Impulsive Control for Exponential Stability of Neural Networks with Time-varying Delay

Xinni Tu, Jianli Li

Department of Mathematics, Hunan Normal University Changsha, Hunan 410081, China,

Maidan Svobody, 4, office 6-32, Kharkiv 61022, Ukraine.

e-mail: ljianli@sina.com (J. Li) 2574314427@qq.com (X. Tu)

Abstract

In this paper we investigate the exponential stability of impul-sive control for neural networks with time-varying delay by using a Lyapunov-Krasovskii functional. One numerical example is given to demonstrate theeffectiveness of the obtained results.

Keywords: Impulsive control; Exponential stability; Neural networks; Lyapunov-Krasovskii functional.

Mathematics Subject Classification: 34A37; 34K10.

Introduction

Neural networks has been used in various fields, such as pattern recognition, signal processing and other fields [1-2]. In recent years, time delays has been occurred frequently. Therefore, some authors pay more attention to neural networks with time-varying delay [3-6] , such as

$$x'(t) = -Cx(t) + Af(x(t)) + Bf(x(t - d(t))), \tag{1}$$

where $x(.) = [x_1(.), x_2(.), ..., x_n(.)]^T \in R^n$ is the neuron state vector; $f(x(.)) = [f_1(x(.)),$

$vspace5mmf_2(x(.)), ..., f_n(x(.))]^T \in R^n$ denotes the neuron activation function; $C = diag(c_1, c_2, ..., c_n)$ is a diagonal matrix with $c_i > 0$; and A and B are the connection weight matrix and the delayed connection weight matrix. respectively, the time delay $d(t)$ is a time-varying differentiable function .

Impulsive control which reduces the control cost is an effective and ideal control technique [7-9]. Impulsive effect exists widely in many evolutionary processes. The paper investigates a delayed neural networks with impulses, which is neither purely continuous-time nor purely discrete-time ones.

In this paper, we consider the system (1) subjected to certain impulsive state displacements at fixed moments of time:

$$\begin{cases} x^{'}(t) = -Cx(t) + Af(x(t)) + Bf(x(t - d(t))), t \neq t_k, \\ x(t^+) = Rx(t), \qquad t = t_k, \end{cases} \tag{2}$$

where R is positive definite block-diagonal matrix. $x_j(t_k^-) = x_j(t_k)$, which mean $x_j(t)$ is left continuous at each t_k. The moments of impulsive satisfy $t_1 < t_2 < \cdots < t_k < t_{k+1} < ...$ and $\lim\limits_{k\to+\infty} t_k = \infty$.

The paper constructs a new Lyapunov-Krasovskii functional (LKF) firstly. Then, we can obtain the LKF which is non-increasing . And a stability condition is established by integral inequality. What's different from the exists is that we are going to consider the impulsive differential equations. Finally, a numerical example is given to demonstrate the effectiveness of the obtained results.

2. Preliminaries

In this paper, we assume that the time delay $d(t)$ is a time-varying differentiable function that satisfies

$$0 \leq d(t) \leq h, d'(t) \leq \mu, \qquad (3),$$

where h and μ are constants.
In addition, it is assumed that each neuron activation function $f_j(.)$ satisfies the following condition

$$0 \leq \frac{f_j(x_j)}{x_j} \leq L_j, \quad f_j(0) = 0, \forall x_j \neq 0, j = 1, 2, .., n. \tag{4}$$

Definition 1 The system (2) is said to be exponentially stable if there exist constants $k > 0$ and $M \geq 1$ such that

$$||x(t)|| \leq M\phi e^{-kt}, \tag{5}$$

where $\phi = \sup\limits_{-h \leq \theta \leq 0} ||x(\theta)||$, k is called the exponential convergence rate.

Lemma 1[3]*For any vector $a, b \in R^n$, the inequality*

$$2a^T b \leq a^T X a + b^T X^{-1} b \tag{6}$$

holds, in which X is any positive matrix (i. e, $X > 0$).

Lemma 2[3] *Suppose that (4) holds, then*

$$\int_v^u [f_j(s) - f_j(v)]ds \leq [u - v][f_j(u) - f_j(v)], \quad j = 1, 2, ..., n. \tag{7}$$

Lemma 3[4] *Assume that (4) holds, then we have*

$$\int_v^u [f_j(s) - f_j(v)]ds \geq \frac{1}{2L_j}[f_j(u) - f_j(v)]^2, \quad j = 1, 2, ..., n. \tag{8}$$

3. Main results

Theorem 1.*The system (2) with (4) and a time-varying delay satisfying condition (3) is exponential stable and have the exponential convergence rate k, if there exist $P = P^T > 0, M = M^T > 0, W = W^T > 0, N = N^T > 0, D = diag(d_1, d_2, ..., d_n) \geq 0$, such that the following matrix is feasible*

$$\Gamma_1 = \begin{bmatrix} \Theta_1 & & & \\ & \Theta_2 & & \\ & & \Theta_3 & \\ & & & \Theta_4 \end{bmatrix} < 0,$$

$$\Gamma_2 = \begin{bmatrix} \Theta_5 & \\ & \Theta_6 \end{bmatrix} < 0,$$

where

$\Theta_1 = 2kP - 2PC + 2PAL + P^2 + 4kLD + e^{2kh}M + hN,$

$\Theta_2 = -2DCL^{-1} + 2DA + D^2 + e^{2kh}W,$

$\Theta_3 = (\mu - 1)M, \Theta_4 = 2B^T B + (\mu - 1)W,$

$\Theta_5 = R^T PR - P + 2R^T LDR, \Theta_6 = -DL^{-1},$

$L = diag(L_1, L_2, ..., L_n).$

*Proof.*Construct the following Lyapunov-Krasovskii functional

$$V(x(t)) = V_1(x(t)) + V_2(x(t)) + V_3(x(t)),$$

$V_1(x(t)) = e^{2kt}x^T(t)Px(t) + 2\sum\limits_{j=1}^{n} d_j e^{2kt} \int_0^{x_j} f_j(s)ds,$

$V_2(x(t)) = e^{2kh}\int_{t-d(t)}^{t} e^{2ks}[x^T(s)Mx(s) + f^T(x(s))Wf(x(s))]ds,$

$V_3(x(t)) = \int_{-h}^{0}\int_{t+\theta}^{t} e^{2ks}x^T(s)Nx(s)dsd\theta,$

where $P = P^T > 0, M = M^T > 0, W = W^T > 0, N = N^T > 0, D = diag(d_1, d_2, ..., d_n) \geq 0$ are to be determined.

Firstly, we define the following vector

$$\eta_1(t) = [x^T(t) \;\; f^T(x(t)) \;\; x^T(t-d(t)) \;\; f^T(x(t-d(t)))],$$

$$\eta_2(t) = [x^T(t) \;\; f^T(x(t))].$$

For $t \neq t_k$, by using Lemma 1 and 2, calculating the derivative of $V_i(x(t))(i = 1, 2, 3)$ along the trajectories of the system (2) yields.

$$\begin{aligned}
V_1^{'}(x(t)) &= 2ke^{2kt}x^T(t)Px(t) + 2e^{2kt}x^T(t)Px^{'}(t) + 4\sum_{j=1}^{n} kd_je^{2kt}\int_0^{x_j} f_j(s)ds \\
&\quad +2\sum_{j=1}^{n} d_je^{2kt}f_j(x_j(t))x_j^{'}(t) \\
&\leq 2ke^{2kt}x^T(t)Px(t) + 2e^{2kt}x^T(t)Px^{'}(t) + 4ke^{2kt}f^T(x(t))Dx(t) \\
&\quad +2e^{2kt}f^T(x(t))Dx^{'}(t) \\
&\leq 2ke^{2kt}x^T(t)Px(t) - 2e^{2kt}x^T(t)PCx(t) + 2e^{2kt}x^T(t)PALx(t) \\
&\quad +e^{2kt}x^T(t)P^2x(t) + e^{2kt}f^T(x(t-d(t)))B^TBf(x(t-d(t))) \\
&\quad +4ke^{2kt}x^T(t)LDx(t) - 2e^{2kt}f^T(x(t)DCL^{-1}f(x(t)) \\
&\quad +2e^{2kt}f^T(x(t))DAf(x(t)) + e^{2kt}f^T(x(t))D^2f(x(t)) \\
&\quad +e^{2kt}f^T(x(t-d(t)))B^TBf(x(t-d(t))) \\
&= e^{2kt}[x^T(t)(2kP - 2PC + 2PAL + P^2 + 4kLD)x(t) \\
&\quad +f^T(x(t))(-2DCL^{-1} + 2DA + D^2)f(x(t)) \\
&\quad +f^T(x(t-d(t))(2B^TB)f(x(t-d(t)))].
\end{aligned}$$

$$\begin{aligned}
V_2^{'}(x(t)) &= e^{2kh}e^{2kt}(x^T(t)Mx(t) + f^T(x(t)Wf(x(t))) - e^{2kh}e^{2k(t-d(t))} \\
&\quad (1-d^{'}(t))[x^T(t-d(t))Mx(t-d(t))+f^T(x(t-d(t))Wf(x(t-d(t)))] \\
&\leq e^{2kt}[x^T(t)e^{2kh}Mx(t) + f^T(x(t))e^{2kh}Wf(x(t)) + x^T(t-d(t)) \\
&\quad (\mu-1)Mx(t-d(t)) + f^T(x(t-d(t))(\mu-1)Wf(x(t-d(t)))].
\end{aligned}$$

$$V_3^{'}(x(t)) = he^{2kt}x^T(t)Nx(t) - \int_{t-h}^{t} e^{2ks}x^T(s)Nx(s)ds \leq e^{2kt}x^T(t)hNx(t).$$

From the previous expression, we can get

$$\begin{aligned}
V^{'}(x(t)) &\leq e^{2kt}[x^T(t)(2kP - 2PC + 2PAL + P^2 + 4kLD + e^{2kh}M + hN)x(t) \\
&\quad +f^T(x(t))(-2DCL^{-1}+2DA+D^2+e^{2kh}W)f(x(t))+x^T(t-d(t))(\mu-1) \\
&\quad Mx(t-d(t)) + f^T(x(t-d(t))(2B^TB + (\mu-1)W)f(x(t-d(t)))] \\
&= e^{2kt}\eta_1(t)\Gamma_1\eta^T(t).
\end{aligned}$$

From $\Gamma_1 < 0$, we have $V^{'}(x(t)) < 0$.

When $t = t_k(k = 1, 2, ...)$, by using Lemmas 2 and 3 , we can obtain

$$\begin{aligned}
V_1(x(t_k^+)) - V_1(x(t_k)) &= e^{2kt_k^+}x^T(t_k^+)Px(t_k^+) + 2\sum_{j=1}^{n} d_je^{2kt_k^+}\int_0^{x_j^+} f_j(s)ds \\
&\quad -e^{2kt_k}x^T(t_k)Px(t_k) - 2\sum_{j=1}^{n} d_je^{2kt_k}\int_0^{x_j} f_j(s)ds
\end{aligned}$$

$$\begin{aligned} &\leq e^{2kt_k}x^T(t_k)(R^TPR-P)x(t_k)+2e^{2kt_k}f^T(x(t_k^+))Dx(t_k^+) \\ &\quad -e^{2kt_k}f^T(x(t_k))DL^{-1}f(x(t_k)) \\ &= e^{2kt_k}[x^T(t_k)(R^TPR-P+2R^TLDR)x(t_k)-f^T(x(t_k))DL^{-1}f(x(t_k))], \end{aligned}$$

$$\begin{aligned} V_2(x(t_k^+))-V_2(x(t_k)) &= e^{2kh}\int_{t_k^+-d(t_k^+)}^{t_k^+}e^{2ks}(x^T(s)Mx(s)+f^T(x(s))Wf(x(s)))ds \\ &\quad -e^{2kh}\int_{t_k-d(t_k)}^{t_k}e^{2ks}(x^T(s)Mx(s)+f^T(x(s))Wf(x(s)))ds \\ &= 0, \end{aligned}$$

$$\begin{aligned} V_3(x(t_k^+))-V_3(x(t_k)) &= \int_{-h}^{0}\int_{t_k^++\theta}^{t_k^+}e^{2ks}x^T(s)Nx(s)dsd\theta-\int_{-h}^{0}\int_{t_k+\theta}^{t_k}e^{2ks}x^T(s)Nx(s)dsd\theta \\ &= 0, \end{aligned}$$

then, we can obtain

$$\begin{aligned} V(x(t_k^+))-V(x(t_k)) &\leq e^{2kt_k}[x^T(t_k)(R^TPR-P+2R^TLDR)x(t_k)-f^T(x(t_k))DL^{-1}f(x(t_k))] \\ &= e^{2kt_k}\eta_2(t_k)\Gamma_2\eta_2^T(t_k), \end{aligned}$$

$\Gamma_2<0$ implies $V(x(t_k^+))\leq V(x(t_k))$.

It follows from $V^{'}(x(t))<0$ and $V(x(t_k^+))\leq V(x(t_k))$ that $V(x(t))\leq V(x(0))$ for any $t\geq 0$.

However, from Lemma 2 we have

$$\begin{aligned} V(x(0)) &= x^T(0)Px(0)+2\sum_{j=1}^{n}d_j\int_0^{x_j}f_j(s)ds+e^{2kh}\int_{-d(0)}^{0}e^{2ks}[x^T(t)Mx(t) \\ &\quad +f^T(x(t))Wf(x(t))]ds+\int_{-h}^{0}\int_{\theta}^{0}e^{2ks}x^T(t)Nx(t)dsd\theta \\ &\leq \lambda_{max}(P)||\phi||^2+2\sum_{j=1}^{n}d_jx_j(0)f_j(x_j(0))+e^{2kh}\lambda_{max}(M)\int_{-d(0)}^{0}x^T(s)x(s)ds \\ &\quad +e^{2kh}\lambda_{max}(W)\int_{-d(0)}^{0}f^T(x(s))f(x(s))ds+\lambda_{max}(N)\int_{-h}^{0}\int_{\theta}^{0}x^T(s)x(s)dsd\theta \\ &\leq \lambda_{max}(P)||\phi||^2+2\lambda_{max}(DL)||\phi||^2+he^{2kh}\lambda_{max}(M)||\phi||^2 \\ &\quad +he^{2kh}\lambda_{max}(W)\lambda_{max}(L^2)||\phi||^2+h^2\lambda_{max}(N)||\phi||^2 \\ &= \Lambda||\phi||^2, \end{aligned}$$

where

$$\Lambda=\lambda_{max}(P)+2\lambda_{max}(DL)+he^{2kh}\lambda_{max}(M)+he^{2kh}\lambda_{max}(W)\lambda_{max}(L^2)+h^2\lambda_{max}(N),$$

$$\phi=\sup_{-h\leq\theta\leq 0}||x(\theta)||.$$

On the other hand, we get

$$V(x(t))\geq e^{2kt}x^T(t)Px(t)\geq e^{2kt}\lambda_{min}(P)||x(t)||^2.$$

Therefore

$$||x(t)||\leq\sqrt{\frac{\Lambda}{\lambda_{min}(P)}}||\phi||e^{-kt},$$

which shows that system (2) is exponentially stable and has the exponential convergence rate k. The proof is completed.

4. Examples

Consider the following neural networks system

$$\begin{cases} x'(t) = -Cx(t) + Af(x(t)) + Bf(x(t-d(t))), t \neq t_k \\ x(t^+) = Rx(t), \qquad t = t_k, \end{cases}$$

where

$$A = \begin{bmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & -1 \end{bmatrix}, \qquad C = diag(\ 2,\ 3\ ),$$

$$B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}, \qquad L_1 = L_2 = 1,$$

$$R = \begin{bmatrix} \frac{1}{2} & \\ & \frac{1}{2} \end{bmatrix}.$$

.

Then we can construct a LKF. Let

$$M = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \qquad P = D = diag(\ 1,\ 1\ ),$$

$$N = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}, \qquad k = \frac{1}{8},$$

$$W = \begin{bmatrix} \frac{8}{3} & 1 \\ 1 & \frac{8}{3} \end{bmatrix}, \qquad \mu = \frac{1}{4}.$$

It follows from $\Gamma_1 < 0$ and $\Gamma_2 < 0$ that the all conditions of Theorem 1 are satisfied, so the neural networks system is exponential stability.

Conclusions

We investigate the impulsive neural networks with time-varying delay. By constructing a new Lyapunov-Krasovskii functional (LKF) , a exponential stability condition is established, which extend some previous results. One numerical example is given to demonstrate the effectiveness of the obtained results.

Conflicts of Interest

Authors declare that there is no conflict of interest.

Funding Statement

This work is supported by the NSFC(11571088, 11471109), the Zhejiang Provincial Natural Science Foundation of China (LY14A010024), and Scientific Research Fund of Hunan Provincial Education Department (14A098).

References

[1] L. Chua, L. Yang, Cellular neural network: theory, *IEEE Trans. Circuits Syst.*, **35**(1988), 1257-1272.

[2] L. Chua, T. Roska, Cellular Neural Networks and Visual Computing, Univ. Press, Cambridge,U.K., 2000.

[3] Y. He, M. W, and J .H. She, Delay-dependent exponential stability of delayed neural networks with time-varying delay, *IEEE Trans.Circuits Systems-II:Exp.Briefs,* **53**(2006)553-557.

[4] Q. Zhang, X.P. Wei, J. Xu, Delay-dependent exponential stability of cellular neural networks with time-varying delay, *Chaos Solitons and Fractals*, **23**(2005)1363-1369

[5] M. D. Ji, Y. He ,M. Wu, C. K. Zhang, New exponential stability criterion of delayed neural networks with time-varying delay, *Proceeding of the 33rd chinese control conference*, (2014)6119-6123

[6] M. D. Ji, Y. He, M. Wu, C.K. Zhang, Further results on exponential stability criterion of delayed neural networks with time-varying delay *Appl. Math. Comput* **256** (2015) 175-182.

[7] H. Zhao, L. X. Li, H. P. Peng, J. H. Xiao, Y. X. Yang, M. W. Zheng, Impulsive control for synchronization and parameters identification of uncertain multi-links complex network, *Nonlinear Dyn.*, **83** (2016)1437-1451.

[8] Y. Q. Guo, Y. Y. Wang, L .H. Xie, Robust stability of reset control systems with uncertain output matrix, *Automatica.* **48** (2012) 1879-1884.

[9] M. Bohner, I. Stamova, G. Stamov, Impulsive control function differential systems of fractional order: stability with respect to manifolds, *The Euorpean Physical Journal Special Topics*, **226** (2017) 3591-3607.