Increasing stability of the inverse source problem for one dimensional domain

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Abstract

In this paper, we are investigating the one dimensional inverse source problem for Helmholtz equation where the source function is compactly supported in our domain. We show that increasing stability possible using multi-frequency wave at the two endpoints. Our main result is to obtain a stability estimate consists of two parts: the data discrepancy and the high frequency tail.

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Introduction and statement of problem

We consider with the one dimensional Helmholtz equation in a one layered medium:

$$u(x,\omega)'' + k^2 u(x,\omega) = f, \quad x \in (-1,1), \tag{0.1}$$

where the wave field u is required to satisfy the outgoing wave conditions:

$$u'(-1,\omega) + iku(-1,\omega) = 0, \qquad u'(1,\omega) - iku(1,\omega) = 0 \tag{0.2}$$

Given $f \in L^2(-1,1)$, it is well-known that the problem (0.1)-(0.2) has a unique solution:

$$u(x,\omega) = \int_{-1}^{1} G(x-y) f(y) dy, \tag{0.3}$$

where $G(x)$ is the Green function given as follows

$$G(x) = \frac{ie^{ik|x|}}{2k}. \tag{0.4}$$

This work concerns the inverse source problem when the source function f is a complex function with a compact support contained in $(-1,1)$. In this paper, our goal is to recover the source function f using the boundary data $u(1,\omega)$ and $u(-1,\omega)$ with $\omega \in (0,k)$ where $K > 1$ is a positive constant.

Inverse source problem areas in many area of science. It has numerous applications in acoustical and biomedical/medical imaging, antenna synthesis, geophysics, and material science ([2, 3]). It has been known that the data of the inverse source problems for Helmholtz equations with single frequency can not guarantee the uniqueness ([13], Ch.4). On the other hand, various studies, for instance in [4], showed that the uniqueness can be regained by taking multifrequency boundary measurement in a non-empty frequency interval $(0, K)$ noticing the analyticity of wave-field on the frequency [13, 17]. On the other hand, various studies, for instance in [12], showed that the uniqueness can be regained by taking multi-frequency boundary measurement in a non-empty frequency interval (0,K) noticing the analyticity of wave-field on the frequency. Because of the wide applications, these problems have been attracted considerable attention. These kinds of problems have been extensively investigated by many researchers such as In the paper [1, 5, 6, 7, 8, 9, 10, 14, 15, 16, 18, 19] and [20]. We also have to mention that these types of problems and techniques can apply to systems. For an example, in [11], inverse source problems was considered for classical elasticity system.

In this paper, In this paper, we assume that the medium is homogeneous in the whole space. Here we try to establish a stability estimate to recover of the source functions for the inverse source problem for the one-dimensional Helmholtz equation. In this paper both functions $f \in H^2((-1,1))$ are assumed to be zero outside our domain and $suppf \subset (-1,1)$. The main result is the following theorem.

Theorem 0.1. *There exists a generic constant C depending on the domain $(-1,1)$ such that*

$$\| f \|^2_{(0)} (-1,1) \leq C\Big(\epsilon^2 + \frac{M^2}{K^{\frac{2}{3}} E^{\frac{1}{4}} + 1}\Big), \tag{0.5}$$

for all $u \in H^2((-1,1))$ solving (0.1) with $K > 1$. Here

$$\epsilon^2 = \int_0^K \omega^2 \big(|u(1,\omega)|^2 + |u(-1,\omega)|^2\big) d\omega,$$

$E = -ln\epsilon$ and $M = \max\big\{ \| f \|^2_{(1)} (-1,1), 1\big\}$ where $\| \, . \, \|_{(l)} (\Omega)$ is the standard Sobolev norm in $H^l(\Omega)$.

$$f_1 = \begin{cases} f & \text{if } x > 0, \\ 0 & \text{if } x < 0, \end{cases} \quad f_2 = \begin{cases} 0 & \text{if } x > 0, \\ f & \text{if } x < 0. \end{cases}$$

Remark 1.1: The estimate in (0.5) consists of two parts: the data discrepancy and the high frequency part. The first part is of the LIpschitz type. The second part is of logarithmic type. The second part decrease as K increases which makes the problem more stable. The estimate (0.5) also implies the uniqueness of the inverse source problem.

1 Proof of Theorem 1.1

1.1 Increasing Stability of Continuation to higher frequencies

Let

$$I(k) = I_1(k) + I_2(k)$$

where

$$I_1(k) = \int_0^k \omega^2 |u(-1,\omega)|^2 d\omega, \qquad I_2(k) = \int_0^k \omega^2 |u(1,\omega)|^2 d\omega, \tag{1.1}$$

using (0.3) and a simple calculation shows that

$$\omega u(1,\omega) = \int_0^1 \frac{i}{2} e^{i\omega(1-y)} f_1(y) dy, \qquad \omega u(-1,\omega) = \int_{-1}^0 \frac{i}{2} e^{i\omega(-1-y)} f_2(y) dy, \tag{1.2}$$

where $y \in (-1,1)$. Functions I_1 and I_2 are both analytic with respect to the wave number $k \in \mathbb{C}$ and play important roles in relating the inverse source problems of the Helmholtz equation and the Cauchy problems for the wave equations.

Lemma 1.1. *Let supp$f \in (-1,1)$ and $f \in H^1(-1,1)$. Then*

$$|I_1(k)| \le C\Big(|k| \parallel f \parallel^2_{(0)} (-1,1)\Big) e^{2|k_2|}, \tag{1.3}$$

$$|I_2(k)| \le C\Big(|k| \parallel f \parallel^2_{(0)} (-1,1)\Big) e^{2|k_2|}. \tag{1.4}$$

Proof. Since we have $k = k_1 + k_2 i$ is complex analytic on the set $\mathbb{S} \setminus [0,k]$, where $\mathbb{S}$ is the sector $S = \{k \in \mathbb{C} : |\arg k| < \frac{\pi}{4}\}$ with $k = k_1 + ik_2$. Since the integrands in (1.1) are analytic functions of k in $\mathbb{S}$, their integrals with respect to ω can be taken over any path in $\mathbb{S}$ joining points 0 and k in the complex plane. Using the change of variable $\omega = ks$, $s \in (0,1)$ in the line integral (0.3), the fact that $y \in (-1,1)$.

$$I_1(k) = \int_0^1 ks \Big| \int_0^1 \frac{1}{2} e^{i(ks)(1-y)} f_1(y) dy \Big|^2 ds, \tag{1.5}$$

and

$$I_2(k) = \int_0^1 ks \Big| \int_{-1}^0 \frac{1}{2} e^{i(ks)(-1-y)} f_2(y) dy \Big|^2 ds. \tag{1.6}$$

Noting

$$|e^{iks(-1-y)}| \le e^{2|k_2|}, \quad |e^{iks(1-y)}| \le e^{2|k_2|},$$

using the Schwartz inequality and integrating with respect to s, using the bound for $|k|$ in $\mathbb{S}$, we complete the proof of (1.3). Using the same technique, we can prove the (1.4).

□

Noticing that functions $I_1(k), I_2(k)$ are analytic functions of $k = k_1 + ik_2 \in \mathbb{S}$ and $|k_2| \le k_1$. The following steps are essential to link the unknown $I_1(k)$ and $I_2(k)$ for $k \in [K,\infty)$ to the known value ϵ in (0.1).
Obviously

$$|I_1(k)e^{-2k}| \le C\Big(|k_1| \parallel f \parallel^2_{(0)} (-1,1)\Big) e^{-2k_1} \le CM^2, \tag{1.7}$$

where $M = \max\big\{ \parallel f \parallel_{(0)}^2 (-1,1), 1\big\}$. With the similar argument bound (1.7) is true for $I_2(k)$. Observing that

$$|I_1(k)e^{-2k}| \leq \epsilon^2, \quad |I_2(k)e^{-2k}| \leq \epsilon^2 \ \ on \ [0,K].$$

Let $\mu(k)$ be the harmonic measure of the interval $[0,K]$ in $\mathbb{S}\backslash[0,K]$, then as known (for example see [13], p.67), from two previous inequalities and analyticity of the function $I_1(k)e^{-2k}$ and $I_2(k)e^{-2k}$ we conclude that

$$|I_1(k)e^{-2k}| \leq C\epsilon^{2\mu(k)}M^2, \tag{1.8}$$

when $K < k < +\infty$. Similarly it also yields for

$$|I_2(k)e^{-2k}| \leq C\epsilon^{2\mu(k)}M^2, \tag{1.9}$$

consequently

$$|I(k)e^{-2k}| \leq C\epsilon^{2\mu(k)}M^2. \tag{1.10}$$

To achieve a lower bound of the harmonic measure $\mu(k)$, we use the following technical lemma. The proof can be found in [7].

Lemma 1.2. *Let $\mu(k)$ be the harmonic measure of the interval $[0,K]$ in $\mathbb{S}\backslash[0,K]$, then*

$$\begin{cases} \frac{1}{2} \leq \mu(k), & if \quad 0 < k < 2^{\frac{1}{4}}K, \\ \frac{1}{\pi}\Big(\big(\frac{k}{K}\big)^4 - 1\Big)^{\frac{-1}{2}} \leq \mu(k), & if \quad 2^{\frac{1}{4}}K < k \ . \end{cases} \tag{1.11}$$

Lemma 1.3. *Let source function $f \in L^2(-1,1)$ with supp$f \subset (-1,1)$, then*

$$\parallel f \parallel_{(0)}^2 (-1,1) \leq C\int_0^\infty \omega^2\big(|u(-1,\omega)|^2 + |u(1,\omega)|^2\big)d\omega.$$

Proof. Using the result of [19] by applying the Green function (0.4) and letting $k_1 = k_2 = k$. □

Lemma 1.4. *Let source function $f \in L^2(-1,1)$, then*

$$\omega^2|u(-1,\omega)|^2 \leq C\Big|\int_{-1}^0 e^{2\omega y}f_2(y)dy\Big|^2$$

$$\omega^2|u(1,\omega)|^2 \leq C\Big|\int_0^1 e^{2\omega y}f_1(y)dy\Big|^2$$

Proof. It follows from (1.2) and $y \in (-1,1)$. □

1.2 Increasing stability for inverse source problem

To continue the estimate for reminders in (1.5) and (1.6) for (k,∞), we need the following lemma.

Lemma 1.5. *Let u be a solution to the forward problem (0.1) with $f_1 \in H^1(\Omega)$ with supp$f \subset (-1,1)$, then*

$$\int_k^\infty \omega^2|u(-1,\omega)|^2 d\omega + \int_k^\infty \omega^2|u(1,\omega)|^2 d\omega \leq Ck^{-1}\Big(\parallel f \parallel_{(1)}^2 (-1,1)\Big) \tag{1.12}$$

Proof. Using (1.2), we obtain

$$\int_k^\infty \omega^2 |u(-1,\omega)|^2 d\omega + \int_k^\infty \omega^2 |u(1,\omega)|^2 d\omega \tag{1.13}$$

$$\leq C\Big(\int_k^\infty \Big| \int_0^1 e^{i\omega y} f_1(y) dy \Big|^2 d\omega + \int_k^\infty \Big| \int_{-1}^0 e^{i\omega y} f_2(y) dy \Big|^2 d\omega \Big). \tag{1.14}$$

Using integration by parts and the fact that $suppf_1 \subset (0,1)$ and $suppf_2 \subset (0,1)$, we have

$$\int_0^1 e^{-i\omega y} f_1(y) dy = \frac{1}{i\omega} \int_0^1 e^{-i\omega y} (\partial_y f_1(y)) dy,$$

and

$$\int_{-1}^0 e^{-i\omega y} f_2(y) dy = \frac{1}{i\omega} \int_{-1}^0 e^{-i\omega y} (\partial_y f_2(y)) dy,$$

consequently for the first and second terms in (1.14) we obtain

$$\Big| \int_0^1 e^{i\omega y} f_1(y) dy \Big|^2 \leq \frac{C}{\omega^2} \parallel f_1 \parallel^2_{(1)} (0,1) \leq \frac{C}{\omega^2} \parallel f_1 \parallel^2_{(1)} (-1,1)$$

$$\leq \frac{C}{\omega^2} \parallel f \parallel^2_{(1)} (-1,1),$$

utilizing the same argument for the second term in (1.14) and integrating with respect to ω the proof is complete. □

Now, we are ready to proof Theorem 0.1.

Proof. We can assume that $\epsilon < 1$ and $3\pi E^{-\frac{1}{4}} < 1$, otherwise the bound (0.1) is obvious. Let

$$k = \begin{cases} K^{\frac{2}{3}} E^{\frac{1}{4}} & \text{if} \quad 2^{\frac{1}{4}} K^{\frac{1}{3}} < E^{\frac{1}{4}} \\ K & \text{if} \quad E^{\frac{1}{4}} \leq 2^{\frac{1}{4}} K^{\frac{1}{3}}, \end{cases} \tag{1.15}$$

if $E^{\frac{1}{4}} \leq 2^{\frac{1}{4}} K^{\frac{1}{3}}$, then $k = K$, using the (1.8) and (1.10), we can conclude

$$|I(k)| \leq 2\epsilon^2. \tag{1.16}$$

If $2^{\frac{1}{4}} K^{\frac{1}{3}} < E^{\frac{1}{4}}$, we can assume that $E^{-\frac{1}{4}} < \frac{1}{4\pi}$, otherwise $C < E$ and hence $K < C$ and the bound (0.5) is straightforward. From (1.15), Lemma 2.2, (1.8) and the equality $\epsilon = \frac{1}{e^E}$ we obtain

$$|I(k)| \leq CM^2 e^{4k} e^{\frac{-2E}{\pi}\left((\frac{k}{K})^4 - 1\right)^{\frac{-1}{2}}}$$

$$\leq CM^2 e^{-\frac{2}{\pi} K^{\frac{2}{3}} E^{\frac{1}{2}} (1 - \frac{5\pi}{2} E^{\frac{-1}{4}})},$$

using the trivial inequality $e^{-t} \leq \frac{6}{t^3}$ for $t > 0$ and our assumption at the beginning of the proof, we obtain

$$|I(k)| \leq CM^2 \frac{1}{K^2 E^{\frac{3}{2}} \left(1 - \frac{5\pi}{2} E^{-\frac{1}{4}}\right)^3}. \tag{1.17}$$

Due to the (1.5), (1.16), (1.17), and Lemma 2.5. we can conclude

$$\int_0^{+\infty} \omega^2 |u(-1,\omega)|^2 d\omega + \int_0^{+\infty} \omega^2 |u(1,\omega)|^2 d\omega \tag{1.18}$$

$$\leq I(k) + \int_k^{\infty} \omega^2 |u(-1,\omega)|^2 d\omega + \int_k^{\infty} \omega^2 |u(1,\omega)|^2 d\omega$$

$$\leq 2\epsilon^2 + \frac{CM^2}{K^2 E^{\frac{3}{2}}} + \frac{\| f \|^2_{(2)} (-1,1)}{K^{\frac{2}{3}} E^{\frac{1}{4}} + 1}\Big).$$

Using the inequalities in (1.18) and Lemma 2.3., we finally obtain

$$\| f \|^2_{(0)} (\Omega) \leq C\Big(\epsilon^2 + \frac{M^2}{K^2 E^{\frac{3}{2}}} + \frac{\| f \|^2_{(1)} (-1,1)}{K^{\frac{2}{3}} E^{\frac{1}{4}} + 1}\Big)$$

Due to the fact that $K^{\frac{2}{3}} E^{\frac{1}{4}} < K^2 E^{\frac{3}{2}}$ for $1 < K, 1 < E$, the proof is complete.

□

2 Conclusion

In this paper, we studied the inverse source problem with many frequencies in a one dimensional domain. The result showed that if K grows the estimate improves. It also showed that if we have date exists for all wave number $k \in (0, \infty)$, the estimate will be a Lipschitz estimate.

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