

# Norms over anti fuzzy $G$ -submodules

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## Abstract

In this study, we define anti fuzzy  $G$ -submodules with respect to  $t$ -conorms and investigate some of their algebraic properties. Later we introduce the union and direct sum of them and finally, we prove that the union, direct sum, homomorphic images and pre images of them are also anti fuzzy  $G$ -submodules with respect to  $t$ -conorms.

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## 1. Introduction

In mathematics, given a group  $G$ , a  $G$ -module is an abelian group  $M$  on which  $G$  acts compatibly with the abelian group structure on  $M$ . This widely applicable notion generalizes that of a representation of  $G$ . Group (co)homology provides an important set of tools for studying general  $G$ -modules. The term  $G$ -module is also used for the more general notion of an  $R$ -module on which  $G$  acts linearly (i.e. as a group of  $R$ -module automorphisms). Undoubtedly the notion of fuzzy set theory initiated by Zadeh [42] in 1965 in a seminal paper, plays the central role for further development. This notion tries to show that an object corresponds more or less to the particular category we want to assimilate it to; that was how the idea of defining the membership of an element to a set not on the Aristotelian pair  $\{0, 1\}$  any more but on the continuous interval  $[0, 1]$  was born. The notion of a fuzzy set is completely non-statistical in nature and the concept of fuzzy set provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership rather than the presence of random variables. In fact the idea of describing all shades of reality was for long the obsession of some logicians [19, 37]. During last four decades the fuzzy set theory has rapidly developed into an area which scientifically as well as from the application point of view, is recognized as a very valuable contribution to the existing knowledge [3, 4, 5, 6, 7, 8, 11, 12, 14, 17, 18, 23, 24, 43, 44]. The triangular conorm ( $T$ -conorm) originated from the studies of probabilistic metric spaces [22, 38] in which triangular inequalities were extended using the theory of  $T$ -conorm. Later, Hohle [13], Alsina et al. [2], etc. introduced the  $T$ -conorm into fuzzy set theory and suggested that the  $T$ -conorm be used for the intersection and union of fuzzy sets. Since then, many other researchers have presented various types of  $T$ -operators for the same purpose [41, 10, 40] and even proposed some methods to generate the variations of these operators [16]. The author by using norms, investigated some properties of fuzzy algebra [25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]. In this paper, we use the triangular conorm ( $T$ -conorm) to define the anti fuzzy  $G$ -submodules and we obtain some related results. Next we show that the union, direct sum, homomorphic images and pre images of them are also anti fuzzy  $G$ -submodules with respect to  $t$ -conorms.

## 1 Preliminaries

The following definitions and preliminaries are required in the sequel of our work and hence presented in brief. For details we refer readers to [20, 9, 15, 21, 39, 1, 36]. Throughout the paper,  $\mathbb{Q}, \mathbb{R}, \mathbb{C}$  will always be rational, real and complex numbers, respectively.

**Definition 1** Let  $R$  be a ring. A commutative group  $(M, +)$  is called a left  $R$ -module or a left module over  $R$  with respect to a mapping

$$\cdot : R \times M \rightarrow M$$

if for all  $r, s \in R$  and  $m, n \in M$ ,

$$(1) r.(m + n) = r.m + r.n,$$

$$(2) r.(s.m) = (rs).m,$$

$$(3) (r + s).m = r.m + s.m.$$

If  $R$  has an identity  $1$  and if  $1.m = m$  for all  $m \in M$ , then  $M$  is called a unitary or unital left  $R$ -module. A right  $R$ -module can be defined in a similar fashion.

**Definition 2** Let  $M$  be an  $R$ -module and  $N$  be a nonempty subset of  $M$ . Then  $N$  is called a submodule of  $M$  if  $N$  is a subgroup of  $M$  and for all  $r \in R, a \in N$ , we have  $ra \in N$ .

**Definition 3** Let  $G$  be a finite group. A vector space  $M$  over a field  $K$  is called a  $G$ -module if for every  $g \in G$  and  $m \in M$ , there exist a product (called the action of  $G$  on  $M$ )  $m.g \in M$  satisfying the following axioms:

$$(1) m.1_G = m, \forall m \in M \text{ ( } 1_G \text{ being the identity element in } G),$$

$$(2) m.(g.h) = (m.g).h, \forall m \in M : g, h \in G \text{ and}$$

$$(3) (k_1m_1 + k_2m_2).g = k_1(m_1.g) + k_2(m_2.g) \forall m_1, m_2 \in M : g \in G : k_1, k_2 \in K.$$

**Example 4** Let  $G = \{1, -1\}$  and  $M = \mathbb{C}$ . Then  $M$  is a vector space over  $\mathbb{R}$  and under the usual addition and multiplication of real numbers, we can show that  $M$  is a  $G$ -module.

**Remark 5** The operation  $(m, g) \rightarrow m.g$  defined above may be called a right-action of  $G$  on  $M$  and  $M$  may be said to be a right  $G$ -module. In a similar way, we can define left-action and left  $G$ -module. We shall consider all  $G$ -modules as left  $G$ -modules.

**Definition 6** Let  $M$  be a  $G$ -module. A vector subspace  $N$  of  $M$  is a  $G$ -submodule if  $N$  is also a  $G$ -module under the same action of  $G$ . Thus  $N$  is  $G$ -submodule of  $G$ -module  $M$  if and only if  $N$  is submodule of  $M$  and  $N$  be a  $G$ -module.

**Example 7** Let  $\mathbb{Q}$  be the field of rationals and  $G = \{1, -1\}$  and  $M = \mathbb{R}$ . Then  $M$  is a  $G$ -module over  $\mathbb{Q}$ . Now for each  $r \notin \mathbb{Q}$  we get that  $N = \mathbb{Q}(r)$  is a  $G$ -submodule of  $M$ .

**Definition 8** Let  $M$  and  $N$  be  $G$ -modules. A mapping  $f : M \rightarrow N$  is a  $G$ -module homomorphism if

$$(1) f(k_1m_1 + k_2m_2) = k_1f(m_1) + k_2f(m_2)$$

$$(2) f(gm) = gf(m)$$

for all  $m_1, m_2 \in M$  and  $k_1, k_2 \in K$  and  $g \in G$ .

**Definition 9** Let  $X$  a non-empty sets. A fuzzy subset  $\mu$  of  $X$  is a function  $\mu : X \rightarrow [0, 1]$ . Denote by  $[0, 1]^X$ , the set of all fuzzy subset of  $X$ .

**Definition 10** Let  $f$  be a mapping from  $R$ -module  $M$  into  $R$ -module  $N$ . Let  $\mu \in [0, 1]^M$  and  $\nu \in [0, 1]^N$ . For all  $y \in N$  define  $f(\mu) \in [0, 1]^N$  and  $f^{-1}(\nu) \in [0, 1]^M$  as

$$f(\mu)(y) = \begin{cases} \sup\{\mu(x) \mid x \in M, f(x) = y\} & \text{if } f^{-1}(y) \neq \emptyset \\ \emptyset & \text{if } f^{-1}(y) = \emptyset. \end{cases}$$

Also for all  $x \in M$ ,  $f^{-1}(\nu)(x) = \nu(f(x))$ .

**Definition 11** A  $t$ -conorm  $C$  is a function  $C : [0, 1] \times [0, 1] \rightarrow [0, 1]$  having the following four properties:

- (C1)  $C(x, 0) = x$  (neutral element),  
 (C2)  $C(x, y) \leq C(x, z)$  if  $y \leq z$  (monotonicity),  
 (C3)  $C(x, y) = C(y, x)$  (commutativity),  
 (C4)  $C(x, C(y, z)) = C(C(x, y), z)$  (associativity),

for all  $x, y, z \in [0, 1]$ .

Properties (C1), (C2) and (C3) give  $C(1, 1) = C(0, 1) = C(1, 0) = 1$ ,  $C(0, 0) = 0$ .

We say that  $C$  is idempotent if for all  $x \in [0, 1]$ , we have  $C(x, x) = x$ .

**Example 12** For all  $x, y \in [0, 1]$  the basic  $t$ -conorms are

$$\begin{aligned} C_m(x, y) &= \max\{x, y\}, \\ C_b(x, y) &= \min\{1, x + y\}, \\ C_p(x, y) &= x + y - xy, \\ C^*(x, y) &= \begin{cases} x & \text{if } y = 0 \\ y & \text{if } x = 0 \\ 1 & \text{otherwise} \end{cases} \end{aligned}$$

which are called standard union, bounded sum, algebraic sum and drastic union respectively.

We can see that

$$C_m(x, y) \leq C_p(x, y) \leq C_b(x, y) \leq C^*(x, y).$$

**Definition 13** Define

$$C_n(x_1, x_2, \dots, x_n) = C(x_i, C_{n-1}(x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n))$$

for all  $1 \leq i \leq n, n \geq 2, C_2 = C$ . Also define

$$C_\infty(x_1, x_2, \dots) = \lim_{n \rightarrow \infty} C(x_1, x_2, \dots, x_n).$$

**Definition 14** The union of fuzzy subsets  $\mu_1$  and  $\mu_2$  in a set  $X$  over  $t$ -conorm  $C$  we mean the fuzzy subset  $\mu = \mu_1 \cup \mu_2$  in the set  $X$  such that for any  $x \in X$

$$\mu(x) = (\mu_1 \cup \mu_2)(x) = C(\mu_1(x), \mu_2(x)).$$

**Definition 15** Define the union of a collection of fuzzy subsets  $\{\mu_1, \mu_2, \dots\}$  in a set  $X$  over  $t$ -conorm  $C$  as fuzzy subset  $\cup_C \mu_i$  such that for any  $x \in X$ ,  $(\cup_C \mu_i)(x) = C_\infty(\mu_1(x), \mu_2(x), \dots)$ .

**Lemma 16** Let  $C$  be a  $t$ -conorm. Then

$$C(C(x, y), C(w, z)) = C(C(x, w), C(y, z)),$$

for all  $x, y, w, z \in [0, 1]$ .

## 2 Anti fuzzy $G$ -modules under $t$ -conorms

Firstly, we define anti fuzzy  $G$ -modules on  $M$  under  $t$ -conorm  $C$ .

**Definition 17** Let  $G$  be a finite group and  $M$  be a  $G$ -module over  $K$ , which is a subfield of  $\mathbb{C}$ . Then an anti fuzzy  $G$ -module on  $M$  under  $t$ -conorm  $C$  is a fuzzy subset  $\mu : M \rightarrow [0, 1]$  such that

$$(1) \mu(ax + by) \leq C(\mu(x), \mu(y))$$

$$(2) \mu(gm) \leq \mu(m)$$

for all  $a, b \in K : x, y \in M : m \in M$  and  $g \in G$ .

Denote by  $AFGC(M)$ , the set of all anti fuzzy  $G$ -modules on  $M$  under  $t$ -conorm  $C$ .

**Example 18** Let  $G = \{1, -1\}$  and  $M = \mathbb{R}^2$  is a vector space over real field  $\mathbb{R}$ . Then  $M$  is a  $G$ -module over  $\mathbb{R}$ . Define  $\mu : \mathbb{R}^2 \rightarrow [0, 1]$  by

$$\mu(x_1, x_2) = \begin{cases} (0, 0) & \text{if } (x_1, x_2) = (0, 0), \\ 0.45 & \text{if } (x_1, x_2) \neq (0, 0). \end{cases}$$

If  $C$  be standard union  $t$ -conorm  $C(a, b) = C_m(a, b) = \max\{a, b\}$  for all  $a, b \in [0, 1]$ , then  $\mu \in AFGC(M)$ .

In the followin proposition, we investigate relationship between  $AFGC(M)$  and  $G$ -submodules of  $M$ .

**Proposition 19** Let  $M$  be a  $G$ -module over  $K$  and  $\mu$  be a fuzzy set of  $M$ . If  $\mu \in AFGC(M)$  and  $C$  be idempotent  $t$ -conorm, then  $L(\mu, \alpha) = \{x \in M : \mu(x) \leq \alpha\}$  will be  $G$ -submodule of  $M$ .

**Proof** If  $L(\mu, \alpha) = \emptyset$ , then nothing to prove. Therefore, suppose that  $L(\mu, \alpha) \neq \emptyset$ , and let  $x, y \in U(\mu, \alpha)$  and  $a, b \in K$ . Then

(1)  $\mu(x) \leq \alpha$  and  $\mu(y) \leq \alpha$  and as  $\mu \in AFGC(M)$  so  $\mu(ax+by) \leq C(\mu(x), \mu(y)) \leq C(\alpha, \alpha) = \alpha$  and then  $\mu(ax+by) \leq \alpha$  so  $ax + by \in L(\mu, \alpha)$ .

(2) If  $g \in G$ , then  $\mu(gx) \leq \mu(x) \leq \alpha$  and then  $gx \in L(\mu, \alpha)$ .

Thus (1) and (2) give us that  $L(\mu, \alpha)$  is  $G$ -submodule of  $M$ . ■

**Corollary 20** Let  $\mu \in AFGC(M)$ . Then  $N = \{x \mid x \in M, \mu(x) = 0\}$  is a  $G$ -submodule of the module  $M$ .

**Proof** In the Proposition 3.3 set  $\alpha = 0$ . ■

In the following propositions, we prove that the union and direct sum of all anti fuzzy  $G$ -modules on  $M$  under  $t$ -conorm  $C$  are also anti fuzzy  $G$ -modules on  $M$  under  $t$ -conorm  $C$ .

**Proposition 21** The union of any collection of anti fuzzy  $G$ -submodules is an anti fuzzy  $G$ -submodule.

**Proof** Let  $x, y \in M$  and  $a, b \in K$ . Then

$$\begin{aligned} (1) \quad (\cup_C \mu_i)(ax + by) &= C_\infty(\mu_1(ax + by), \mu_2(ax + by), \dots) \leq C_\infty(C(\mu_1(x), \mu_1(y)), C(\mu_2(x), \mu_2(y)), \dots) \\ &= C(C_\infty(\mu_1(x), \mu_2(x), \dots), C_\infty(\mu_1(y), \mu_2(y), \dots)) = C((\cup_C \mu_i)(x), (\cup_C \mu_i)(y)). \end{aligned}$$

(2) Let  $g \in G$  then

$$(\cup_C \mu_i)(gx) = C_\infty(\mu_1(gx), \mu_2(gx), \dots) \leq C_\infty(\mu_1(x), \mu_2(x), \dots) = (\cup_C \mu_i)(x).$$

Thus  $(\cup_C \mu_i) \in AFGC(M)$ . ■

**Definition 22** By the direct sum of fuzzy sets  $\{\mu_1, \mu_2, \dots, \mu_n\}$  over a  $t$ -conorm  $C$  we mean the fuzzy subset  $\mu = \bigoplus_{i=1}^n \mu_i$  such that

$$\mu(x_1, x_2, \dots, x_n) = (\bigoplus_{i=1}^n \mu_i)(x_1, x_2, \dots, x_n) = C_n(\mu_1(x_1), \mu_2(x_2), \dots, \mu_n(x_n)).$$

**Proposition 23** Let  $\{M_1, M_2, \dots, M_n\}$  be a collection of  $G$ -submodules and  $M = \bigoplus_{i=1}^n M_i$  be its direct sum. If  $\mu_i \in AFGC(M_i)$ , then  $\mu = \bigoplus_{i=1}^n \mu_i \in AFGC(M)$ .

**Proof** Let  $x, y \in M$ ,  $x = (x_1, x_2, \dots, x_n)$ ,  $y = (y_1, y_2, \dots, y_n)$  and  $a, b \in K$  and  $g \in G$ . Then

(1)

$$\begin{aligned} \mu(ax + by) &= \mu(a(x_1, x_2, \dots, x_n) + b(y_1, y_2, \dots, y_n)) \\ &= \mu((ax_1, ax_2, \dots, ax_n) + (by_1, by_2, \dots, by_n)) \\ &= \mu(ax_1 + by_1, ax_2 + by_2, \dots, ax_n + by_n) \\ &= C_n(\mu_1(ax_1 + by_1), \mu_2(ax_2 + by_2), \dots, \mu_n(ax_n + by_n)) \\ &\leq C_n(C(\mu_1(x_1), \mu_1(y_1)), C(\mu_2(x_2), \mu_2(y_2)), \dots, C(\mu_n(x_n), \mu_n(y_n))) \\ &= C(C_n(\mu_1(x_1), \mu_2(x_2), \dots, \mu_n(x_n)), C_n(\mu_1(y_1), \mu_2(y_2), \dots, \mu_n(y_n))) \\ &= C(\mu(x), \mu(y)). \end{aligned}$$

(2)

$$\begin{aligned} \mu(gx) &= \mu(g(x_1, x_2, \dots, x_n)) = \mu(gx_1, gx_2, \dots, gx_n) \\ &= C_n(\mu_1(gx_1), \mu_2(gx_2), \dots, \mu_n(gx_n)) \leq C_n(\mu_1(x_1), \mu_2(x_2), \dots, \mu_n(x_n)) \\ &= \mu(x). \end{aligned}$$

Therefore  $\mu \in AFGC(M)$ . ■

In the following propositions, we investigate  $AFGC(-)$  under  $G$ -module homomorphisms.

**Proposition 24** Let  $f : M \rightarrow N$  be a  $G$ -module epimorphism. If  $\mu \in AFGC(M)$ , then  $f(\mu) \in AFGC(N)$ .

**Proof** Let  $y_1, y_2 \in N$  and  $a, b \in K$ .

(1)

$$\begin{aligned} f(\mu)(ay_1 + by_2) &= \sup\{\mu(ax_1 + bx_2) \mid x_1, x_2 \in M, f(ax_1) = ay_1, f(bx_2) = by_2\} \\ &= \sup\{\mu(ax_1 + bx_2) \mid x_1, x_2 \in M, af(x_1) = ay_1, bf(x_2) = by_2\} \\ &\leq \sup\{C(\mu(x_1), \mu(x_2)) \mid x_1, x_2 \in M, f(x_1) = y_1, f(x_2) = y_2\} \\ &= C(\sup\{\mu(x_1) \mid f(x_1) = y_1\}, \sup\{\mu(x_2) \mid f(x_2) = y_2\}) \\ &= C(f(\mu)(y_1), f(\mu)(y_2)). \end{aligned}$$

(2) Let  $y \in N$  and  $g \in G$ .

$$\begin{aligned} f(\mu)(gy) &= \sup\{\mu(gx) \mid x \in M, f(gx) = gy\} \\ &= \sup\{\mu(gx) \mid x \in M, gf(x) = gy\} \\ &\leq \sup\{\mu(x) \mid x \in M, f(x) = y\} \\ &= f(\mu)(y). \end{aligned}$$

Thus  $f(\mu) \in AFGC(N)$ . ■

**Proposition 25** Let  $f : M \rightarrow N$  be a  $G$ -module homomorphism. If  $\nu \in AFGC(N)$ , then  $f^{-1}(\nu) \in AFGC(M)$ .

**Proof** Let  $x_1, x_2 \in M$  and  $a, b \in K$ . Then

(1)

$$\begin{aligned} f^{-1}(\nu)(ax_1 + bx_2) &= \nu(f(ax_1 + bx_2)) = \nu(f(ax_1) + f(bx_2)) \\ &= \nu(af(x_1) + bf(x_2)) \leq C(\nu(f(x_1)), \nu(f(x_2))) \\ &= C(f^{-1}(\nu)(x_1), f^{-1}(\nu)(x_2)). \end{aligned}$$

(2) Let  $x \in M$  and  $g \in G$ . Then

$$f^{-1}(\nu)(gx) = \nu(f(gx)) = \nu(gf(x)) \leq \nu(f(x)) = f^{-1}(\nu)(x).$$

Then  $f^{-1}(\nu) \in AFGC(M)$ . ■

**Proposition 26** Let  $M$  be a  $G$ -module and  $N$  be a subset of  $M$ . Let

$$\mu(x) = \begin{cases} 0 & \text{if } x \in N \\ \alpha & \text{if } x \notin N \end{cases}$$

with  $\alpha \in (0, 1]$  and  $C$  be an idempotent  $t$ -conorm. Then  $\mu \in AFGC(M)$  if and only if  $N$  is a  $G$ -submodule of  $M$ .

**Proof** Let  $\mu \in AFGC(M)$  and we prove that  $N$  is a submodule of  $M$ . Let  $x, y \in N \subseteq M$  and  $a, b \in K$ . Now

$$\mu(ax + by) \leq C(\mu(x), \mu(y)) = C(0, 0) = 0$$

so  $\mu(ax + by) = 0$  and then  $ax + by \in N$ .

Also let  $g \in G$  and then  $\mu(gx) \leq \mu(x) = 0$  so  $\mu(gx) = 0$  and then  $gx \in N$ .

Therefore  $N$  is a submodule of  $M$  and since  $N$  be a subset of  $M$  so  $N$  will be a  $G$ -submodule of  $M$ .

Conversely, let  $N$  is a submodule of  $M$  and we prove that  $\mu \in AFGC(M)$ . Suppose  $x, y \in M$  and  $a, b \in K$  and we investigate the following conditions:

(1) If  $x, y \in N$ , then  $ax + by \in N$  and then

$$\mu(ax + by) = 0 \leq 0 = C(0, 0) = C(\mu(x), \mu(y)).$$

(2) For any  $x \in N$  and  $y \notin N$  then  $ax + by \notin N$  and so

$$\mu(ax + by) = \alpha \leq \alpha = C(\alpha, 0) = C(0, \alpha) = C(\mu(x), \mu(y)).$$

(3) Let  $x \notin N$  and  $y \in N$  then  $ax + by \notin N$  and then

$$\mu(ax + by) = \alpha \leq \alpha = C(\alpha, 0) = C(\mu(x), \mu(y)).$$

(4) Finally, if  $x, y \notin N$ ,  $ax + by \notin N$  and so

$$\mu(ax + by) = \alpha \leq \alpha = C(\alpha, \alpha) = C(\mu(x), \mu(y)).$$

Therefore from (1)-(4) we have that

$$\mu(ax + by) \leq C(\mu(x), \mu(y)).$$

Now let  $x \in M$  and  $g \in G$ . Then we have:

(1) If  $x \in N$  then  $gx \in N$  and then  $\mu(gx) = 0 \leq \mu(x)$ .

(2) If  $x \notin N$ , then  $gx \notin N$  and so  $\mu(gx) = \alpha \leq \alpha = \mu(x)$ .

Therefore from (1) and (2) we have that  $\mu(gx) \leq \mu(x)$ .

Thus  $\mu \in AFGC(M)$ . ■

**Corollary 27** *Let  $M$  be an  $R$ -module and  $N$  be a subset of  $M$ . If  $\mu : N \rightarrow \{0, 1\}$  be the anti characteristic function as*

$$\mu(x) = \begin{cases} 0 & \text{if } x \in N \\ 1 & \text{if } x \notin N \end{cases}$$

*then  $\mu \in AFGC(M)$  if and only if  $N$  is a submodule of  $M$ .*

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