# Markov Moment Problem in Concrete Spaces Revisited

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**Abstract**

This review paper starts by recalling two main results on abstract Markov moment problem. A corresponding application involving concrete spaces of functions is proved in detail. In the end, using polynomial approximation on special unbounded closed subsets, some multidimensional Markov moment problem on such subsets are recalled, without repeating the proofs. Our approximation results solve the difficulty arising from the fact that there exist positive polynomials on $\mathbb{R}^n$, $n \geq 2$ which cannot be written as sums of squares of polynomials. However, the upper constraint of the solution is written in terms of products of quadratic forms. The solutions are operators having as codomain an order complete Banach lattice. The latter space might be a commutative algebra of self-adjoint operators. All solutions obtained in this paper are continuous, and thanks to the density of polynomials in the involved domain function spaces, their uniqueness follows too. Operator valued solutions for classical moment problem are pointed out.

**Keywords**: Markov moment problem; concrete spaces; polynomial approximation on unbounded subsets

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## 1. Introduction

We recall the classical formulation of the moment problem, under the terms of T. Stieltjes, given in 1894-1895 (see the basic book of N.I. Akhiezer [1] for details): find the repartition of the positive mass on the nonnegative semi-axis, if the moments of arbitrary orders $j$ ( $j = 0,1,2,\ldots$ ) are given. Precisely, in the Stieltjes moment problem, a sequence of real numbers $(y_j)_{j\geq 0}$ is given and one looks for a nondecreasing real function $\sigma(t)$ $(t \geq 0)$, which verifies the moment conditions

$$\int_0^\infty t^j d\sigma = y_j \qquad (j = 0,1,2,\ldots)$$

This is a one dimensional moment problem, on an unbounded interval. Namely, is an interpolation problem with the constraint on the positivity of the measure dσ. The existence, the uniqueness and the construction of the solution $\sigma$ are studied. It is a classical moment problem, since the values $y_j, j \in \mathbb{N}$ of the linear form defined by $d\sigma$ on basic polynomials are prescribed. Passing to an example of the multidimensional real classical moment problem, let denote

$$\varphi_j(t_1,\ldots,t_n) = t_1^{j_1} \cdots t_n^{j_n}, \quad j = (j_1,\ldots,j_n) \in \mathrm{N}^n, \ t = (t_1, \ldots, t_n) \in \mathbb{R}_+^n, n \in \mathbb{N}, n \geq 2 \qquad (1)$$

If a sequence $(y_j)_{j\in\mathbb{N}^n}$ is given, one studies the existence, uniqueness and construction of a linear positive form $F$ defined on a function spaces containing polynomials, such that the moment conditions

$$F(\varphi_j) = y_j, \qquad j \in \mathbb{N}^n \qquad (2)$$

be accomplished. Usually, the positive linear form $F$ can be represented by means of a positive regular Borel measure on $\mathbb{R}_+^n$. When un upper constraint on the solution $F$ is required too, we have a Markov moment problem. This requirement is formulated as $F$ being dominated by a convex functional, which might be a norm and its aim is to control the continuity and the norm of the solution. All these aspects motivate the study sketched in the next section, which is mainly devoted to the abstract moment problem. Clearly, the classical moment problem is an extension problem for linear functionals, from the subspace of polynomials to a function space which contains both polynomials as well as the continuous compactly supported real functions on $\mathbb{R}_+^n$. From solutions linear functionals, many authors considered solutions linear operators. Of course, in this case the moments $y_j, j \in \mathbb{N}^n$ are elements of an ordered vector space $Y$ (usually $Y$ is an order complete Banach lattice). The order completeness is necessary in order to apply Hahn-Banach type results for operators defined on polynomials and having $Y$ as codomain. Various aspects of the classical moment problem have been studied [1]-[9], [11]-[16], [18]-[23]. The paper [2] discusses connections of the moment problem with fixed point theory. On the other side, as it is well-known, for natural $n \geq 2$ there exist positive polynomials on $\mathbb{R}^n$ which cannot be expressed as a sum of squares of polynomials. Because the form of positive polynomials on an unbounded closed subset of $\mathbb{R}^n, n \geq 2$ is not known, the multidimensional classical moment problem on such a subset is much more difficult than that of the one-dimensional case. Many of the published papers deal with the moment problem for semi-algebraic compact subsets of $\mathbb{R}^n$. The form of positive polynomials on such compacts is known (cf. [3], [21], [22]). Markov moment problem was studied in [5], [8], [9], [11]-[16], [18], [19], [20] and many other papers. Connection of the moment problem with operator theory has been pointed out in [2], [8], [21], [22] and other articles/monographs. General results in functional analysis applied along this work, including extension of linear operators with two constraints, can be found in [1], [10], [11]. See [4], [6], [23] and the references there for the uniqueness of a solution . A construction of a solution is proposed in [8]. The main purpose of this paper is to find necessary and sufficient (or only sufficient) conditions for the existence of the solutions of Markov moment problems in concrete spaces. The uniqueness of the solution follows too, thanks to continuity of the solution, also using the density of polynomials in the domain-space. In some cases, solving such problems requires polynomial approximation of nonnegative compactly supported continuous functions, in $L_1$ spaces associated to moment determinate measures. Studying such problems is the second aim of this work. In [17], polynomial approximation on unbounded subsets of $\mathbb{R}^n$ is applied to characterize invariance of the unit ball of some $L_1$ spaces. The rest of the paper is organized as follows. Section 2 is devoted to recalling some methods used along this paper. In Section 3, the abstract moment problem and related remarks are recalled. Section 4 refers to concrete Markov moment problems and related polynomial approximation on unbounded subsets. The multidimensional Markov moment problem is pointed out. Section 5 concludes the paper.

## 2. Methods

The basic used methods in the sequel are

1) Extension of linear operators with two constraints.
2) Measure theory results.
3) Polynomial approximation on unbounded subsets.

## 3. Results and Discussion: on the abstract moment problem

**Theorem 3.1.** ([11])

*Let X be a preordered vector space with its positive cone* $X_+$, *Y an order complete vector lattice,* $T : X \to Y$ *a convex operator.* $\{x_j\}_{j \in J} \subset X, \{y_j\}_{j \in J} \subset Y$ *given families. The following assertions are equivalent*

(a) *there exists a linear positive operator* $F : X \to Y$ *such that*

$$F(x_j) = y_j \;\forall j \in J,\;\; F(x) \le T(x) \;\forall x \in X$$

(b) *for any finite subset* $J_0 \subset J$ *and any* $\{\lambda_j\}_{j \in J_0} \subset R$, *we have*

$$\sum_{j \in J_0} \lambda_j x_j \le x \Rightarrow \sum_{j \in J_0} \lambda_j y_j \le T(x)$$

A clearer sandwich-moment problem variant is the following one.

**Theorem 3.2.** ([11])

*Let* $X, Y, \{x_j\}_{j \in J}, \{y_j\}_{j \in J}$ *be as in Theorem 3.1 and* $F_1, F_2 \in L(X,Y)$ *two linear operators. The following statements are equivalent*

(a)*there exists a linear operator* $F \in L(X,Y)$ *such that*

$$F_1(x) \le F(x) \le F_2(x), \quad \forall x \in X_+, \quad F(x_j) = y_j, \quad \forall j \in J;$$

(b)*for any finite subset* $J_0 \subset J$ *and any* $\{\lambda_j\}_{j \in J_0} \subset R$, *we have*

$$\left( \sum_{j \in J_0} \lambda_j x_j = \varphi_2 - \varphi_1,\; \varphi_1, \varphi_2 \in X_+ \right) \Rightarrow \sum_{j \in J_0} \lambda_j y_j \le F_2(\varphi_2) - F_1(\varphi_1)$$

The next theorem of this section is an earlier extension result, called Lemma of the majorizing subspace, for positive linear operators on subspaces in ordered vector spaces $(X, X_+)$, for which the positive cone $X_+$ is generating $(X = X_+ - X_+)$. Recall that in such an ordered vector space X, a vector subspace S is called a majorizing subspace if for any $x \in X$, there exists $s \in S$ such that $x \le s$.

**Theorem 3.3.**

*Let* $X$ *be an ordered vector space whose positive cone is generating,* $S \subset X$ *a majorizing vector subspace,* $Y$ *an order complete vector lattice,* $F_0: S \to Y$ *a linear positive operator. Then* $F_0$ *has a linear positive extension* $F: X \to Y$ *at least.*

*Remark 3.1.* In the statements of theorems 3.1, 3.2 the basic implication is (b)⇒(a), while the converse is obvious. Similarly, theorem 3.3 gives a sufficient condition for the existence of a positive extension. For application of this last general result to the classical moment problem, one can take the space $X$ of all continuous functions on $\mathbb{R}_+^n$, whose modulus is dominated by a polynomial, $S$ the subspace of polynomials, $F_0: S \to Y, F_0\left(\sum_{j \in J_0} \alpha_j t^j\right) = \sum_{j \in J_0} \alpha_j y_j, t^j = t_1^{j_1} \cdots t_n^{j_n}, t = t = (t_1, \dots, t_n) \in \mathbb{R}_+^n, J_0 \subset \mathbb{N}^n, \alpha_j \in \mathbb{R}, j \in J_0$ where $J_0$ is a finite subset. If $F_0$ is a (linear) positive operator, then, according to theorem 3.3, it has a positive (linear) extension to the whole space $X$. Usually, the space $X$ is dense in a classical Banach lattice of integrable functions on a subset of $\mathbb{R}_+^n$. Note that in theorem 3.3 no upper bound for the extension $F$ is required. Obviously, $F$ verifies the interpolation moment conditions (2), thanks to the definition of $F_0$.

## 4. Main Text

### 4.1. Solving Markov moment problems in concrete spaces

The next result gives a sufficient, as well as a necessary condition for the existence of a solution for a Markov moment problem (see [8]).

**Theorem 4.1.** Let *$T$ be a measurable space, $\nu$ a positive $\sigma$ –finite measure on $T, X := L_{1,\nu}(T)$ endowed with the natural ordering and norm $\|\cdot\|_1$ . Let $\{x_j; j \in J\} \subset X, \{y_j; j \in J\} \subset \mathbb{R}$, where $J$ is an arbitrary set of indexes. Consider the following statements*

(a) *there exists $h \in L_{\infty,\nu}(T)$, such that $-1 \le h(t) \le 1$ a. e. in $T$ and $\int_T x_j(t)h(t)d\nu = y_j \;\; \forall j \in J$;*

(b) *for any finite subset $J_0 \subset J$ and any $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$, the following inequality holds*

$$\sum_{i,j\in J_0} \lambda_i\lambda_j\, y_i y_j \le \sum_{i,j\in J_0} \lambda_i\lambda_j \int_T x_i(t)d\nu \int_T x_j(t)d\nu$$

(c) *for any finite subset $J_0 \subset J$ and any $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$, the following relation holds*

$$\sum_{i,j\in J_0} \lambda_i\lambda_j\, y_i y_j \le \sum_{i,j\in J_0} |\lambda_i||\lambda_j| \int_T |x_i(t)|\, d\nu \int_T |x_j(t)|\, d\nu$$

*Then (b)⇒(a)⇒(c)*

**Proof.** (b)⇒(a). We apply Theorem 3.2 (b)⇒(a). Namely, if

$$\sum_{j=0}^{n} \lambda_j x_j = g_2 - g_1, g_1, g_2 \in X_+ = \{g \in X; g(t) \ge 0 \;\; a.e. in\, T\},$$

then

$$-g_1 \le \sum_{j=0}^{n} \lambda_j x_j \le g_2 \Rightarrow -\int_T g_1(t)d\nu \le \sum_{j=0}^{n} \lambda_j \int_T x_j(t)d\nu \le \int_T g_2(t)d\nu \Rightarrow$$

$$\left|\sum_{j=0}^{n} \lambda_j \int_T x_j(t)d\nu\right| \le \int_T g_2(t)d\nu + \int_T g_1(t)d\nu =$$

$$\int_T g_2(t)d\nu - \left(-\int_T g_1(t)d\nu\right) = F_2(g_2) - F_1(g_1), F_2(\varphi) := \int_T \varphi(t)d\nu\,, F_1 := -F_2$$

On the other side, condition from (b) can be written as

$$\left(\sum_{j=0}^{n} \lambda_j y_j\right)^2 \le \left(\sum_{j=0}^{n} \lambda_j \int_T x_j(t)d\nu\right)^2 \Leftrightarrow \left|\sum_{j=0}^{n} \lambda_j y_j\right| \le \left|\sum_{j=0}^{n} \lambda_j \int_T x_j(t)d\nu\right|$$

The above relations yield

$$\sum_{j=0}^{n} \lambda_j x_j = g_2 - g_1, g_1, g_2 \in X_+ \Rightarrow \sum_{j=0}^{n} \lambda_j y_j \le \left|\sum_{j=0}^{n} \lambda_j y_j\right| \le$$

$$\left|\sum_{j=0}^{n} \lambda_j \int_T x_j(t)dv\right| \leq F_2(g_2) - F_1(g_1), F_2(\varphi) := \int_T \varphi(t)dv, F_1 := -F_2$$

Thus the implication of point (b) in Theorem 2.2 holds true, and, the implication (b)⇒(a) of the latter theorem leads to the existence of a linear operator $F: X \to \mathbb{R}$ such that $F(x_j) = y_j, j \in J$,

$$-\int_T g(t)dv \leq F(g) \leq \int_T g(t)dv \Leftrightarrow |F(g)| \leq \int_T g(t)dv = \|g\|_1 \ \forall g \in X_+$$

For an arbitrary function $\varphi \in X$, it results

$$|F(\varphi)| \leq |F(\varphi^+)| + |F(\varphi^-)| \leq \int_T \varphi^+(t)dv + \int_T \varphi^-(t)dv = \int_T |\varphi(t)|dv = \|\varphi\|_1$$

Thus $\|F\| \leq 1$, so that $F$ is continuous (and linear) on $L_{1,\nu}(T)$. Hence, there exists $h \in L_{\infty,\nu}(T), \|h\|_\infty \leq 1$ (i. e. $-1 \leq h(t) \leq 1$ a. e. in $T$), and

$$y_j = F(x_j) = \int_T h(t)x_j(t)dv, j \in J$$

The proof of (b)⇒(a) is complete. The implication (a)⇒(c) is obvious. Indeed,

$$|y_j| = \left|\int_T h(t)x_j(t)dv\right| \leq \int_T |h(t)|\ |x_j(t)|dv \leq \int_T |x_j(t)|dv, j \in J$$

The conclusion follows. This completes the proof. □

### 4.2. Polynomial approximation on unbounded subsets and the multidimensional Markov moment problem

In this section we recall some known results on the subject, without giving their proofs. The interested Reader can find the details in the corresponding references. The idea of the proofs of Theorems 4.2.1, 4.2.3 4.2.4, 4.2.5 and of the Corollary 4.2.1 was to apply Theorem 3.3 and the approximation results from lemmas stated in the sequel.

**Lemma 4.2.1.**

*Let $\psi : [0,\infty) \to R_+$ be a continuous function, such that $\lim_{t\to\infty} \psi(t) \in R_+$ exists. Then there is a decreasing sequence $(h_l)_l$ in the linear hull of the functions*

$$\varphi_k(t) = \exp(-kt), \quad k \in \mathrm{N}, \quad t \geq 0,$$

*such that $h_l(t) > \psi(t)$, $t \geq 0$, $l \in \mathrm{N}$, $\lim h_l = \psi$ uniformly on $[0,\infty)$. There exists a sequence of polynomial functions $(\tilde{p}_l)_{l\in\mathbb{N}}, \tilde{p}_l \geq h_l > \psi, \lim \tilde{p}_l = \psi$, uniformly on compact subsets of $[0,\infty)$.*

**Lemma 4.2.2.**

*Let $\nu$ be a M-determinate positive regular measure on $[0,\infty)$, with finite moments of all natural orders. If $\psi$, $(\tilde{p}_l)_l$ are as in Lemma 4.2.1, then there exists a subsequence $(\tilde{p}_{lm})_m$, such that $\tilde{p}_{lm} \to \psi$ in $L_{1,\nu}([0,\infty))$ and uniformly on compact subsets. In particular, it follows that the positive cone $P_+$ of positive polynomials is dense in the positive cone $(L_{1,\nu}([0,\infty)))_+$ of $L_{1,\nu}([0,\infty))$.*

**Lemma 4.2.3.** ([9], [12], [14], [15], [16], [19], [20])

*Let $A \subset R^n$ be an unbounded closed subset, and $\nu$ an M-determinate positive regular Borel measure on A, with finite moments of all natural orders. Then for any $x \in (C_0(A))_+$, there exists a sequence $(p_m)_m$, $p_m \in P_+$, $p_m \geq x$, $p_m \to x$ in $L_{1,\nu}(A)$. In particular, we have*

$$\lim \int_A p_m(t)d\nu = \int_A x(t)d\nu,$$

*$P_+$ is dense in $\left(L_{1,\nu}(A)\right)_+$, and $P$ is dense in $L_{1,\nu}(A)$.*

**Lemma 4.2.4.** ([14], [19])

*Let $\nu = \nu_1 \times \nu_2 \times \cdots \times \nu_n$ be a product of n $M-$ determinate positive regular Borel measures on $\mathbb{R}_+ = [0,\infty)$, with finite moments of all natural orders. Then we can approximate any nonnegative continuous compactly supported function in $X := L_{1,\nu}(R_+^n)$ by means of sums of tensor products $p_1 \otimes p_2 \otimes \cdots \otimes p_n$, $p_j$ positive polynomial on the real nonnegative semi axis, in variable $t_j \in [0,\infty), j = 1, \dots, n$.*

Recall that a determinate (M−determinate) measure is uniquely determinate by its moments, or, equivalently, by its values on polynomials. The following statement holds for any closed unbounded subset $A \subset \mathbb{R}^n$, hence does not depend on the form of positive polynomials on $A$. We denote by $\varphi_j$, $\varphi_j(t) := t_1^{j_1} \cdots t_n^{j_n}$, $j = (j_1, \dots, j_n) \in \mathbb{N}^n, t = (t_1, \dots, t_n) \in A$.

**Theorem 4.2.1.**

*Let $A$ be a closed unbounded subset of $\mathbb{R}^n$, $Y$ an order complete Banach lattice, $(y_j)_{j\in\mathbb{N}^n}$ a given sequence in $Y$, $\nu$ a positive regular $M$ −determinate Borel measure on $A$, with finite moments of all orders. Let $F_2 \in B\left(L_{1,\nu}(A), Y\right)$ be a linear positive bounded operator from $L_{1,\nu}(A)$ to $Y$. The following statements are equivalent*

(a)*there exists a unique linear operator $F \in B\left(L_{1,\nu}(A), Y\right)$ such that $F(\varphi_j) = y_j, j \in \mathbb{N}^n$, $F$ isbetween 0 and $F_2$ on the positive cone of $L_{1,\nu}(A)$, and $\|F\| \leq \|F_2\|$;*

(b) *for any finite subset $J_0 \subset \mathbb{N}^n$, and any $\{a_j\}_{j\in J_0} \subset \mathbb{R}$, we have*

$$\sum_{j\in J_0} a_j \varphi_j \geq 0 \text{ on } A \Rightarrow 0 \leq \sum_{j\in J_0} a_j y_j \leq \sum_{j\in J_0} a_j F_2(\varphi_j)$$

We go on by recalling a result on the form of non-negative polynomials in a strip [7], which leads to a simple solution for the related Markov moment problem.

**Theorem 4.2.2.**

*Suppose that $p(t_1,t_2) \in \mathbb{R}[t_1,t_2]$ is non – negative on the strip $A = [0,1] \times \mathbb{R}$. Then $p(t_1,t_2)$ is expressible as*

$$p(t_1,t_2) = \sigma(t_1,t_2) + \tau(t_1,t_2)t_1(1-t_1),$$

*where $\sigma(t_1,t_2),\tau(t_1,t_2)$ are sums of squares in $\mathbb{R}[t_1,t_2]$.*

Let $A=[0,1]\times\mathbb{R}$, $\nu$ a positive $M-$ determinate regular Borel measure on $A$, with finite moments of all orders, $X:=L_{1,\nu}(A),\varphi_j(t_1,t_2):=t_1^{j_1}t_2^{j_2},j=(j_1,j_2)\in\mathbb{N}^2,(t_1,t_2)\in A$. Let $Y$ be on order complete Banach lattice, $(y_j)_{j\in\mathbb{N}^2}$ a sequence of given elements in $Y$.

**Theorem 4.2.3.**

*Let $F_2\in B_+(X,Y)$ be a linear bounded positive operator from $X$ to $Y$. The following statements are equivalent*

(a) *there exists a unique bounded linear operator $F:X\to Y$, such that*

$$F(\varphi_j)=y_j,\ \forall j\in\mathrm{N}^2,$$

*$F$ is between zero and $F_2$ on the positive cone of $X$, $\|F\|\le\|F_2\|$;*

(b) *for any finite subset $J_0\subset\mathbb{N}^2$, and any $\{\lambda_j;j\in J_0\}\subset\mathbb{R}$, we have*

$$0\le\sum_{i,j\in J_0}\lambda_i\lambda_j\,y_{i+j}\le\sum_{i,j\in J_0}\lambda_i\lambda_j\,F_2(\varphi_{i+j});$$

$$0\le\sum_{i,j\in J_0}\lambda_i\lambda_j\left(y_{i_1+j_1+1,i_2+j_2}-y_{i_1+j_1+2,i_2+j_2}\right)\le$$

$$\sum_{i,j\in J_0}\lambda_i\lambda_j\left(F_2(\varphi_{i_1+j_1+1,i_2+j_2}-\varphi_{i_1+j_1+2,i_2+j_2})\right),i=(i_1,i_2),j=(j_1,j_2)\in J_0$$

**Theorem 4.2.4.**

*Le t $X$ be as in Lemma 4.2.4, $(y_j)_{j\in\mathrm{N}^n}$ be a sequence in $Y$, where $Y$ is an order complete Banach lattice, $F_2\in B_+(X,Y)$. The following statements are equivalent*

(a) *there exists a unique (bounded) linear operator $F\in B(X,Y)$ such that, $F(\varphi_j)=y_j,j\in\mathbb{N}^n$,, $F$ is between zero and $F_2$ on the positive cone of X, $\|F\|\le\|F_2\|$;*

(b) for *any finite subset $J_0\subset\mathrm{N}^n$ and any $\{\lambda_j;j\in J_0\}\subset\mathrm{R}$, we have*

$$\sum_{j\in J_0}\lambda_j\varphi_j(t)\ge 0\ \forall t\in\mathrm{R}_+^n\Rightarrow\sum_{j\in J_0}\lambda_j y_j\in Y_+;$$

*for any finite subsets $J_k\subset\mathrm{N}$, $k=1,\dots,n$ and any $\{\lambda_{j_k}\}_{j_k\in J_k}\subset R$, $k=1,\dots,n$, the following relations hold*

$$\sum_{i_1,j_1\in J_1}\left(\cdots\left(\sum_{i_n,j_n\in J_n}\lambda_{i_1}\lambda_{j_1}\cdots\lambda_{i_n}\lambda_{j_n}y_{i_1+j_1+l_1,\dots,i_n+j_n+l_n}\right)\cdots\right)\le$$

$$\sum_{i_1,j_1\in J_1}\left(\cdots\left(\sum_{i_n,j_n\in J_n}\lambda_{i_1}\lambda_{j_1}\cdots\lambda_{i_n}\lambda_{j_n}F_2(\varphi_{i_1+j_1+l_1,\dots,i_n+j_n+l_n}\right)\cdots\right),\quad(l_1,\dots,l_n)\in\{0,1\}^n.$$

Let $\nu=\nu_1\times\nu_2\times\cdots\times\nu_n$, where $\nu_j,\ j=1,\ldots,n$ are positive Borel regular *M*-determinate measures on *R*, with finite moments of all natural orders. Let

$$\varphi_j(t_1,\ldots,t_n)=t_1^{j_1}\cdots t_n^{j_n},\quad j=(j_1,\ldots,j_n)\in\mathrm{N}^n,\quad (t_1,\ldots,t_n)\in R^n.$$

Obviously, a statement similar to that of Lemma 4.2.4 holds true when we replace $\mathbb{R}_+^n$ by $\mathbb{R}^n$. In the latter case, the polynomials $p_j, j = 1,\ldots,n$ are nonnegative on the whole real axis, so that they are sums of squares. Applying such an approximation result and Theorem 3.3, one obtains the following theorem.

**Theorem 4.2.5.**

Let $\nu$ be as above, $X=L_{1,\nu}(R^n)$, $Y$ an order complete Banach lattice, and $(y_j)_{j\in\mathrm{N}^n}$ a multi-indexed sequence in $Y$. *Let $F_2:X\to Y$ be a positive linear bounded operator. The following statements are equivalent*

(a) *there exists a unique bounded linear operator $F:X\to Y$, such that $F(\varphi_j)=y_j,\quad\forall j\in\mathrm{N}^n$, F is between zero and $F_2$ on the positive cone of X, $\|F\|\le\|F_2\|$;*

(b) *for any finite subset $J_0\subset\mathrm{N}^n$, and any $\{\lambda_j;\ j\in J_0\}\subset\mathrm{R}$, we have*

$$\sum_{j\in J_0}\lambda_j\varphi_j(t)\ge 0\ \forall t\in\mathrm{R}^n\Rightarrow\sum_{j\in J_0}\lambda_j y_j\in Y_+,$$

*for any finite subsets $J_k\subset\mathrm{N}$, $k=1,\ldots,n$ and any*

$$\{\lambda_{j_k}\}_{j_k\in J_k}\subset R,\quad k=1,\ldots,n,$$

*the following relations hold*

$$\sum_{i_1,j_1\in J_1}\left(\cdots\left(\sum_{i_n,j_n\in J_n}\lambda_{i_1}\lambda_{j_1}\cdots\lambda_{i_n}\lambda_{j_n}y_{i_1+j_1,\ldots,i_n+j_n}\right)\cdots\right)\le$$
$$\sum_{i_1,j_1\in J_1}\left(\cdots\left(\sum_{i_n,j_n\in J_n}\lambda_{i_1}\lambda_{j_1}\cdots\lambda_{i_n}\lambda_{j_n}F_2(\varphi_{i_1+j_1,\ldots,i_n+j_n}\right)\cdots\right).$$

In what follows we solve an operator valued one dimensional classical Markov moment problem.

Let $H$ be an arbitrary complex or real Hilbert space and $\mathcal{A}$ the real order vector space of all self-adjoint operators acting on $H$. The positive cone of $\mathcal{A}$ consists in all operators $U\in\mathcal{A}$, having the property: $< U(h),h >\ge 0\ \forall h\in H$. Let $A\in\mathcal{A}$. Define

$$Y_1 := \{V\in\mathcal{A}; VA = AV\}, Y = Y(A) := \{U\in Y_1; UV = VU\ \ \forall V\in Y_1\},\tag{3}$$

$$Y_+ = \{U\in Y; < U(h),h >\ge 0\,\forall h\in H\}$$

As it is well-known $Y(A)$ is an order complete Banach lattice. Let $X = C_{\mathbb{R}}(\sigma(A))$, where $\sigma(A)\subset[0,\infty)$ is the spectrum of the fixed positive self-adjoint operator $A$ acting on a complex (or real) Hilbert space $H$. Consider the space $Y(A)$ defined by (3).The following corollary of the above results holds, thanks to the form of non-negative polynomials on $[0,\infty)$ [1].

**Corollary 4.2.1.**

*Let $A, X, Y = Y(A)$ be as above, $(U_n)_{n\geq 0}$ be a sequence of operators in $Y$. The following statements are equivalent*

(a) *there exists a unique linear bounded operator $F: X \to Y$ such that the moment interpolation conditions $F(\varphi_n) = U_n, n \in \mathbb{N}$ are verified and $0 \leq F(\psi) \leq \psi(A), \forall \psi \in X_+, \|F\| \leq 1$;*

(b) *for any finite subset $J_0 \subset \mathbb{N}$ and any $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$, the following implication holds true*

$$\sum_{j\in J_0} \lambda_j t^j \geq 0, \forall t \in \sigma(A) \Rightarrow 0 \leq \sum_{j\in J_0} \lambda_j U_j \leq \sum_{j\in J_0} \lambda_j A^j;$$

(c) *for any finite subset $J_0 \subset \mathbb{N}$ and any $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$, the following relations hold*

$$0 \leq \sum_{i,j\in J_0} \lambda_i \lambda_j U_{i+j+k} \leq \sum_{i,j\in J_0} \lambda_i \lambda_j A^{i+j+k}, k \in \{0,1\}$$

## 5. Conclusions

We hope that the results stated along this review paper are in accordance with the aims claimed in the Abstract and in the Introduction. Only one result is proved (in Section 4), because of its importance in illustrating the first aim of thhe paper: application of constrained extension theorems for linear operators to the Markov moment problem in a space of integrable functions. The other results of subsection 4.2 recall the importance of polynomial approximation on unbounded subsets in solving Markov moment problems in terms of quadratic forms. This is the second aim of this work. Here the proofs have been omitted, since the Reader can find them in recently published articles. Both purposes are actual and illustrate the relationship between different areas in functional analysis, having as common target solving existence (and sometimes uniqueness) of the solutions of Markov moment problem. The latter problem has old roots and modern solutions, according to the References.

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