Markov moment problems and Mazur-Orlicz theorems in concrete spaces

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Abstract

One solves Markov moment and Mazur-Orlicz problems in concrete spaces of functions and respectively operators. This is the main purpose of this review paper. To do this, one uses earlier results as well as recent theorems on the subject. One characterizes the existence of a solution, or one gives sufficient conditions for it does exist. Sometimes the uniqueness of the solution of some moment problems follows too. Spaces of continuous, of integrable and respectively analytic functions are considered as domain space of the solution. Usually, an order complete Banach lattice of self-adjoint operators (the bicommutant) is the target-space. Results on the abstract Markov moment problem, the abstract version of Mazur-Orlicz theorem and appropriate knowledge in functional analysis are applied. Basic elements of measure theory and Cauchy inequalities are used as well.

Keywords: interpolation with two constraints; classical Markov moment problem; abstract Markov moment problem; Mazur-Orlicz theorem; concrete spaces

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1 Introduction

We start by recalling the classical moment problem. Let $A \subset \mathbb{R}^n$ $(n \in \mathbb{N}, n \geq 1)$ be a closed subset,

$$\varphi_j(t) = t^j := t_1^{j_1} \cdots t_n^{j_n}, t = (t_1, \dots, t_n) \in A, j = (j_1, \dots, j_n) \in \mathbb{N}^n \quad (1)$$

On the other hand, being given a sequence $(y_j)_{j \in \mathbb{N}^n}$ of real numbers, one considers the following problem: find necessary and sufficient (or only sufficient) conditions on the elements of the sequence $(y_j)_{j \in \mathbb{N}^n}$ for the existence of a positive Borel measure μ on A such that

$$\int_A \varphi_j \, d\mu = y_j, j \in \mathbb{N}^n \quad (2)$$

For each , $j \in \mathbb{N}^n$, the number y_j is called the moment of order j (with respect to the measure μ). We say that μ is a solution for the moment problem (2). If $n = 1$, then one says that we have a one-dimensional moment problem, while for $n \geq 2$ we have a multidimensional moment problem. Defining the linear form F_0 on the space of P of polynomials by

$$F_0\left(\sum_{j \in J_0} \alpha_j \varphi_j\right) := \sum_{j \in J_0} \alpha_j y_j \quad (3)$$

where $J_0 \subset \mathbb{N}^n$ is an arbitrary finite subset, the moment problem can be reformulated as: find necessary and sufficient conditions on the sequence of moments $y_j, j \in \mathbb{N}^n$, for the existence of a linear extension $F: X \to \mathbb{R}$ of F_0, where X is a function space containing both polynomials and continuous compactly supported real functions defined on A, such that F to be a positive form on X $(F(x) \geq 0, \forall x \in X_+ := \{x \in X; x(t) \geq 0 \ \forall t \in A\})$. If such a linear positive extension F of F_0 does exist, due to Riesz representation theorem, it can be represented by a positive Borel measure μ on A and obviously (2) holds true (cf. (3)). The connection with the analytic form of nonnegative polynomials on A is obvious. For the one-dimensional case, see firstly [1], while the multidimensional case has been treated in many other works, such as [1]-[19]. When an additional upper constraint on the solution F is imposed, we have a Markov moment problem [9]-[16] (see Section 2 for abstract versions). If the moment-numbers $y_j, j \in \mathbb{N}^n$ are replaced by moments-operators $U_j, j \in \mathbb{N}^n$, we have an operator-valued moment problem, etc. For the background related to this paper see [20]-[22]. For uniqueness of the solutions of some moment problems we refer to [17], [23], [24], [25]. Connections of the moment problem with some other fields (such as operator theory and polynomial approximation on unbounded subsets) are pointed out in [5], [15]-[19], The rest of the paper is organized as follows. In Section 2, main results on the abstract moment problem. and a statement related to Mazur-Orlicz theorem are recalled, in the ordered vector spaces setting, according to [10]. Applications to concrete spaces are stated as well. Section 3 contains the proofs (and implicitly the methods) used in justifying the results of section 2. Section 4 concludes the paper.

2 The results

The results of this section have been motivated in the Introduction. The order of the statements follows two criteria: chronology and generality of the corresponding results. The proofs will be omitted. They can be partially found by means of the references.

Theorem 2.1 (Haviland [8]). *Let $A \subset \mathbb{R}^n$ and $L: P \coloneqq \mathbb{R}[t = (t_1, \dots, t_n)] \to \mathbb{R}$ be a linear form. Then L is given by a positive Borel measure μ on A (i.e. $L(p) = \int_A p d\mu$ for all $p \in P$) if and only if $L(p) \geq 0$ for all nonnegative p on A $(p(t) \geq 0, \forall t \in A \Rightarrow L(p) \geq 0)$.*

The next result of this section is a well-known extension result, sometimes called Lemma of the majorizing subspace, for positive linear operators on subspaces in ordered vector spaces (X, X_+), for which the positive cone X_+ is generating ($X = X_+ - X_+$). Recall that in such an ordered vector space X, a vector subspace S is called a majorizing subspace if for any $x \in X$, there exists $s \in S$ such that $x \leq s$.

Theorem 2.2. *Let X be an ordered vector space whose positive cone is generating, $S \subset X$ a majorizing vector subspace, Y an order complete vector lattice, $F_0: S \to Y$ a linear positive operator. Then F_0 has a linear positive extension $F: X \to Y$ at least.*

Observe that if $A \subset \mathbb{R}^n$ is a closed subset, and $S = P$ is the space of polynomial functions, then one can define X as the vector (ordered) space of all real functions dominated in absolute value by a polynomial. Then P is a majorizing subspace of X, while X contains P and the subspace of all continuous compactly supported functions on A. Theorem 2.2 was proved or/and applied in [1], [2], [5], [15], [16] and in many other works. The next two statements refer to the abstract Markov moment problem.

Theorem 2.3 (see [10]). *Let $X, Y, T: X \to Y$ be as in Theorem 3.1.2, $\{x_j\}_{j \in J} \subset X$, $\{y_j\}_{j \in J} \subset Y$ given families. The following assertions are equivalent*

(a) *there exists a linear positive operator $F: X \to Y$ such that*

$$F(x_j) = y_j \ \forall j \in J, \quad F(x) \leq T(x), \ \forall x \in X;$$

(b) *for any finite subset $J_0 \subset J$ and any $\{\lambda_j\}_{j \in J_0} \subset R$, we have*

$$\sum_{j \in J_0} \lambda_j x_j \le x \Rightarrow \sum_{j \in J_0} \lambda_j y_j \le T(x)$$

A clearer sandwich-moment problem variant is the following one.

Theorem 2.4 (see [10]) *.Let X be an ordered vector space, Y an order complete vector lattice, $\{x_j\}_{j \in J} \subset X, \{y_j\}_{j \in J} \subset Y$ given families and $F_1, F_2 \in L(X,Y)$ two linear operators. The following statements are equivalent*

(a)*there is a linear operator $F \in L(X,Y)$ such that*

$$F_1(x) \le F(x) \le F_2(x) \forall x \in X_+, F(x_j) = y_j \ \forall j \in J;$$

(b)*for any finite subset $J_0 \subset J$ and any $\{\lambda_j\}_{j \in J_0} \subset R$, we have*

$$\left(\sum_{j \in J_0} \lambda_j x_j = \psi_2 - \psi_1, \psi_1, \psi_2 \in X_+ \right) \Rightarrow \sum_{j \in J_0} \lambda_j y_j \le F_2(\psi_2) - F_1(\psi_1).$$

Finally, we recall the statement of Mazur-Orlicz Theorem.

Theorem 2.5. (see [10]). *Let $\boldsymbol{X}$ be an ordered vector space, $\boldsymbol{Y}$ an order complete vector lattice, $\{\boldsymbol{x_j}\}_{j \in J}, \{\boldsymbol{y_j}\}_{j \in J}$ arbitrary families in $\boldsymbol{X}$, respectively in $\boldsymbol{Y}$ and $\boldsymbol{T}: \boldsymbol{X} \to \boldsymbol{Y}$ a sublinear operator. The following statements are equivalent*

(a) $\exists F \in L(X,Y)$ *such that* $F(x_j) \ge y_j, \forall j \in J, F(x) \ge 0, \forall x \in X_+,\ F(x) \le T(x), \forall x \in X;$

(b) *for any finite subset $J_0 \subset J$ and any $\{\lambda_j\}_{j \in J_0} \subset \mathbb{R}, \lambda_j \ge 0, \forall j \in J_0$, we have*

$$\sum_{j \in J_0} \lambda_j x_j \le x \in X \Rightarrow \sum_{j \in J_0} \lambda_j y_j \le T(x)$$

Comparing Theorem 2.5 with Theorem 2.3, observe that in the former a "half" of the moment interpolation conditions from the latter appears at point (a), and consequently, the implication at (b) must be accomplished only for nonnegative scalars λ_j in case of the former Theorem 2.5. Recall that an important space which might stand for Y in the above statements is given by the following construction [21]. Let H be a complex Hilbert space, $\mathcal{A}$ the real vector space of all self-adjoint operators acting on $H, A \in \mathcal{A}$. Let

$$Y_1 := \{U \in \mathcal{A}; UA = AU\}, Y = Y(A) := \{V \in Y_1; UV = VU, \forall U \in Y_1\}, \quad (4)$$

$$Y_+ := \{V \in Y; < V(h), h > \ge 0, \forall h \in H\}$$

The space Y is an order complete Banach lattice (endowed with the operatorial norm) and a commutative (real) Banach algebra, as discussed in [21].

For applications to concrete spaces, one first aim is to state a result on the existence of the solution of the Markov moment problem on a semi-algebraic compact K. Let

$$K=\{t\in\mathbb{R}^n;\ r_j(t)\geq 0,\ \ j=1,\ldots,m\},\ \ r_1,\ldots,r_m\in\mathbb{R}[t_1,\ldots,t_n],$$

and assume that K is compact. Such a compact subset is called a semi - algebraic compact. Let $\sum$ denote the set of all finite sums of elements p^2 and $p^2 r_{j_1,\ldots,}r_{j_q}$, where $p\in\mathbb{R}[t_1,\ldots,t_n]$ and $j_1,\ldots,j_q\in\{1,\ldots,m\}$. Note that $\sum$ is a convex cone in $\mathbb{R}[t_1,\ldots,t_n]$ and it is closed with respect to multiplication-operation. We recall the real variant of the main result on the moment problem of [4]. Using geometric form of Hahn-Banach principle, the explicit representation of the positive polynomials on K is deduced in [4] (see also [3]).

Theorem 2.6. ([4]) *Let $(y_j)_{j\in\mathbb{N}^n}$ be a multi-sequence of real number. The following statements are equivalent:*

(a) *there exists a unique positive linear form F on $X=C(K)$ such that the moment conditions*

$$F(\varphi_j)=y_j\ ,\ \ j\in\mathbf{N}^n$$

are accomplished (where φ_j are the basic polynomials given by (1));

(b) *for any finite subset $J_0\subset\mathbb{N}^n$, any $\{\lambda_j; j\in J_0\}\subset\mathbb{R}$, and any*

$$p_q(t)=r_{i_1}(t)\ldots r_{i_q}(t)=\sum_{k\in S}\alpha_k t^k,\{i_1,\ldots,i_q\}\subset\{1,\ldots,m\},$$

(*where $i_{s_1}\neq i_{s_2}$ for $s_1\neq s_2$, and $S\subset\mathbb{N}^n$ is a finite subset*), *we have*

$$0\leq\sum_{i,j\in J_0}\lambda_i\lambda_j y_{i+j},\ \ \ 0\leq\sum_{i,j\in J_0}\lambda_i\lambda_j\left(\sum_{k\in S}\alpha_k y_{i+j+k}\right).$$

Corollary 2.1. ([4]) *If a polynomial function p is positive on K, then* $p\in\sum$.

Using the form of positive polynomials on semi-algebraic compacts given by Corollary 2.1, we proved the following applications to the Markov moment problem for operators. Let Y be an order-complete Banach lattice. Let $F_2\in L_+(C(K),Y)$ be a linear positive operator (note that any such operator is also bounded). Finally, let $(y_j)_{j\in\mathbb{N}^n}$ be a sequence in Y.

Theorem 2.7. ([12]) *The following statements are equivalent*

(a) *there exists a unique linear bounded operator $F\in B(C(K),Y)$ such that*

$$F(\varphi_j)=y_j,\ \ \ j\in N^n,\ \ 0\leq F(\varphi)\leq F_2(\varphi),\ \ \ \forall\varphi\in(C(K))_+,\|F\|\leq\|F_2\|$$

(b) *for any finite subset $J_0\subset\mathbb{N}^n$, any $\{\lambda_j; j\in J_0\}\subset\mathbb{R}$, and any polynomial p_q written above, we have*

$$0 \le \sum_{i,j \in J_0} \lambda_i \lambda_j y_{i+j} \le \sum_{i,j \in J_0} \lambda_i \lambda_j F_2(\varphi_{i+j})$$

$$0 \le \sum_{i,j \in J_0} \lambda_i \lambda_j \left(\sum_{k \in S} \alpha_k y_{i+j+k} \right) \le \sum_{i,j \in J_0} \lambda_i \lambda_j \left(\sum_{k \in S} \alpha_k F_2\left(\varphi_{i+j+k}\right) \right).$$

Next, we consider the "*non-compact case*", which can be solved by solving "*countable many compact cases*". Let B be a Borel subset in R^n such that $B = \bigcup_{m \in Z} K_m$, where $K_m \subset R^n$ is a compact subset $\forall m \in Z$. Assume that a positive regular Borel measure ν on B is given, such that

$$\nu(K_{m_1} \cap K_{m_2}) = 0 \quad \text{if} \quad m_1 \ne m_2$$

Theorem 2.8. ([12]) *Let* $\{y_j; j \in N^n\} \subset R \setminus \{0\}$. *The following statements are equivalent*

(a) *there exists a Borel function h on B such that*

$$\int_S t^j h(t) d\nu = y_j, \quad j \in N^n, \quad 0 \le h(t) \le 1 \quad \nu\text{ - a.e. in } B;$$

(b) *for any* $m \in Z$, *there exists a Borel function* h_m *on* K_m *and* $\varepsilon_m \in R$ *such that*

$$\int_{K_m} t^j h_m(t) d\nu_m = \varepsilon_m y_j, \quad j \in N^n, \quad m \in Z, \quad \sum_{m \in Z} \varepsilon_m = 1,$$

where $\nu_m(B) := \nu(B)$ *for any Borel subset* $B \subset K_m$, $m \in Z$.

In the next theorem *X* will be the space of all absolutely convergent power series in the closed disk $\{|z| \le b\}$,

$$\varphi(z) = \sum_{j=0}^{\infty} \alpha_j z^j, \quad |z| \le b, \quad X_+ = \{\varphi \in X; \alpha_j \in R_+ \quad \forall j \in Z_+\}$$

Let *H* be a Hilbert space, A a self - adjoint operator acting on *H*, and $Y = Y(A)$ the order complete vector lattice and commutative Banach algebra of self-adjoint operators defined by (4). We denote $\varphi_j(z) = z^j$, $|z| \le b$, $j \in Z_+$.In the next theorems, the notion of a resolvent appears naturally during the proofs. This result gives a sufficient condition for the existence a unique solution.

Theorem 2.9. *Let* $b > 1$, $X, Y, \{\varphi_j\}_j$ *be as above. Let*

$$B \in Y, \quad \sigma(B) \subset (0, b), \quad \{B_j\}_{j \in Z_+} \subset Y, \quad \varepsilon > 0.$$

Assume that

$$0 \le B_j \le B^j + \varepsilon \cdot I, \quad \forall j \in Z_+.$$

Then there exists a unique linear bounded positive operator $F \in L_+(X,Y)$, *such that*

$$F(\varphi_j) = B_j \; \forall j \in Z_+, \quad |F(\varphi)| \le ||\varphi|| \left[(I - b^{-1}B)^{-1} + \varepsilon \frac{b}{b-1} I \right] \Rightarrow ||F|| \le \frac{b}{b - ||B||} + \varepsilon \frac{b}{b-1}$$

Our next goal is to consider some Markov moment problems in terms of nonnegative sequences with respect to an interval. We recall this well-known notion.

Definition 2.1. A sequence $(u_n)_{n=0}^{\infty}$ in an ordered vector space *Y* is called nonnegative with respect to the interval $I \subset R$ *if for any* $n \in Z_+$, we have

$$\lambda_0 + \lambda_1 t + \ldots + \lambda_n t^n \ge 0 \;\; \forall t \in I \Rightarrow \lambda_0 u_0 + \lambda_1 u_1 + \ldots + \lambda_n u_n \ge 0 \;\text{ in } Y$$

It is clear that for compact intervals and for $Y = R$, this condition is necessary and sufficient for the existence of a unique positive solution of the moment problem associated to the moments $y_j = u_j \in R$, $j \in Z_+$. The continuity with respect to the sup – norm is also obvious. But no information concerning dominating L^1 norm for the solution holds. Such information would be useful for integral representation of the extension of the solution to the space $L^1(I)$ (see the next results).

Theorem 2.10. *Let* $b \in (0, \infty)$. *Consider the following statements*:

(a) *there exists a unique* $h \in L^{\infty}([0,b])$ *such that*

$$0 \le h(t) \le 1 \;\text{ i.e., } \int_0^b t^j h(t)dt = y_j, \quad \forall j \in Z_+ \;;$$

(b) *the sequence*

$$(1, y_0, 2y_1, \ldots, ny_{n-1}, \ldots)$$

is nonnegative with respect to $[0,b]$.

Then (b) $\Rightarrow$ *(a).*

The next result is an application of Theorem 2.5 (Mazur-Orlicz) to the space *X* of all absolutely convergent power series in the disc $|z| < r$, with real coefficients, continuous up to the boundary. The order relation is given by the coefficients: we write

$$\sum_{n \in N} \lambda_n z^n \prec \sum_{n \in N} \gamma_n z^n \Leftrightarrow (\lambda_n \le \gamma_n, \quad \forall n \in N)$$

Denote $\varphi_n(z) = z^n$, $n \in N$, $|z| \le r$. Let *Y* be the space defined by (4), $(B_n)_{n \in N}$ a sequence in *Y*, and $U \in Y_+$.

Theorem 2.11. *The following statements are equivalent*

(a) *there exists a linear positive operator* $F \in L_+(X,Y)$ *such that*

$$F(\varphi_n) \ge B_n, \quad n \in \mathrm{N}, \quad |F(\varphi)| \le \sum_{n\in\mathrm{N}} |a_n| U^n, \quad \forall \varphi = \sum_{n\in\mathrm{N}} a_n\varphi_n \in X;$$

(b) *the following relations hold*

$$B_n \le U^n, \quad n \in \mathrm{N};$$

Theorem 2.12. *Let* $A \in Y_+$, $||A|| < b$, $b > 1$, $\varepsilon > 0$. *Consider the following statements*:

(a) *there exists a linear positive operator* $F \in L_+(X,Y)$ *such that*

$$F(\varphi_j) \ge U_j, \quad j \in \mathrm{N}, \quad F(\psi) \le b(bI - A)^{-1} ||\psi||_\infty + \varepsilon\psi(I), \quad \psi \in X$$

(b) *the following inequalities holds*:

$$U_j \le A^j + \varepsilon \cdot I, \quad j \in \mathrm{N};$$

(c) *there exists a linear positive operator* $F \in L_+(X,Y)$ *such that*

$$F(\varphi_j) \ge U_j, \quad j \in \mathrm{N}, \quad F(\psi) \le \sum_{n\in\mathrm{N}} |\gamma_n| \cdot A^n + \varepsilon|\psi|(I), \quad \psi = \sum_{n\in\mathrm{N}} \gamma_n\varphi_n \in X.$$

Then (c) $\Leftrightarrow$ (b) $\Rightarrow$ (a).

In the sequel, we go further with Mazur-Orlicz theorem in spaces $L_\mu^p, 1 \le p < \infty$. The first preliminary result is an abstract version.

Theorem 2.13. *Let* X *be a Banach lattice,* Y *an order complete Banach lattice,* $\{\varphi_j\}_{j\in J} \subset X_+, \{y_j\}_{j\in J} \subset Y$, G *a linear positive bounded operator from* X *into* Y, α *a positive number. The following statements are equivalent*

(a) *there exists a linear positive bounded operator* $F \in B_+(X,Y)$, *such that*

$$F(\varphi_j) \ge y_j, \forall j \in J, F(x) \le \alpha G(|x|), \forall x \in X, \|F\| \le \alpha\|G\|;$$

(b) $y_j \le \alpha G(\varphi_j), \forall j \in J.$

Corollary 2.2. *Let* M *be a measure space,* μ *a positive measure on* $M, \mu(M) < \infty, X = L_\mu^p(M), 1 \le p < \infty, g \ge 0$ *an element of* $L_\mu^q(M)$, *where* $q \in (1,\infty]$ *is the conjugate of* p $(1/p + 1/q = 1)$, α *a positive number. Let* $\{\varphi_j\}_{j\in J}, \{y_j\}_{j\in J}$ *be as in Theorem 2.2, where* $Y = \mathbb{R}$. *The following statements are equivalent*

(a) *there exists* $h \in L_\mu^q(M)$, $0 \le h \le \alpha g$ *a.e.,* $\int_M h\varphi_j d\mu \ge y_j, \forall j \in J;$

(b)$y_j \le \alpha \int_M g\varphi_j d\mu, \forall j \in J.$

Corollary 2.3. *Let consider the measure space* $M = \mathbb{R}_+^n, n \in \{1,2\dots\}$, *endowed with the measure* $d\mu = exp\left(-\sum_{j=1}^n p_j t_j\right) dt_1 \cdots dt_n, p_j > 0, \forall j \in \{1,\dots,n\}$, α *a positive number. The following statements are equivalent*

(a) *there exists* $h \in L_\mu^\infty(\mathbb{R}_+^n), \int_{\mathbb{R}_+^n} ht^j d\mu \ge y_j, \forall j \in \mathbb{N}^n, 0 \le h \le \alpha$ *a.e.;*

(b) $y_j \leq \alpha \frac{j_1! \cdots j_n!}{p_1^{j_1+1} \cdots p_n^{j_n+1}}, \forall j = (j_1, \dots, j_n) \in \mathbb{N}^n$.

Theorem 2.14. Let $X = L_\mu^p(M), 1 < p, \mu \geq 0, \mu(M) < \infty, \{\varphi_j\}_{j\in J} \subset X, \{y_j\}_{j\in J} \subset \mathbb{R}, \alpha > 0, \alpha \in \mathbb{R}$, q the conjugate of p. *Consider the following statements*

(a) *there exists* $h \in \left(L_\mu^q(M)\right)_+$ *such that*

$$\int_M h\varphi_j d\mu \geq y_j, \forall j \in J, \int_M h\psi d\mu \leq \alpha \|\psi\|_p \left(\mu(M)\right)^{1/q}, \qquad \forall \psi \in X;$$

(b) *we have* $y_j \leq \alpha \int_M \varphi_j \, d\mu, \forall j \in J$.

Then (b)⇒(a).

The following theorem represents an application of the general result stated in Theorem 2.5 to some other concrete spaces X, Y. Let H be an arbitrary Hilbert space, $n \in \mathbb{N}$, $n \geq 1$, $A_1, \dots, A_n$ positive commuting self - adjoint operators acting on $H, (B_j)_{j\in\mathbb{N}^n}$ a sequence in Y, where $Y = Y(A_1, \dots, A_n)$ is defined by

$$Y_1 := \{U \in \mathcal{A}(H); UA_j = A_j U, j = 1, \dots, n\}, Y := \{V \in Y_1; UV = VU, \forall U \in Y_1\}, \quad (5)$$

$$Y_+ = \{V \in Y; < V(h), h > \geq 0, \forall h \in H\}$$

Here $\mathcal{A}(H)$ is the real vector space of all self – adjoint operators. One can prove that Y is an order complete Banach lattice with respect to the usual structures induced by those defined on the real space of self – adjoint operators (see [21], p. 303 - 305]), and a commutative real Banach algebra. Notice that the properties of $Y = Y(A_1, \dots, A_n)$, where $A_1, \dots, A_n$ are as mentioned above can be proved in a similar way to those of a $Y(A)$, where A is a self – adjoint operator. Actually, one repeats the proofs from [21], but for several commuting self – adjoint operators. Denote by $\varphi_j, j \in \mathbb{N}^n$ the basic polynomials $\varphi_j(t_1, \dots, t_n) = t_1^{j_1} \cdots t_n^{j_n}, j = (j_1, \dots, j_n) \in \mathbb{N}^n, t = (t_1, \dots, t_n) \in \Sigma_A$, $X := C(\Sigma_A)$, (where Σ_A is the joint spectrum associated to $A = (A_1, A_2, \dots, A_n)$) and by $E_A: Bor(\Sigma_A) \to \mathcal{A}(H)$ the corresponding joint spectral measure.

Theorem 2.15. *The following statements are equivalent*

(a) *there exists a linear bounded positive operator* $F \in B_+(X, Y)$ *such that*

$$F(\varphi_j) \geq B_j, j \in \mathbb{N}^n, F(\varphi) \leq \int_{\Sigma_A} |\varphi| \, dE_A, \forall \varphi \in X, \|F\| \leq 1;$$

(b) $B_j \leq A^j := A_1^{j_1} \cdots A_n^{j_n}, \forall j = (j_1, \dots, j_n) \in \mathbb{N}^n$.

Remark. If in Theorem 2.15 one additionally assumes that $\|A_k\| < 1, k = 1,2, \dots, n$, then for any self - adjoint operators satisfying (b) one has

$$\sum_{j\in\mathbb{N}^n} B_j \leq \prod_{k=1}^{n} (I - A_k)^{-1}.$$

Going back to the multidimensional Markov moment problem, here is a last application of Theorem 2.3. The space Y is defined by (4). The space X consists in all absolutely convergent power series in the closed unit closed polydisc

$$\bar{D}_1 = \left\{z = (z_1, \dots, z_n);\ |z_p| \le 1, p \in \{1, \dots, n\}\right\},$$

with real coefficients. The positive cone of X consists in all such power series having all nonnegative coefficients. The norm on X is defined by

$$\|h\|_{\infty} = sup\{|h(z)|; z \in \bar{D}_1\}, h \in X.$$

As we have mentioned above, one denotes by h_k the basic polynomials

$$h_k(z) = z_1^{k_1} \cdots z_n^{k_n}, k = (k_1, \dots, k_n) \in \mathbb{N}^n, z \in \bar{D}_1.$$

Let $A_1, \dots, A_n$ be positive operators in Y, such that $\|A_p\| < 1, \forall p \in \{1, \dots, n\}$, $a > 0$ a real number and $\{B_k\}_{k \in \mathbb{N}^n}$ a multi- indexed sequence in Y.

Theorem 2.16. *Assume that* $0 \le B_k \le a\, A_1^{k_1} \cdots A_n^{k_n}$ *for all* $k = (k_1, \dots, k_n) \in \mathbb{N}^n$. *Then there exists a unique linear bounded positive operator* $F \in B_+(X, Y)$ *such that*

$$F(h_k) = B_k, k \in \mathbb{N}^n, F(\varphi) \le a\|\varphi\|_{\infty} \prod_{p=1}^{n} (I - A_p)^{-1}, \varphi \in X, \|F\| \le a \prod_{p=1}^{n} (1 - \|A_p\|)^{-1}$$

3 Proofs and related methods

Proof of Theorem 2.9. The following implications hold true

$$\sum_{j \in J_0} \lambda_j \varphi_j \le \varphi = \sum_{j \in Z_+} a_j \varphi_j \Rightarrow \lambda_j \le a_j \le |a_j| \le \frac{||\varphi||_\infty}{b^j}, \quad j \in J_0 \Rightarrow$$

$$\sum_{j \in J_0} \lambda_j B_j \le \sum_{j \in J_{0,+}} \lambda_j (B^j + \varepsilon \cdot I) \le ||\varphi|| \cdot \left[\sum_{j \in Z_+} b^{-j} B^j + \varepsilon \cdot \left(\sum_{j \in Z_+} b^{-j}\right) \cdot I\right] =$$

$$= ||\varphi|| \cdot \left[(I - b^{-1}B)^{-1} + \varepsilon \cdot \frac{b}{b-1} I\right] =: T(\varphi), \quad \varphi \in X.$$

Here $J_0 \subset \mathbb{Z}_+$ is a finite subset and $\sigma(B)$ is the spectrum of B. In the preceding relations the Cauchy inequalities for the function φ have been used. Application of Theorem 2.3 leads to the existence of a linear positive operator F from X to Y, satisfying the moment condition, such that $F \le T$ on X. Now the conclusion follows easily, by using the monotony of the norm on Y:

$$||F|| \le ||(I - b^{-1}B)^{-1}|| + \varepsilon \cdot \frac{b}{b-1} \le 1 + \frac{||B||}{b} + \frac{||B||^2}{b^2} + \dots + \varepsilon \cdot \frac{b}{b-1} = \frac{b}{b - ||B||} + \varepsilon \cdot \frac{b}{b-1}.$$

This concludes the proof. □

Proof of Theorem 2.10. We have:

$$\sum_{j=0}^{n} \lambda_j \tau^j = \varphi_2(\tau) - \varphi_1(\tau), \ \ \varphi_1, \varphi_2 \in X_+, \quad X = L^1([0,b]) \Rightarrow$$

$$\lambda_0 t + \lambda_1 \frac{t^2}{2} + \mathrm{K} + \lambda_n \frac{t^{n+1}}{n+1} \le \int_0^b \varphi_2(\tau) d\tau = F_2(\varphi_2) - F_1(\varphi_1),$$

$$(F_1 \equiv 0), \ \ \forall t \in [0,b] \Rightarrow \int_0^b \varphi_2 d\tau - \lambda_0 t - \lambda_1 \frac{t^2}{2} - \mathrm{K} - \lambda_n \frac{t^{n+1}}{n+1} \ge 0 \quad \forall t \in [0,b] \Rightarrow$$

$$\int_0^b \varphi_2(\tau) d\tau - \lambda_0 y_0 - \lambda_1 \frac{2y_1}{2} - \mathrm{K} - \lambda_n \frac{(n+1) y_n}{n+1} \ge 0 \Rightarrow \sum_{j=0}^{n} \lambda_j y_j \le F_2(\varphi_2) - F_1(\varphi_1).$$

Application of Theorem 2.4, (b) $\Rightarrow$ (a), leads to the existence of a linear functional F on $X = L^1([0,b])$ such that

$$F(x_j) = y_j, \quad j \in Z_+, \quad 0 = F_1(\varphi) \le F(\varphi) \le F_2(\varphi) := \int_0^b \varphi dt, \quad \forall \varphi \in X_+ .$$

Using the representation of a linear positive functional on L^1, there exists

$$h \in L^\infty([0,b]), \quad 0 \le h, \quad F(\varphi) = \int_0^b \varphi \cdot h dt, \quad \forall \varphi \in L^1([0,b]) .$$

From the last equality written for $\varphi = \chi_B$, $B \subset [0,b]$ being a Borel subset, we infer (via measure theory) that

$$h \le 1 \ a.e., \quad \int_0^b t^j h(t) dt = F(x_j) = y_j, \quad \forall j \in Z_+ .$$

This concludes the proof. □

Proof of Theorem 2.11. (b) $\Rightarrow$ (a). One applies theorem 2.5 to $x_j = \varphi_j, \ \ j \in \mathrm{N}$. If

$$\sum_{j \in J_0} \lambda_j \varphi_j \le \psi := \sum_{n \in \mathrm{N}} \alpha_n \varphi_n \ (\Rightarrow \alpha_n \in R_+, \ \ n \in \mathrm{N}, \ \ \lambda_j \le \alpha_j, \ \ j \in J_0),$$

then the hypothesis and the above relations yield:

$$\lambda_j B_j \le \lambda_j U^j \le \alpha_j \cdot U^j, \quad j \in \mathrm{N} \Rightarrow$$
$$\sum_{j \in J_0} \lambda_j B_j \le \sum_{j \in J_0} \alpha_j \cdot U^j \le \sum_{n \in \mathrm{N}} \alpha_n \cdot U^n = \sum_{n \in \mathrm{N}} |\alpha_n| \cdot U^n =$$
$$|\psi|(U) = T(\psi), \quad T(\varphi) := \sum_{n \in \mathrm{N}} |\alpha_n| \cdot U^n = T(-\varphi), \quad \psi = \sum_{n \in \mathrm{N}} a_n \varphi_n \in X.$$

Notice that the definition of the order relation on the space X implies

$$\left| \sum_{n \in \mathrm{N}} \alpha_n \varphi_n \right| = |\psi| = \sum_{n \in \mathrm{N}} |\alpha_n| \varphi_n .$$

Hence, the assertions from (b), Theorem 2.5 are accomplished and the conclusion follows from a direct application of the latter theorem. On the other hand, (a) $\Rightarrow$ (b) is almost obvious, since $B_n \le F(\varphi_n)$ for all $n \in \mathrm{N}$ lead to

$$B_n \le F(\varphi_n) \le |F(\varphi_n)| \le T(\varphi_n) = U^n, \quad n \in \mathrm{N}.$$

This concludes the proof. □

Proof of Theorem 2.12. In order to prove that (b) $\Rightarrow$ (a), we apply the implication (b) $\Rightarrow$ (a) of Theorem 2.5. Verifying the conditions (b) of the latter theorem, and using Cauchy's inequalities, one deduces:

$$\sum_{j \in J_0} \lambda_j \varphi_j \le \psi = \sum_{n \in \mathrm{N}} \gamma_n \varphi_n, \quad \lambda_j \ge 0, \ U_j \le A^j + \varepsilon \cdot I \Rightarrow \gamma_n \ge 0, \quad \forall n \in \mathrm{N},$$

$$\lambda_j \le \gamma_j \le \frac{\|\psi\|}{b^j}, \quad j \in J_0 \Rightarrow \sum_{j \in J_0} \lambda_j U_j \le \|\psi\| \cdot \left(\sum_{j \in J_0} \frac{A^j}{b^j} \right) + \varepsilon \left(\sum_{j \in J_0} \gamma_j \right) \cdot I \le$$
$$\left(\sum_{n \in \mathrm{N}} \frac{A^n}{b^n} \right) \cdot \|\psi\| + \varepsilon \psi(1) \cdot I = \left(I - \frac{A}{b} \right)^{-1} \|\psi\| + \varepsilon \psi(1) I =$$
$$b(bI - A)^{-1} \|\psi\| + \varepsilon \psi(1) I = b(bI - A)^{-1} \|\psi\| + \varepsilon \psi(I) =: T(\psi), \quad \psi \in X.$$

Now the first conclusion follows via Theorem 1.5. On the other hand, the implication (c) $\Rightarrow$ (b) is almost obvious, since

$$U^j \le F(\varphi_j) \le A^j + \varepsilon \varphi_j(I) = A^j + \varepsilon I, \quad j \in \mathrm{N}.$$

It remains to prove the converse implication, that is (b) $\Rightarrow$ (c). To this end, we apply (b) $\Rightarrow$ (a) of Theorem 2.5 once more. The following implications hold true

$$\sum_{j \in J_0} \lambda_j \varphi_j \ \pi \ \psi = \sum_{n \in \mathrm{N}} \gamma_n \varphi_n, \quad \lambda_j \in R_+, \quad j \in J_0 \Rightarrow$$
$$\lambda_j U_j \le \lambda_j A^j + \varepsilon \lambda_j I \le \gamma_j A^j + \varepsilon \cdot \gamma_j I, \quad j \in J_0 \Rightarrow$$
$$\sum_{j \in J_0} \lambda_j U_j \le \sum_{n \in \mathrm{N}} \gamma_n A^n + \varepsilon \left(\sum_{n \in \mathrm{N}} \gamma_n \right) \cdot I \le$$
$$\sum_{n \in \mathrm{N}} |\gamma_n| A^n + \varepsilon |\psi|(I) = |\psi|(A) + \varepsilon |\psi|(I) =: T(\psi).$$

Application of Theorem 2.5 yields the existence of a linear operator *F* with the properties mentioned at point (c). This concludes the proof. □

Proof of Theorem 2.13. (a)⇒(b) is obvious, because of $y_j \le F(\varphi_j) \le \alpha G(|\varphi_j|) = \alpha G(\varphi_j), \forall j \in J$. For the converse, we apply Theorem 2.5, (b)⇒(a). Let $J_0 \subset J$ be a finite subset, $\{\lambda_j\}_{j \in J_0} \subset \mathbb{R}_+, x \in X$, such that $\sum_{j \in J_0} \lambda_j \varphi_j \le x$. Then using (b) and the fact that the scalars λ_j are nonnegative, as well as the positivity of G, we derive

$$\sum_{j\in J_0}\lambda_j y_j \le \alpha \sum_{j\in J_0}\lambda_j G(\varphi_j) = \alpha G\left(\sum_{j\in J_0}\lambda_j\varphi_j\right) \le \alpha G(x) \le \alpha G(|x|) := T(x).$$

Application of Theorem 2.5 leads to the existence of a linear positive operator F from X into Y such that

$$F(\varphi_j) \ge y_j, \forall j \in J, F(x) \le \alpha G(|x|), \forall x \in X.$$

From the last relation, also using the fact that the norms on X and Y are solid ($|u| \le |v| \Rightarrow \|u\| \le \|v\|$), we deduce

$$|F(x)| \le \alpha G(|x|) \Rightarrow \|F(x)\| \le \alpha\|G\|\||x|\| = \alpha\|G\|\|x\|, \forall x \in X.$$

It follows that $\|F\| \le \alpha\|G\|$. This concludes the proof. □

Proof of Corollary 2.2. One applies Theorem 2.13 for $G(\psi) = \int_M g\psi d\mu, \psi \in X$, $Y = \mathbb{R}$, as well as the representation of linear positive continuous functionals on L^p spaces by means of nonnegative elements from L^q spaces. In order to prove (b)⇒(a), from the preceding results it follows that there exists $h \in \left(L^q_\mu(M)\right)_+$ such that $\int_M h\varphi_j d\mu \ge y_j, \forall j \in J$ and

$$\int_M h\psi d\mu \le \alpha \int_M g\psi d\mu$$

for all nonnegative functions $\psi \in L^p_\mu(M)$. Now we choose $\psi = \chi_B$, where B is an arbitrary measurable subset of M. Then the last relation can be rewritten as

$$\int_B (h - \alpha g)\, d\mu \le 0$$

for all such subsets B. A straightforward application of Theorem 1.40 [22], leads to $h - \alpha g \le 0$ a.e. in M. Since (a)⇒(b) is obvious, this concludes the proof. □

Proof of Corollary 2.3. One applies Corollary 2.2 to $p = 1, q = \infty, g = 1$ a.e. The notation t^j is the multi–index notation $t^j = t_1^{j_1}\cdots t_n^{j_n}$. The conclusion follows via Fubini 's theorem and Gamma function properties. □

Proof of Theorem 2.14. *Let* $J_0 \subset J$ be a finite subset, $\{\lambda_j\}_{j\in J_0} \subset \mathbb{R}_+$. Hölder inequality and using also (b), lead to the following implications

$$\sum_{j\in J_0}\lambda_j\varphi_j \le \psi \Rightarrow \int_M \left(\sum_{j\in J_0}\lambda_j\varphi_j\right) d\mu \le \int_M \psi d\mu \le \|\psi\|_p (\mu(M))^{1/q} \Rightarrow$$

$$\sum_{j\in J_0}\lambda_j y_j \le \alpha \int_M \left(\sum_{j\in J_0}\lambda_j\varphi_j\right) d\mu \le \alpha\, \|\psi\|_p (\mu(M))^{1/q} := T(\psi).$$

Application of Theorem 2.1 and measure theory arguments, as discussed in [22, theorem 6.16, p. 122-124], yield the existence of $h \in L^q_\mu(M)$ such that

$$F(\varphi_j) = \int_M h\varphi_j d\mu \geq y_j, \forall j \in J, F(\psi) = \int_M h\psi d\mu \leq \alpha \|\psi\|_p (\mu(M))^{1/q}, \psi \in X.$$

Moreover, since $F(\psi) \geq 0, \forall \psi \in X_+$, we have

$$\int_M h\psi d\mu \geq 0, \forall \psi \in X_+.$$

Taking $\psi = \chi_B$, where $B \subset M$ is a measurable set such that $\mu(B) > 0$, one obtains

$$\int_B h d\mu \geq 0$$

for all such subsets B. Application of theorem 1.40 [22] leads to $h \geq 0\, \mu - a.e.$ From the previous relations we also derive that $\|h\|_q \leq \alpha (\mu(M))^{1/q}$. This concludes the proof. □

Proof of Theorem 2.15. The implication (a)⇒(b) is obvious:

$$B_j \leq F(\varphi_j) \leq \int_{\Sigma_A} |\varphi_j| dE_{(A_1,\ldots,A_n)} = \int_{\Sigma_A} \varphi_j dE_{(A_1,\ldots,A_n)} = A_1^{j_1} \cdots A_n^{j_n},$$

$j \in \mathbb{N}^n$ (we have used the positivity of the operators A_k which leads to $|\varphi_j| = \varphi_j$ on Σ_A). For the converse, one applies Theorem 2.1, (b)⇒(a), where $\mathbb{N}^n$ stands for J, φ_j stands for x_j and B_j stands for $y_j, \forall j \in \mathbb{N}^n$. Let J_0 and $\{\lambda_j\}_{j \in J_0}$ be as mentioned at point (b) of Theorem 2.5. The following implications hold:

$$\sum_{j \in J_0} \lambda_j \varphi_j \leq \varphi \in X \Rightarrow \sum_{j \in J_0} \lambda_j \int_{\Sigma_A} \varphi_j dE_A = \sum_{j \in J_0} \lambda_j A_1^{j_1} \cdots A_n^{j_n} \leq \int_{\Sigma_A} \varphi dE_A \leq \int_{\Sigma_A} |\varphi|\, dE_A := T(\varphi).$$

The positivity of the spectral measure dE_A has been used. On the other hand, the hypothesis (b), the fact that the scalars λ_j are nonnegative and the preceding evaluation yield

$$\lambda_j B_j \leq \lambda_j A^j, \forall j \Rightarrow \sum_{j \in J_0} \lambda_j B_j \leq \sum_{j \in J_0} \lambda_j A^j = \sum_{j \in J_0} \lambda_j A_1^{j_1} \cdots A_n^{j_n} \leq T(\varphi),$$

where $T(\varphi)$ was defined above. Thus, the implication at (b) Theorem 2.5 is accomplished. Application of the latter theorem leads to the existence of a "feasible solution" F having the property mentioned at point (a) of the present theorem. The last property is a consequence of the preceding one, using the fact that the norm on Y is solid. This concludes the proof.

Proof of Theorem 2.16. We apply Theorem 2.3 (b)⇒(a), to $x_j = h_j, y_j = B_j, j \in J := \mathbb{N}^n$. To this end, we verify the implication mentioned at point (b) of the latter theorem. Cauchy's inequalities lead to

$$\sum_{j \in J_0} \lambda_j h_j \leq \varphi = \sum_{k \in \mathbb{N}^n} \beta_k h_k \Rightarrow \lambda_j \leq \beta_j \leq |\beta_j| \leq \frac{\|\varphi\|_\infty}{(1-\varepsilon)^{j_1+\cdots+j_n}}$$

for all positive $\varepsilon < min\left\{1 - \|A_p\|; p \in \{1, \ldots n\}\right\}$ and all $j \in J_0$. Making $\varepsilon \downarrow 0$, we get $\lambda_j \leq \|\varphi\|_\infty, \forall j \in J_0$. Now application of the relations verified by the operators $B_j, j \in J_0$ yields

$$\lambda_j B_j \leq \|\varphi\|_\infty B_j \leq a\|\varphi\|_\infty A_1^{j_1} \cdots A_n^{j_n}, j \in J_0 \Rightarrow$$

$$\sum_{j\in J_0} \lambda_j B_j \leq a\|\varphi\|_\infty \sum_{j\in J_0} A_1^{j_1} \cdots A_n^{j_n} \leq a\|\varphi\|_\infty \left(\sum_{k_1\in\mathbb{N}} A_1^{k_1}\right) \cdots \left(\sum_{k_n\in\mathbb{N}} A_n^{k_n}\right) =$$

$$a\|\varphi\|_\infty \prod_{p=1}^{n} \left(I - A_p\right)^{-1} =: T(\varphi).$$

Thus, the implication (b) from Theorem 2.3 is verified. Application of the latter theorem, leads to the existence of a linear positive operator $F: X \to Y$, such that the interpolation conditions

$$F(h_k) = B_k, k \in \mathbb{N}^n$$

are verified and $F(\varphi) \leq T(\varphi) = a\|\varphi\|_\infty \prod_{p=1}^{n} \left(I - A_p\right)^{-1}$ for all φ in X. In particular, the last relations yield

$$\|F\| \leq a \prod_{p=1}^{n} \left\| \sum_{m=0}^{\infty} A_p^m \right\| \leq a \prod_{p=1}^{n} \left(\sum_{m=0}^{\infty} \|A_p\|^m \right) = a \prod_{p=1}^{n} \left(1 - \|A_p\|\right)^{-1}.$$

The uniqueness of the solution follows from its continuity, also using the density of polynomials in the space X. This concludes the proof. □

4 Conclusions

We have recalled earlier and mainly actual results on the subject, as well as their relationship. Necessary and sufficient (or only sufficient) conditions for the existence of a solution are formulated and proved. This is the purpose of this review article. In the case of Markov moment problems, sometimes the uniqueness of the solution follows from the proof of its existence. The recently published results are presented accompanied by their proofs. The technical methods of proving such theorems are: the extension results for linear operators, Cauchy inequalities for complex analytic functions, operator inequalities, measure theory arguments, earlier related results. A relationship between these fields is implicitly illustrated.

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