hemiccontractive semigroup. The class of  $\alpha$ -demiccontractive semigroup reduces to the class of demiccontractive semigroup with  $\alpha = 1$ . Every strictly pseudocontractive semigroup with a nonempty fixed point set is a demiccontractive semigroup.

From the implications above, it follows that the class of  $\alpha$ -hemicontractive semigroup is the most general of all the classes of semigroup defined above.

On the other hand, the convergence problems of semigroup has been considered by many authors in the past three decades. Several implicit and explicit iterative schemes have been introduced and studied by many researchers in nonlinear analysis for approximating the common fixed points of nonexpansive semigroup, strictly pseudocontractive semigroup, pseudocontractive semigroup and demicontractive semigroup (see for example, [1, 2, 3, 4, 9, 10, 11, 14, 17, 19, 21, 22] and the references there in).

In 1998, Shioji and Takahashi [9] first introduced and rigorously studied a Halpern-type implicit iterative scheme for approximating the common fixed point of a family of asymptotically nonexpansive semigroup in a Hilbert space.

In [10], Suzuki introduced the following implicit iteration process for finding the common fixed point of nonexpansive semigroup in a Hilbert space.

$$\begin{cases} \psi_0 \in C, \\ \psi_n = m_n u + (1 - m_n)S(s_n)\psi_n, \end{cases} \quad \forall n \geq 1. \quad (17)$$

He proved the strong convergence of (17) to a common fixed point of nonexpansive semigroup by imposing some appropriate conditions on  $\{m_n\}$  and  $\{s_n\}$ .

In 2005, Xu [14] extended the result of Suzuki [10] from Hilbert space to the more general uniformly convex Banach space with a weakly continuous duality mapping.

In 2005, Aleyner and Reich [1] first introduced and studied the following explicit Halpern-type iteration process for approximating the common fixed point of a family  $\{S(s) : s \geq 0\}$  of nonexpansive semigroup in a reflexive Banach space with a uniformly Gâteaux differentiable norm:

$$\begin{cases} \psi_0 \in C, \\ \psi_{n+1} = m_n u + (1 - m_n)S(s_n)\psi_n, \end{cases} \quad \forall n \geq 1. \quad (18)$$

In 2007, Zhang et al. [19] introduced the following two steps iteration process for approximating the common fixed point of nonexpansive semigroup  $\{S(s) : s \geq 0\}$  in a reflexive Banach space with a uniformly Gâteaux differentiable norm, uniformly convex smooth space and uniformly convex Banach with a weakly continuous normalized duality mapping:

$$\begin{cases} \psi_0 \in C, \\ \psi_{n+1} = m_n u + (1 - m_n)\zeta_n, \\ \zeta_n = m'_n \psi_n + (1 - m'_n)S(s_n)\psi_n, \end{cases} \quad \forall n \geq 1, \quad (19)$$

where  $u$  is an arbitrary (but fixed) element in  $C$ ,  $\{m_n\}$ ,  $\{m'_n\}$  and  $\{s_n\}$  are sequences in  $(0, 1)$ ,  $[0, 1]$  and  $\mathfrak{R}^+$  respectively.

In [21]-[22], Zhang considered the following implicit iteration scheme and proved that it converges strongly the common fixed point of strictly pseudocontractive semigroups in reflexive Banach spaces:

$$\begin{cases} \psi_0 \in C, \\ \psi_n = m_n \psi_{n-1} + (1 - m_n)S(s_n)\psi_n, \end{cases} \quad \forall n \geq 1. \quad (20)$$

where  $\{m_n\}$  is a sequence in  $[0,1]$  and  $\{s_n\}$  is an increasing sequence in  $[0, \infty)$ . Many authors have studied (20) and proved its convergence to the common fixed points of nonexpansive semigroup, strictly pseudocontractive semigroup and pseudocontractive semigroup respectively (see for example Kim [5], Quan et al. [8], Thong [12]–[13] and the references there in).

Motivated by the above results, the aim of this article is to study and prove the convergence of the implicit iteration process (20) to the common fixed point of  $\alpha$ -hemiccontractive semigroup. Since the class of  $\alpha$ -hemiccontractive semigroup is more general than the classes of semigroups which has been considered by Zhang [21]–[22], Kim [5], Quan et al. [8], Thong [12]–[13], hence, our results extend, generalized and complement the corresponding results of Zhang [21]–[22], Kim [5], Quan et al. [8], Thong [12]–[13] and several others in the existing literature.

## 2 Preliminaries

The following definitions will be useful in proving our main results.

**Definition 2.1** (see [5]). A mapping  $S : C \rightarrow C$  is said to be *demiclosed* with respect to  $\zeta \in C$  if for any sequence  $\{\psi_n\}$  in  $C$ , it follows from  $(\psi_n \rightharpoonup \psi)$  and  $S\psi_n \rightarrow \zeta$  that  $\psi \in C$  and  $S(\psi) = \zeta$ . If  $I - S$  is *demiclosed* at zero, i.e., for any sequence  $\{\psi_n\}$  in  $C$ , the condition  $(\psi_n \rightharpoonup \psi)$  and  $\psi_n - S\psi_n \rightarrow 0$  imply  $\psi - S\psi = 0$ .

**Definition 2.2** (see [5]). A Banach space  $E$  is said to satisfy Opial’s condition if whenever  $\{\psi_n\}$  is a sequence in  $E$  which converges weakly to  $\psi$  as  $n \rightarrow \infty$ , then

$$\limsup_{n \rightarrow \infty} \|\psi_n - \psi\| < \limsup_{n \rightarrow \infty} \|\psi_n - \zeta\|, \quad \forall \zeta \in E, \zeta \neq \psi.$$

## 3 Main results

**Theorem 3.1.** Let  $E$  be a real Banach space and let  $C$  be a nonempty compact convex subset of  $E$ . Let  $\mathfrak{S} = \{S(s) : s \geq 0\}$  be a strongly continuous semigroup of Lipschitz  $\alpha$ -hemiccontractive mapping on  $C$  for some  $\alpha > 1$  such that  $F(\mathfrak{S}) = \bigcap_{s \geq 0} F(S(s)) \neq \emptyset$ . Let  $\{m_n\}, \{s_n\}$  be sequences of real numbers satisfying  $\{m_n\} \subset (0, b) \subset (0, 1), s_n > 0$  and  $\lim_{n \rightarrow \infty} m_n = 0$ . Then the sequence  $\{\psi_n\}$  defined (20) converges strongly to an element in  $F(\mathfrak{S})$ .

*Proof.* We divide the proof into four steps.

Step 1. Since the the common fixed point set  $F(\mathfrak{S})$  is nonempty, let  $p \in F(\mathfrak{S})$ . For each  $p \in F(\mathfrak{S})$ , we prove that  $\lim_{n \rightarrow \infty} \|\psi_n - \alpha p\|$  exists.

From (20), we have

$$\begin{aligned} \|\psi_n - \alpha p\|^2 &= \langle \psi_n - \alpha p, j(\psi_n - \alpha p) \rangle \\ &= \langle [m_n \psi_{n-1} + (1 - m_n)S(s_n)\psi_n] - \alpha p, j(\psi_n - \alpha p) \rangle \\ &= \langle m_n(\psi_{n-1} - \alpha p) + (1 - m_n)(S(s_n)\psi_n - \alpha p), j(\psi_n - \alpha p) \rangle \\ &= m_n \langle \psi_{n-1} - \alpha p, j(\psi_n - \alpha p) \rangle \\ &\quad + (1 - m_n) \langle (S(s_n)\psi_n - \alpha p), j(\psi_n - \alpha p) \rangle. \end{aligned} \tag{21}$$

Since  $\mathfrak{S} = \{S(s) : s \geq 0\}$  is an  $\alpha$ -hemicontractive semigroup, then from (21) we have

$$\begin{aligned} \|\psi_n - \alpha p\|^2 &\leq m_n \|\psi_{n-1} - \alpha p\| \|\psi_n - \alpha p\| + (1 - m_n) \|\psi_n - \alpha p\|^2 \\ &= m_n \|\psi_{n-1} - \alpha p\| \|\psi_n - \alpha p\| \\ &\quad + \|\psi_n - \alpha p\|^2 - m_n \|\psi_n - \alpha p\|^2. \end{aligned} \tag{22}$$

From simplifying (22), we obtain

$$\|\psi_n - \alpha p\|^2 \leq \|\psi_{n-1} - \alpha p\| \|\psi_n - \alpha p\|. \tag{23}$$

If  $\|\psi_n - \alpha p\| = 0$ , the result follows trivially. Let  $\|\psi_n - \alpha p\| > 0$ ; it follows from (23) that

$$\|\psi_n - \alpha p\| \leq \|\psi_{n-1} - \alpha p\|, \tag{24}$$

which implies that the sequence  $\{\|\psi_n - \alpha p\|\}$  is monotone and non-increasing. Hence  $\lim_{n \rightarrow \infty} \|\psi_n - \alpha p\|$  exists. It follows that  $\{\psi_n\}$  is bounded.

Step 2. We prove  $\lim_{n \rightarrow \infty} \|S(s_n)\psi_n - \psi_n\| = 0$ .

The sequence  $\{\|\psi_n - \alpha p\|_{n \in \mathbb{N}}\}$  is a bounded since  $\lim_{n \rightarrow \infty} \|\psi_n - \alpha p\|$  exists, so the sequence  $\{\psi_n\}$  is bounded. From (20), we obtain

$$\begin{aligned} \|S(s_n)\psi_n\| &= \left\| \frac{1}{1 - m_n} \psi_n - \frac{m_n}{1 - m_n} \psi_{n-1} \right\| \\ &\leq \frac{1}{1 - m_n} \|\psi_n\| + \frac{m_n}{1 - m_n} \|\psi_{n-1}\| \\ &\leq \frac{1}{1 - b} \|\psi_n\| + \frac{b}{1 - b} \|\psi_{n-1}\| \end{aligned}$$

This implies that  $\{S(s_n)\psi_n\}$  is bounded. Again, from (20) we have

$$\begin{aligned} \|\psi_n - S(s_n)\psi_n\| &= \|m_n \psi_{n-1} + (1 - m_n) S(s_n)\psi_n - S(s_n)\psi_n\| \\ &= \|m_n \psi_{n-1} + S(s_n)\psi_n - m_n S(s_n)\psi_n - S(s_n)\psi_n\| \\ &= \|m_n \psi_{n-1} - m_n S(s_n)\psi_n\| \\ &= m_n \|\psi_{n-1} - S(s_n)\psi_n\|, \end{aligned}$$

and from the condition  $\lim_{n \rightarrow \infty} m_n = 0$ , we have

$$\lim_{n \rightarrow \infty} \|S(s_n)\psi_n - \psi_n\| = 0. \tag{25}$$

Step 3. Now we prove that for all  $s > 0$ ,  $\lim_{n \rightarrow \infty} \|S(s)\psi_n - \psi_n\| = 0$ .

Since the  $\alpha$ -hemicontraction semigroup  $\mathfrak{S} = \{S(s) : s \geq 0\}$  is Lipschitz, for any  $k \in \mathbb{N}$ ,

$$\begin{aligned} \|S((k + 1)s_n)\psi_n - S(ks_n)\psi_n\| &= \|S(ks_n)S(s_n)\psi_n - S(ks_n)\psi_n\| \\ &\leq L(ks_n) \|S(s_n)\psi_n - \psi_n\| \\ &\leq M \|S(s_n)\psi_n - \psi_n\|, \end{aligned}$$

where  $M = \sup_{s > 0} L(s) < \infty$ . Since from (25),  $\lim_{n \rightarrow \infty} \|S(s_n)\psi_n - \psi_n\| = 0$ , so for any  $k \in \mathbb{N}$ , it follows that

$$\lim_{n \rightarrow \infty} \|S((k + 1)s_n)\psi_n - S(ks_n)\psi_n\| = 0. \tag{26}$$

Notice that

$$\begin{aligned} \left\| S(s)\psi_n - S\left(\left[\frac{s}{s_n}\right]s_n\right)\psi_n \right\| &= \left\| S\left(s - \left[\frac{s}{s_n}\right]s_n + \left[\frac{s}{s_n}\right]s_n\right)\psi_n - S\left(\left[\frac{s}{s_n}\right]s_n\right)\psi_n \right\| \\ &= \left\| S\left(\left[\frac{s}{s_n}\right]s_n\right)S\left(s - \left[\frac{s}{s_n}\right]s_n\right)\psi_n - S\left(\left[\frac{s}{s_n}\right]s_n\right)\psi_n \right\| \\ &\leq L\left(\left[\frac{s}{s_n}\right]s_n\right)\left\| S\left(s - \left[\frac{s}{s_n}\right]s_n\right)\psi_n - \psi_n \right\| \\ &\leq M\left\| S\left(s - \left[\frac{s}{s_n}\right]s_n\right)\psi_n - \psi_n \right\|. \end{aligned}$$

Since  $S(\cdot)$  is strongly continuous, we have

$$\lim_{n \rightarrow \infty} \left\| S\left(\left[\frac{s}{s_n}\right]s_n\right)\psi_n - S(s)\psi_n \right\| = 0. \tag{27}$$

Now for all  $s > 0$  and  $s > s_n$ , we have

$$\begin{aligned} \|\psi_n - S(s)\psi_n\| &\leq \sum_{k=0}^{\left[\frac{s}{s_n}\right]-1} \left\| S((k+1)s_n)\psi_n - S(ks_n)\psi_n + S\left(\left[\frac{s}{s_n}\right]s_n\right)\psi_n - S(s)\psi_n \right\| \\ &\leq \sum_{k=0}^{\left[\frac{s}{s_n}\right]-1} \|S((k+1)s_n)\psi_n - S(ks_n)\psi_n\| + \left\| S\left(\left[\frac{s}{s_n}\right]s_n\right)\psi_n - S(s)\psi_n \right\|. \end{aligned}$$

It follows from (26) and (27) that

$$\lim_{n \rightarrow \infty} \|\psi_n - S(s)\psi_n\| = 0. \tag{28}$$

Step 4. We prove  $\{\psi_n\}$  converges strongly to an element of  $F(\mathfrak{S})$ .

Since  $C$  is a compact convex subset of  $E$ , we know that there exists a subsequence  $\{\psi_{n_j}\} \subset \{\psi_n\}$ , such that  $\psi_{n_j} \rightarrow \alpha\psi \in C$ . So we have  $\lim_{j \rightarrow \infty} \|S(s)\psi_{n_j} - \psi_{n_j}\| = 0$  from  $\lim_{n \rightarrow \infty} \|S(s)\psi_n - \psi_n\| = 0$ , and

$$\|\alpha\psi - S(s)\alpha\psi\| = \lim_{j \rightarrow \infty} \|S(s)\psi_{n_j} - \psi_{n_j}\| = 0. \tag{29}$$

This implies that  $\alpha\psi \in F(\mathfrak{S})$ . Since for any  $\alpha p \in F(\mathfrak{S})$ ,  $\lim_{n \rightarrow \infty} \|\psi_n - \alpha p\|$  exists, and  $\lim_{n \rightarrow \infty} \|\psi_n - \alpha\psi\| = \lim_{j \rightarrow \infty} \|\psi_{n_j} - \alpha\psi\| = 0$ , we have that  $\{\psi_n\}$  converges strongly to an element in  $F(\mathfrak{S})$ . This complete the prove of theorem 3.1.  $\square$

**Theorem 3.2.** *Let  $E$  be a reflexive Banach space satisfying Opial condition and let  $C$  be a nonempty closed convex subset of  $E$ . Let  $\mathfrak{S} = \{S(s) : s \geq 0\}$  be a strongly continuous semigroup of Lipschitz  $\alpha$ -hemicontractive mapping on  $C$  for some  $\alpha > 1$  such that  $F(\mathfrak{S}) = \bigcap_{s \geq 0} F(S(s)) \neq \emptyset$ . Let  $\{m_n\}, \{s_n\}$  be sequences of real numbers satisfying  $\{m_n\} \subset (0, b] \subset (0, 1), s_n > 0$  and  $\lim_{n \rightarrow \infty} m_n = 0$ . Let  $(I - S(s))$  be demiclosed at 0. Then the sequence  $\{\psi_n\}$  defined (20) converges weakly to an element in  $F(\mathfrak{S})$ .*

*Proof.* From the same method of prove in Theorem 3.1, then we have that for each  $p \in F(\mathfrak{S})$ , the limit  $\lim_{n \rightarrow \infty} \|\psi_n - \alpha p\|$  exists and  $\{S(s_n)\psi_n\}$  is bounded, for all  $s > 0$ ,  $\lim_{n \rightarrow \infty} \|S(s)\psi_n - \psi_n\| = 0$ .

In order to show the weak convergence of the iteration process (20) to a common fixed point of  $\mathfrak{S} = \{S(s) : s \geq 0\}$ , we will prove that  $\{\psi_n\}$  has a unique sequential limit in  $F(\mathfrak{S})$ . For this, let  $\{\psi_{n_j}\}$  and  $\{\psi_{n_k}\}$  be two subsequences of  $\{\psi_n\}$  which converges weakly to  $u$  and  $v$  respectively. Since for all  $s > 0$ ,  $\lim_{n \rightarrow \infty} \|S(s)\psi_n - \psi_n\| = 0$  and  $(I - S(s))$  is

demiclosed at 0. Thus,  $u, v \in F(\mathfrak{S})$ .

Now, we show the uniqueness. Since  $u, v \in F(\mathfrak{S})$ , so  $\lim_{n \rightarrow \infty} \|\psi_n - \alpha u\|$  and  $\lim_{n \rightarrow \infty} \|\psi_n - \alpha v\|$  exists. Let  $u \neq v$ . Then, by Opial's condition, we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\psi_n - \alpha u\| &= \lim_{j \rightarrow \infty} \|\psi_{n_j} - \alpha u\| \\ &< \lim_{j \rightarrow \infty} \|\psi_{n_j} - \alpha v\| \\ &= \lim_{n \rightarrow \infty} \|\psi_n - \alpha v\| \\ &= \lim_{k \rightarrow \infty} \|\psi_{n_k} - \alpha v\| \\ &< \lim_{k \rightarrow \infty} \|\psi_{n_k} - \alpha u\| \\ &= \lim_{n \rightarrow \infty} \|\psi_n - \alpha u\|. \end{aligned}$$

This is a contradiction, so  $u = v$ . Hence,  $\{\psi_n\}$  converges weakly to an element in  $F(\mathfrak{S})$ .

This completes the proof of Theorem 3.2. □

Since the classes of pseudocontraction semigroup and  $\alpha$ -demictractive semigroup are proper subclasses of the classes of  $\alpha$ -hemicontractive semigroup, the following results are obtain immediately from Theorem 3.1 and 3.2.

**Corollary 3.3.** *If  $\mathfrak{S} = \{S(s) : s \geq 0\}$  is a strongly continuous semigroup of pseudocontraction mapping with nonempty common fixed point set, then under the hypothesis of Theorem 3.1, the sequence  $\{\psi_n\}_{n=1}^{\infty}$  iteratively defined by (20) converges strongly to an element in  $F(\mathfrak{S})$ .*

**Corollary 3.4.** *If  $\mathfrak{S} = \{S(s) : s \geq 0\}$  is a strongly continuous semigroup of pseudocontractive mappings with nonempty common fixed point set  $F(\mathfrak{S})$ , then under the hypothesis of Theorem 3.2, the sequence  $\{\psi_n\}_{n=1}^{\infty}$  iteratively defined by (20) converges weakly to an element in  $F(\mathfrak{S})$ . Hence, the Theorem of Kim [5] is a special case of Theorem 3.2.*

**Corollary 3.5.** *If  $\mathfrak{S} = \{S(s) : s \geq 0\}$  is a strongly continuous semigroup of  $\alpha$ -demictractive mappings, then under the hypothesis of Theorem 3.1, the sequence  $\{\psi_n\}_{n=1}^{\infty}$  iteratively defined by (20) converges strongly to an element in  $F(\mathfrak{S})$ .*

**Corollary 3.6.** *If  $\mathfrak{S} = \{S(s) : s \geq 0\}$  is a strongly continuous semigroup of  $\alpha$ -demictractive mappings, then under the hypothesis of Theorem 3.2, the sequence  $\{\psi_n\}_{n=1}^{\infty}$  iteratively defined by (20) converges weakly to an element in  $F(\mathfrak{S})$ .*

## 4 Conclusion

In this paper, we have seen that the class of  $\alpha$ -hemicontractive semigroup is a superclass of the classes of nonexpansive semigroup, strictly pseudocontraction semigroup, pseudocontractive semigroup, hemicontraction semigroup, demicontraction semigroup and  $\alpha$ -demicontraction semigroup and these classes of semigroups have been considered by Zhang [21]-[22], Kim [5], Quan et al. [8], Chang et al. [3] and Thong [12]-[13]. In proving our main results, there is no other condition imposed on  $s_n$  in the Theorems 3.1 and 3.2 except that in the definition of  $\alpha$ -hemicontraction semigroups. So our results improve, extend, generalize and unify the corresponding results of many authors such as Zhang [21]-[22], Kim [5], Quan et al. [8], Chang et al. [3] and Thong [12]-[13] and several other results in the existing literature.

## References

- [1] A. Aleyner and S. Reich, An explicit construction of sunny nonexpansive retractions in Banach spaces, *Fixed Point Theory Appl.*, 3(2005), 295–305.
- [2] S. S. Chang, C. K. Chan, H. W. Joseph Lee and L. Yang, A system of mixed equilibrium problems, fixed point problems of strictly pseudocontractive mappings and nonexpansive semigroups. *Appl Math Comput.* 216(1)(2010),51–60. doi:10.1016/j.amc.2009.12.060.
- [3] S. S. Chang, Y. J. Cho, H. W. Joseph Lee and C. K. Chan, Strong convergence theorems for Lipschitzian demicontraction semigroups in Banach spaces. *Fixed Point Theory Appl* (2011). doi:10.1155/2011/583423.
- [4] S. Li, L. H. Li and F. Su, General iteration methods for a one-parameter nonexpansive semigroups in Hilbert spaces. *Nonlinear Anal.* 70(9)(2009), 3065–3071. doi:10.1016/j.na.2008.04.007.
- [5] G. E. Kim, Approximating common fixed points of Lipschitzian pseudocontraction semigroups, *J. Fixed Point Theory Appl.*, DOI 10.1007/s11784-016-0299-7.
- [6] L. Maruster and Ş. Maruster, Strong convergence of the Mann iteration for  $\alpha$ -demicontractive mappings, *Mathematical and Computer Modelling* 54 (2011), 2486–2492.
- [7] M. O. Osilike and A. C. Onah, Strong convergence of the Ishikawa Iteration for Lipschitz Hemicontractive mappings, *Seria Mathematica Informatica*, 1 (2015) LIII.
- [8] J. Quan, S. Chang and M. Liu, Strong and weak convergence of an implicit iterative process for pseudocontractive semigroups in Banach space, *Fixed Point Theory and Applications* 2012, 2012:16 <http://www.fixedpointtheoryandapplications.com/content/2012>.
- [9] N. Shioji and W. Takahashi, Strong convergence theorems for asymptotically nonexpansive mappings in Hilbert spaces, *Nonlinear Anal.*, 34(1)(1998),87–99. doi:10.1016/S0362-546X(97)00682-2
- [10] T. Suzuki, On strong convergence to a common fixed point of nonexpansive semigroups in Hilbert spaces. *Proc. Am.Math Soc.* 131(7) (2003), 2133–2136. doi:10.1090/S0002-9939-02-06844-2.
- [11] T. Suzuki, Fixed point property for nonexpansive mappings versus that for nonexpansive semigroups. *Nonlinear Anal.*, 70(2009), 3358–3361. doi:10.1016/j.na.2008.05.003.
- [12] D. V. Thong, An implicit iteration process for nonexpansive semigroups, *Nonlinear Analysis*, 74 (2011) 6116–6120.
- [13] D. V. Thong, On Mann type implicit iteration process for strictly pseudocontraction semigroups, *Annals of the University of Craiova, Mathematics and Computer Science Series*, 38(3)(2011), 101–108.
- [14] H. K. Xu, Strong convergence theorem for contraction semigroups in Banach spaces. *Bull. Aust. Math. Soc.* 72(3)(2005), 371–379. doi:10.1017/S000497270003519X.
- [15] Xu, HK: Inequalities in Banach spaces with applications. *Nonlinear Anal.* 16(12)(1991), 1127–1138. doi:10.1016/0362-546X(91)90200-K
- [16] Yang and Zhao *Fixed Point Theory and Applications* 2012, 2012:24 <http://www.fixedpointtheoryandapplications.com/content/2012/1/24> Page 9 of 10.
- [17] H. K. Xu, Inequalities in Banach spaces with applications, *Nonlinear Anal.* 16(1991), 1127–1138.

- [18] S. S. Zhang, L. Yang, H. W. Joseph Lee, C. K. Chan, Strong convergence theorem for nonexpansive semigroups in Hilbert spaces. *Acta Math Sinica*. 52(2)(2009), 337–342.
- [19] S. S. Zhang, L. Yang and J. A. Liu, Strong convergence theorem for nonexpansive semigroups in Banach spaces. *Appl Math. Mech.* 28(10)(2007), 1287–1297. doi:10.1007/s10483-007-1002-x.
- [20] Zhang, SS: Convergence theorem of common fixed points for Lipschitzian pseudocontraction semigroups in Banach spaces. *Appl Math Mech.* 30(2009), 145–152.
- [21] S. S. Zhang, Convergence theorem of common fixed points for Lipschitzian pseudo-contraction semi-groups in Banach spaces, *Appl. Math. Mech. -Engl. Ed.* 30 (2009), 145–152.
- [22] S. S. Zhang, Weak convergence theorem for Lipschitzian pseudocontraction semigroups in Banach spaces, *Acta Mathematica Sinica, English Series* 26 (2010), 337–344.