Numerical Method of Differential Equation by Runge-Kutta Method With Full Fuzzy Initial Values

T. Jayakumar , T. Muthukumar

Department of Mathematics
Sri Ramakrishna Mission Vidyalaya College of Arts and Science
Coimbatore-641 020, Tamilnadu, India
jayakumar.thippan68@gmail.com, vtmuthukumar@gmail.com

Abstract

In this paper, we study differential equation with fully fuzzy initial value. We propose a numerical method to approximate the fuzzy solution by using partition of fuzzy interval and generalization of Hukuhara difference and division. We prove some theorems for differential equation by fuzzy initial value. Finally, we solve numerical example to illustrate our proposed method.

Keywords: Fuzzy differential equations, Hukuhara difference, Full Fuzzy Runge-kutta method.

1. Introduction

The concept of fuzzy set theory was first introduced by Lotfi Zadeh in 1960s which is now used as a powerful tool for modeling uncertainty and for processing vague or subjective information in mathematical models. Fuzzy set theory are used to study a variety of problems fuzzy metric spaces [20], fuzzy linear systems [4, 5, 23], fuzzy differential equations [6, 7, 10, 16, 17] and other topics. The concept of fuzzy derivative was first introduced by Chang and Zadeh [7], and it was followed by Dubois and Prade[8]. The fuzzy differential equations and fuzzy initial value problem were regularly treated by Kaleva [15] and Seikkala[22]. Several authors have produced a wide range of results in both the theoretical and applied fields of fuzzy differential equations [1, 2, 6, 14, 19, 21, 22]. Some of researchers worked for approximate solving the fuzzy initial value problem y = f(x, y) where x_0 is real number and $y(x_0) = y_0$ fuzzy number [1, 2, 6]. We used of definition fuzzy directed line induced by L. Hongliang et al.[17] and extent to fuzzy interval. This paper used of partition of fuzzy interval [17] and generalization of Hukuhara difference and division [24]. Engineers, Computer Scientists and Operations Researchers have taken up fuzzy sets with interest, Mathematicians gave serious interest to fuzzy sets only in the resent years, though they have been involved with the development of fuzzy sets from the very beginning. Many interesting Mathematical problems are coming to the fore front and now fuzzy sets has emerged as an independent branch of Applied Mathematics and was discussed by many authors [9, 10, 11, 12, 13, 15, 16, 18, 26, 27]. Abbasbandy and Allah Viranloo studied Numerical solution of fuzzy differential equation by Runge-Kutta method[3].

Fuzzy sets are taken into consideration with respect to a nonempty base set X of elements of interest. The important idea is that each element $x \in X$ is assigned a membership grade $u(x)$ taking values in [0,1], with $u(x) = 0$ corresponding to non-membership, $0 < u(x) < 1$ to partial membership, and $u(x) = 1$ to full membership. Zadeh says that fuzzy subset of X is a nonempty subset $\{(x, u(x)) : x \in X\}$ of $X \times [0, 1]$ for some function $u : X \to [0, 1]$. The function u itself is often used for the fuzzy set.

In this article, we develop numerical methods for solving fuzzy differential equations by an application of fourth order Runge-Kutta method. In Section 2 we list some basic definitions to fuzzy valued functions. Section 3 contains numerical methods with full fuzzy initial values. Section 4 contains the Runge-Kutta method for solving full fuzzy differential equations. Section 5 contains the numerical examples to illustrate the theory.

2. Preliminaries

First, we review fuzzy numbers and some results about it. There are various definitions for

the concept of fuzzy number. Let E^1 be the set of all functions $u : R \to [0,1]$ such that u is normal, fuzzy convex, upper semicontinuous and closure of $\{x \in R : u(x) > 0\}$, is compact. For any $u \in E$, u is called a fuzzy number in parametric form a pair $(\underline{u}(r), \overline{u}(r))$ of function $\underline{u}(r), \overline{u}(r)$, $0 \le r \le 1$ which satisfies the following requirements:

1. $\underline{u}(r)$ is bounded monotonic increasing left continuous function.
2. $\overline{u}(r)$ is bounded monotonic decreasing left continuous function.
3. $\underline{u}(r) \le \overline{u}(r)$, $0 \le r \le 1$.

In this paper, we used of parametric form of fuzzy numbers. For $u, v \in E^1$, the metric distance is define as

$$D(u,v) = \sup_{r \in [0,1]} max\{|\underline{u}(r) - \underline{v}r|, |\overline{u}(r) - \overline{v}(r)|\}. \tag{1}$$

Theorem 2.1. [25],
(1) (E, D) is a complete metric space,
(2) $D(u + w, \ v + w) = D(u,v)$ here $u, v, w \in E$,
(3) $D(u + v, w + e) \le D(u,w) + D(v,e)$, here $u, v, w, e \in E$. For ranking of $u, v \in E$, $u \ \le \ v$ if and only if $\underline{u}(r) \le \underline{v}(r)$ and $\overline{u}(r) \le \overline{v}(r)$ and $u < v$ if and only if $\underline{u}(r) < \underline{v}(r)$ and $\overline{u}(r) < \overline{v}(r)$ for any $r \in [0,1]$.

Definition 2.1.[17], Let $u_0, v_0 \in E, \overline{u_0}(0) < \underline{v_0}(0)$. The fuzzy number set $\{w_t \in E | w_t = (1 - t)u_0 + tv_0, t \in (-\infty, +\infty)\}$ is called fuzzy directed line induced by u_0, v_0 and denoted by $\overrightarrow{u_0 v_0}$.

Theorem 2.2. [17], Let $w_s, w_t \in \overrightarrow{u_0 v_0}$, then
(1)$s \le t \Leftrightarrow \ w_s \ \le \ w_t$
(2)$s \ = \ t \ \Leftrightarrow \ w_s \ = \ w_t, \ i.e. \ s \ \ne \ t \Leftrightarrow \ w_s \ \ne w_t.$

Definition 2.2.[24] For two fuzzy number $u, v \in E$, then

$$u + v = w \in E \longleftrightarrow \begin{cases} \underline{u} + \underline{v} = \underline{w}, \\ \overline{u} + \overline{v} = \overline{w}, \end{cases}$$

$$\lambda u = \begin{cases} (\lambda \underline{u}, \lambda \overline{u}), & \lambda \ge 0, \\ (\lambda \overline{u}, \lambda \underline{u}), & \lambda < 0, \end{cases}$$

$$u \ominus_g v = w \in E \longleftrightarrow \begin{cases} (i)u = v + w, \\ or \\ (ii)v = u + (-1)w, \end{cases} \tag{2}$$

$$uv = w \longleftrightarrow \begin{cases} \underline{w} = min\{\underline{u} \ \underline{v}, \overline{u} \ \overline{v}, \underline{u} \ \overline{v}, \overline{u} \ \underline{v}\}, \\ \overline{w} = max\{\underline{u} \ \underline{v}, \overline{u} \ \overline{v}, \underline{u} \ \overline{v}, \overline{u} \ \underline{v}\}, \end{cases}$$

$$u \div_g v = w \in E \longleftrightarrow \begin{cases} (i)u = vw, \\ or \\ (ii)v = uw^{-1}. \end{cases} \quad w^{-1} = (\frac{1}{\overline{w}(r)}, \ \frac{1}{\underline{w}(r)})$$

Theorem 2.3.[24] $A \ominus_g B$ exists if and only if $B \ominus_g A$ and $(-B) \ominus_g (-A)$ exist and $A \ominus_g B =$

$$(-B) \ominus_g (-A) = -(B \ominus_g A).$$

Definition 2.3.[17] Let $f : \overrightarrow{u_0 v_0} \longrightarrow E$ be a fuzzy mapping and $x \in R$ fixed. Suppose for $|y| > |x|$,

$xy > 0$, $f(w_y) \ominus f(w_x)$ exists, when $x < 0$, $\frac{f(w_y) \ominus f(w_x)}{w_y \ominus w_x}$ exists, when $x < 0$ $\frac{f(w_x) \ominus f(w_y)}{w_x \ominus w_y}$ exists.

And suppose for $|y| < |x|$, $xy \geq 0$, $f(w_x) \ominus f(w_y)$ exists, and when $x > 0$, $\frac{f(w_x) \ominus f(w_y)}{w_x \ominus w_y}$ exists,

when $x < 0$ $\frac{f(w_y) \ominus f(w_x)}{w_y \ominus w_x}$ exists. If there is a $F'(w_x) \in E$ such that when $x > 0$

$$\lim_{y \to x^+} D\left(\frac{f(w_y) \ominus f(w_x)}{w_y \ominus w_x}, f'(w_x) \right) = 0,$$

and

$$\lim_{y \to x^-} D\left(\frac{f(w_x) \ominus f(w_y)}{w_x \ominus w_y}, f'(w_x) \right) = 0,$$

hold. When $x < 0$

$$\lim_{y \to x^+} D\left(\frac{f(w_x) \ominus f(w_y)}{w_x \ominus w_y}, f'(w_x) \right) = 0,$$

and

$$\lim_{y \to x^-} D\left(\frac{f(w_y) \ominus f(w_x)}{w_y \ominus w_x}, f'(w_x) \right) = 0.$$

hold. Then we say f is fuzzy differentiable at w_x and its fuzzy derivative at w_x is $f'(w_x)$.

3. Numerical Method with Full Fuzzy Initial Values

In this section, we are going to study the differential equation with fully fuzzy initial values as

$$\begin{cases} y' = f(x, y) \\ y(u_0) = U_0, u_0, v_0 \in E \end{cases} \tag{3}$$

where $f : [u_0, v_0] \times E \longrightarrow E$ is such that $u_0, v_0 \in E, \overline{u_0}(0) < \underline{v_0}(0)$ and $D^2((x, y), (\overline{x}, \overline{y}) < \delta$.
(1)For every $\epsilon > 0$, there exists a $\delta > 0$ such that $D(f(x, y), \overline{f}(\overline{x}, \overline{y}) < \epsilon$,
$(x, y), (\overline{x}, \overline{y}) \in [u_0, v_0] \times E$.
(2)There exists a $L > 0$, $D(f(x, y_1), f(x, y_2)) \leq LD(y_1, y_2)$, $x, y_1, y_2 \in E$ For numerically solving equation (3), we approximate $y(v_0)$. Now we define the metric distance in E^2 as follows:

$$D^2((u, v), (u', v')) = max\{D(u, u'), D(v, v')\}, \qquad (u, v), (u', v') \in E^2. \tag{4}$$

Definition 3.1. Let $u_0, v_0 \in E, \overline{u_0}(0) < \underline{v_o}(0)$.The fuzzy number set $\{w_t \in E | w_t = (1 - t)u_0 + tv_0, t \in [0, 1]\}$, is called fuzzy interval$[u_0, v_0]$.

Definition 3.2. Suppose $[u_0, v_0]$is the fuzzy interval. If $p[0, 1] = \{x_0 = 0, x_1, ..., x_n = 1 | x_0 < x_1 < .... < x_n\}$, denote the partition of [0,1]. If $w_i = (1 - x_i)u_0 + x_i v_0; i = 0, 1, ..., n$, then, by Theorem 2.2 $w_0 = u_0 < w_1 < ..... < w_{n-1} < w_n = v_0$, Therefore $\widetilde{p}_{[u_0, v_0]} = \{w_0 = u_0, w_1......, w_{n-1}, w_n = v_0\}$ is the partition of $[u_0, v_0]$.

Theorem 3.1. Let $u \in E, m, n \in > 0$ then $mu \ominus_g nu = (m - n)u$.
Proof:

$$mu \ominus_g nu = (m - n)u \longleftrightarrow \begin{cases} (i)mu = nu + (m - n)u, \\ or \\ (ii)nu = mu + (-1)(m - n)u, \end{cases} \tag{5}$$

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(i)$(m\underline{u}, m\overline{u}) = (n\underline{u}, n\overline{u}) + (m-n)(\underline{u}, \overline{u})$ if $m \geq n$ this case is correct.

(ii) $(n\underline{u}, n\overline{u}) = (m\underline{u}, m\overline{u}) + (n-m)(\underline{u}, \overline{u})$ if $m < n$ this case is correct.

Regarding to generalization of Hukuhara difference and division and definition 2.3, differentiability of F at $w_x$is proved in the following theorem.

Theorem 3.2. Let $F : \overrightarrow{u_0 v_0} \longrightarrow E$ be a fuzzy mapping and $x \in R^+$ if $f'(w_x) \in E$ exist and

$$\lim_{h \longrightarrow 0} D\Big(\{f(w_{x+h}) \ominus_g f(w_x)\} \div_g \{w_{x+h} \ominus_g w_x\}, f'(w_x)\Big) = 0$$

then we say f is fuzzy differentiable at w_x and its fuzzy derivative at w_x is $F'(w_x)$.

Proof:

$$f(w_x) \ominus_g f(w_{x+h}) = -(f(w_{x+h}) \ominus f(w_x))$$
$$w_x \ominus_g w_{x+h} = -(w_{x+h} \ominus_g w_x) \tag{6}$$

Regarding to, Definition 2.3 and $x \in R^+$. If $h \longrightarrow 0^+$ then $|x+h| > |x|$, $(x+h).x > 0$, $f(w_{x+h} \ominus_g f(w_x), \{f(w_{x+h} \ominus_g f(w_x)\} \div_g \{w_{x+h} \ominus_g w_x\}$ exits, If $h \longrightarrow 0^-$ then $|x+h| < |x|$, $(x+h).x > 0$, $f(w_x) \ominus_g f(w_{x+h}), \{f(w_x \ominus_g f(w_{x+h})\} \div_g \{w_x \ominus_g w_{x+h}\}$ exits, and with

$$\lim_{h \longrightarrow 0} D \left(\{f(w_{x+h}) \ominus_g f(w_x)\} \div_g \{w_{x+h} \ominus_g w_x\}, f'(w_x) \right) = 0,$$

then

$$\lim_{h \longrightarrow 0^+} D \left(\{f(w_{x+h}) \ominus_g f(w_x)\} \div_g \{w_{x+h} \ominus_g w_x\}, f'(w_x) \right) = 0,$$

and relations (4,5)

$$\lim_{h \longrightarrow 0^-} D \left(\{f(w_x) \ominus_g f(w_{x+h})\} \div_g \{w_x \ominus_g w_{x+h}\}, f'(w_x) \right) = 0.$$

Hence, proof is complete

For approximation $y(v_0)$ in problem (1), we consider $p_{[0,1]} = \{ \frac{i}{n} | i = 0, 1, ...n\}$ as a partition of [0,1], using definition 3.2 $\widetilde{p}_{[u_0, v_0]} = \{w_i = (1 - \frac{i}{n})u_0 + \frac{i}{n}v_0 | i = 0, 1, ..., n\}$ is the partition of $[u_0, v_0]$. Suppose that $y'(w_i)$exists, hence using Theorem 3.2

$$\lim_{h \longrightarrow 0} D \left(\{y(w_{i+i}) \ominus_g y(w_i)\} \div_g \{w_{i+1} \ominus_g w_i\}, y'(w_i) \right) = 0 \tag{7}$$

then by (1) where $y'(w_i) = f(w_i, y(w_i))$.

4. The Runge-Kutta method

The Runge-Kutta method is a fourth order approximation of $Y_1'(t;r)$and $Y_2'(t;r)$ to develop the Runge-Kutta method, we define

$$\underline{y}_{k,n+1}(r) - \underline{y}_{k,n}(r) = \sum_{i=1}^{4} w_i \underline{k_i}(t_{k,n}; y_{k,n}(r)),$$

$$\overline{y}_{k,n+1}(r) - \underline{y}_{k,n}(r) = \sum_{i=1}^{4} w_i \overline{k_i}(t_{k,n}; y_{k,n}(r)),$$

Where w_1, w_2, w_3, w_4 are constants and

$$
\begin{aligned}
k_{1,1} &= min\{hf(t, u)|u \in [y_1(t, r), y_2(t, r)]\}, \\
k_{1,2} &= max\{hf(t, u)|u \in [y_1(t, r), y_2(t, r)]\}, \\
k_{2,1} &= min\{hf(t + \frac{1}{2}h, u)|u \in [z_{1,1}(t, y(t, r)), z_{1,2}(t, y(t, r))]\}, \\
k_{2,2} &= max\{hf(t + \frac{1}{2}h, u)|u \in [z_{1,1}(t, y(t, r)), z_{1,2}(t, y(t, r))]\}, \\
k_{3,1} &= min\{hf(t + \frac{1}{2}h, u)|u \in [z_{2,1}(t, y(t, r)), z_{2,2}(t, y(t, r))]\}, \\
k_{3,2} &= max\{hf(t + \frac{1}{2}h, u)|u \in [z_{2,1}(t, y(t, r)), z_{2,2}(t, y(t, r))]\}, \\
k_{4,1} &= min\{hf(t + h, u)|u \in [z_{3,1}(t, y(t, r)), z_{3,2}(t, y(t, r))]\}, \\
k_{4,2} &= max\{hf(t + h, u)|u \in [z_{3,1}(t, y(t, r)), z_{3,2}(t, y(t, r))]\},
\end{aligned}
$$

Next we define

$$
\begin{aligned}
z_{1,1}(t, y(t, r)) &= y_1(t, r) + \frac{1}{2}k_{1,1}(t, y(t, r)), \\
z_{1,2}(t, y(t, r)) &= y_2(t, r) + \frac{1}{2}k_{1,2}(t, y(t, r)), \\
z_{2,1}(t, y(t, r)) &= y_1(t, r) + \frac{1}{2}k_{2,1}(t, y(t, r)), \\
z_{2,2}(t, y(t, r)) &= y_2(t, r) + \frac{1}{2}k_{2,2}(t, y(t, r)), \\
z_{3,1}(t, y(t, r)) &= y_1(t, r) + k_{3,1}(t, y(t, r)), \\
z_{3,2}(t, y(t, r)) &= y_2(t, r) + k_{3,2}(t, y(t, r)).
\end{aligned}
$$

Now we define

$$
\begin{aligned}
F[t, y(t, r)] &= k_{1,1}(t, y(t, r)) + 2k_{2,1}(t, y(t, r)) + 2k_{3,1}(t, y(t, r)) + k_{4,1}(t, y(t, r)), \\
G[t, y(t, r)] &= k_{1,2}(t, y(t, r)) + 2k_{2,2}(t, y(t, r)) + 2k_{3,2}(t, y(t, r)) + k_{4,2}(t, y(t, r)).
\end{aligned}
$$

The exact and approximate solution at $t_{n,0} \leq n \leq N$ are denoted by

$$
[Y(t_n)]_r = (Y_1(t_n, r), Y_2(t_n, r)),
$$

$$
[y(t_n)]_r = [y_1(t_n, r), y_2(t_n, r)], \text{ as }
$$

$$
Y_1(t_{n+1}, r) \simeq Y_1(t_n, r) + \frac{1}{6}F[t_n, Y(t_n, r)],
$$

$$
Y_2(t_{n+1}, r) \simeq Y_2(t_n, r) + \frac{1}{6}G[t_n, Y(t_n, r)], \text{ and }
$$

$$
y_1(t_{n+1}, r) = y_1(t_n, r) + \frac{1}{6}F[t_n, Y(t_n, r)],
$$

$$
y_2(t_{n+1}, r) = y_2(t_n, r) + \frac{1}{6}G[t_n, Y(t_n, r)].
$$

With Theorem 3.1 and $w_i = (1 - \frac{i}{n})u_0 + \frac{i}{n}v_0$ then $w_{i+1} \ominus_g w_i = -\frac{1}{n}u_0 + \frac{1}{n}v_0$ hence with division of (2.2). Eq (7) rewritten as

$$
\begin{cases}
y_1(w_{i+1} \ominus_g y_1(w_i)) \simeq (\frac{1}{n}v_0 + (-\frac{1}{n})u_0)\frac{1}{6}(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}) \in E, \\
\\
y_1(w_0) = U_0 \qquad i = 0, 1, ...n - 1,
\end{cases}
\tag{8}
$$

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$$\begin{cases} y_2(w_{i+1}) \ominus_g y_2(w_i)) \simeq (\frac{1}{n}v_0 + (-\frac{1}{n})u_0)\frac{1}{6}\left(k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2}\right) \in E, \\ \\ y_2(w_0) = U_0 \qquad i = 0, 1, ...n - 1. \end{cases} \tag{9}$$

Therefore by difference of (2)

$$\begin{cases} y_1(w_{i+1}) \simeq y_1(w_i) + (\frac{1}{n}v_0 + (-\frac{1}{n})u_0)\frac{1}{6}\left(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}\right), \\ \\ y_1(w_0) = U_0 \qquad i = 0, 1, ...n - 1, \end{cases} \tag{10}$$

$$\begin{cases} y_2(w_{i+1}) \simeq y_2(w_i) + (\frac{1}{n}v_0 + (-\frac{1}{n})u_0)\frac{1}{6}\left(k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2}\right), \\ \\ y_2(w_0) = U_0 \qquad i = 0, 1, ...n - 1. \end{cases} \tag{11}$$

We will replace the exact solution $Y(w_i)$; $i = 0, 1...., n$ by approximated solution $y(w_i)$;i=0,1.....n and then Eq (9) and Eq(10) can be rewriten as

$$\begin{cases} Y_1(w_{i+1}) \simeq Y_1(w_i) + (\frac{1}{n}v_0 + (-\frac{1}{n})u_0)\frac{1}{6}\left(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}\right), \\ \\ Y_1(w_0) = U_0 \qquad i = 0, 1, ...n - 1, \end{cases} \tag{12}$$

and

$$\begin{cases} Y_2(w_{i+1}) \simeq Y_2(w_i) + (\frac{1}{n}v_0 + (-\frac{1}{n})u_0)\frac{1}{6}\left(k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2}\right), \\ \\ Y_2(w_0) = U_0 \qquad i = 0, 1, ...n - 1. \end{cases} \tag{13}$$

For illustrating of our proposed method ,we solve an example as follows:

5. Numerical Examples :
Example 5.1.
Consider the following differential equation with fully fuzzy initial value.

$$\begin{cases} y' = xy \\ \\ y(u_0) = U_0 = (e^{0.005r^2+1}, e^{0.005r^2-0.02r+1.02}) \end{cases} \tag{14}$$

where $u_0 = (0.1r, 0.2 - 0.1r)$ and $F(x, y) = xy$. We want to approximate $y(v_0)$ by $v_0 = (0.3 + 0.1r, 0.5 - 0.1r)$.
We show that $f : [u_0, v_0] \times E \longrightarrow E$ satisfied in condition (1) and (2).
Therefore (1) is satisfied if for fixe $\epsilon > 0, 0 < \delta < \frac{\epsilon}{|\overline{v_0}(0)|+M}$ exits such that

$$M \geq \max_{x \in [\underline{u_0}(0), \overline{v_0}(0)]} \{|y(x)|\}$$

Assume $D^2((x, y), (x', y')) = max\{D(x, x'), D(y, y')\} < \delta$
then, $D(f(x, y), f(x', y')) = \sup_{r \in [0,1]} max\{\left|\underline{xy}(r) - \underline{x'y'}(r)\right|, \left|\overline{xy}(r) - \overline{x'y'}(r)\right|\} = \Phi$
(i) If

$$\begin{aligned} \Phi &= \left|\underline{x}(r^*)\underline{y}(r^*) - \underline{x'}(r^*)\underline{y'}(r^*)\right|, \\ &\leq \left|\underline{x}(r^*)\right| D(y, y') + \left|\underline{y}(r^*)\right| D(x, x'), \\ &< \left|\overline{v_0}(0)\right| \delta + M\delta, \\ &= (\left|\overline{v_0}(0)\right| + M)\delta, \\ &< \epsilon. \end{aligned}$$

(ii) If

$$
\begin{aligned}
\Phi &= \left| \overline{x}(r^*)\overline{y}(r^*) - \overline{x'}(r^*)\overline{y'}(r^*) \right|, \\
&\leq \left| \overline{x}(r^*) \right| D(y, y') + \left| \overline{y}(r^*) \right| D(x, x'), \\
&< \left| \overline{v_0}(0) \right| \delta + M\delta, \\
&= (\left| \overline{v_0}(0) \right| + M)\delta, \\
&< \epsilon.
\end{aligned}
$$

(2) is satisfied,since

$$
D(F(x, y), F(x, y')) = D(xy, xy') = \sup_{r \in [0,1]} max\{ \left| \underline{xy}(r) - \underline{xy'}(r) \right|, \left| \overline{xy}(r) - \overline{xy'}(r) \right| \} = \Psi
$$

I. If $\Psi = \left| \overline{x}(r^*)\overline{y}(r^*) - \overline{x}(r^*)\overline{y'}(r^*) \right| = \left| \overline{x}(r^*) \right| \left| \overline{y}(r^*) - \overline{y'}(r^*) \right| \leq LD(y, y')$,
II. If $\Psi = \left| \underline{x}(r^*)\underline{y}(r^*) - \underline{x}(r^*)\underline{y'}(r^*) \right| = \left| \underline{x}(r^*) \right| \left| \underline{y}(r^*) - \underline{y'}(r^*) \right| \leq LD(y, y')$.

We used of n=4 with $p[0, 1] = \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}\}$ hence

$$
\begin{aligned}
w_0 &= u_0 = (0.1r, 0.2 - 0.1r), \\
w_1 &= \frac{3}{4}u_0 + \frac{1}{4}v_0 = (0.075 + 0.1r, 0.275 + 0.1r), \\
w_2 &= \frac{1}{2}u_0 + \frac{1}{2}v_0 = (0.15 + 0.1r, 0.35 + 0.1r), \\
w_3 &= \frac{1}{4}u_0 + \frac{3}{4}v_0 = (0.225 + 0.1r, 0.425 + 0.1r), \\
w_4 &= v_0 = (0.3 + 0.1r, 0.5 - 0.1r), \\
y_1(w_{i+1}) &\simeq y_1(w_i) + \frac{1}{6}(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}), \\
y_2(w_{i+1}) &\simeq y_2(w_i) + \frac{1}{6}(k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2}),
\end{aligned}
$$

where $i = 0, 1, 2, 3$

By
$$
\frac{1}{4}v_0 + (-\frac{1}{4})u_0 = (0.025 + 0.05r, 0.125 - 0.05r)
$$

$$
y_1(w_{i+1}) \simeq y_1(w_i) + \frac{1}{6}(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}), i = 0, 1, 2, 3,
$$

$$
y_2(w_{i+1}) \simeq y_2(w_i) + \frac{1}{6}(k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2}), i = 0, 1, 2, 3,
$$

$$
y_1(w_4) = y_1(v_0) \simeq y_1(w_3) + \frac{1}{6}(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}),
$$

$$
y_2(w_4) = y_2(v_0) \simeq y_2(w_3) + \frac{1}{6}(k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2}).
$$

Where,

$$
\begin{aligned}
k_{1,1}(w_i, y(w_i)) &= min\{(0.025 + 0.05r, 0.125 - 0.05r)f(w_i, u(w_i))| \\
&\quad u(w_i) \in [y_1(t, r), y_2(w_i, r)]\}, \\
k_{1,2}(w_i, y(w_i)) &= max\{(0.025 + 0.05r, 0.125 - 0.05r)f(w_i, u(w_i))| \\
&\quad u(w_i) \in [y_1(t, r), y_2(w_i, r)]\},
\end{aligned}
$$

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$$
\begin{aligned}
k_{2,1}(w_i, y(w_i)) &= min\{(0.025 + 0.05r , 0.125 - 0.05r)f(w_i + \frac{(0.025 + 0.05r)}{2}, u(w_i))| \\
& u(w_i) \in [z_{1,1}(w_i, y(w_i, r)), z_{1,2}(w_i, y(w_i, r))]\}, \\
k_{2,2}(w_i, y(w_i)) &= max\{(0.025 + 0.05r , 0.125 - 0.05r)f(w_i + \frac{0.125 - 0.05r}{2}, u(w_i))| \\
& u(w_i) \in [z_{1,1}(w_i, y(w_i, r)), z_{1,2}(w_i, y(w_i, r))]\}, \\
k_{3,1}(w_i, y(w_i)) &= min\{(0.025 + 0.05r , 0.125 - 0.05r)f(w_i + \frac{0.025 + 0.05r}{2}, u(w_i))| \\
& u(w_i) \in [z_{2,1}(w_i, y(w_i, r)), z_{2,2}(w_i, y(w_i, r))]\}, \\
k_{3,2}(w_i, y(w_i)) &= max\{(0.025 + 0.05r , 0.125 - 0.05r)f(w_i + \frac{0.125 - 0.05r}{2}, u(w_i))| \\
& u(w_i) \in [z_{2,1}(w_i, y(w_i, r)), z_{2,2}(w_i, y(w_i, r))]\},
\end{aligned}
$$

$$
\begin{aligned}
k_{4,1}(w_i, y(w_i)) &= min\{(0.025 + 0.05r , 0.125 - 0.05r)f(w_i + 0.025 + 0.05r, u(w_i))| \\
& u(w_i) \in [z_{3,1}(w_i, y(w_i, r)), z_{3,2}(w_i, y(w_i, r))]\}, \\
k_{4,2}(w_i, y(w_i)) &= max\{(0.025 + 0.05r , 0.125 - 0.05r)f(w_i + 0.125 - 0.05r, u(w_i))| \\
& u(w_i) \in [z_{3,1}(w_i, y(w_i, r)), z_{3,2}(w_i, y(w_i, r))]\},
\end{aligned}
$$

$$
\begin{aligned}
z_{1,1}(w_i, y(w_i, r)) &= y_1(w_i, r) + \frac{1}{2}k_{1,1}(w_i, y(w_i, r)), \\
z_{1,2}(w_i, y(w_i, r)) &= y_2(w_i, r) + \frac{1}{2}k_{1,2}(w_i, y(w_i, r)), \\
z_{2,1}(w_i, y(w_i, r)) &= y_1(w_i, r) + \frac{1}{2}k_{2,1}(w_i, y(w_i, r)), \\
z_{2,2}(w_i, y(w_i, r)) &= y_2(w_i, r) + \frac{1}{2}k_{2,2}(w_i, y(w_i, r)), \\
z_{3,1}(w_i, y(w_i, r)) &= y_1(w_i, r) + k_{3,1}(w_i, y(w_i, r)), \\
z_{3,2}(w_i, y(w_i, r)) &= y_2(w_i, r) + k_{3,2}(w_i, y(w_i, r)).
\end{aligned}
$$

The exact solution of (11) is $Y = e^{\frac{x^2}{2}+1}$ therefore $Y(v_0) = e^{\frac{v_0^2}{2}+1} = (e^{0.005r^2+0.03r+1.045}, e^{0.005r^2-0.05r+1.125})$
If we used of n=8 with $p[0,1] = \{0, \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8}, \frac{8}{8}\}$ then

$$
\begin{aligned}
w_0 &= u_0 = (0.1r, 0.2 - 0.1r), & w_1 &= (0.0375 + 0.1r, 0.2375 + 0.1r), \\
w_2 &= (0.075 + 0.1r, 0.275 + 0.1r), & w_3 &= (0.1125 + 0.1r, 0.3125 + 0.1r), \\
w_4 &= (0.15 + 0.1r, 0.35 - 0.1r), & w_5 &= (0.1875 + 0.1r, 0.3875 + 0.1r), \\
w_6 &= (0.225 + 0.1r, 0.425 + 0.1r), & w_7 &= (0.2625 + 0.1r, 0.4625 + 0.1r), \\
w_8 &= (0.3 + 0.1r, 0.5 - 0.1r). &
\end{aligned}
$$

$$
\begin{aligned}
y_1(w_{i+1}) &\simeq y_1(w_i) + \frac{1}{6}(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}), \\
y_2(w_{i+1}) &\simeq y_2(w_i) + \frac{1}{6}(k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2}),
\end{aligned}
$$

where $i = 0, 1, ..., 7$
By
$$\frac{1}{8}v_0 + (-\frac{1}{8})u_0 = (0.0125 + 0.025r, 0.0625 - 0.025r),$$

$$y_1(w_{i+1}) \simeq y_1(w_i) + \frac{1}{6}(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}), i = 0, 1, 2, ..., 7,$$

$$y_2(w_{i+1}) \simeq y_2(w_i) + \frac{1}{6}(k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2}), i = 0, 1, 2, ..., 7,$$
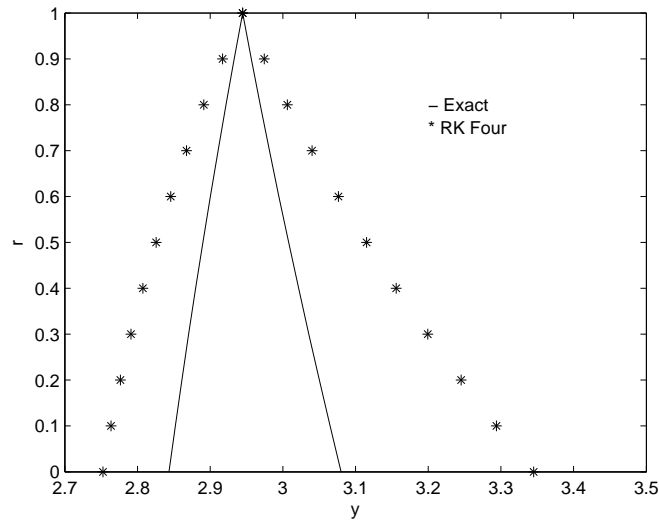
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Figure 1: (for n=4)

$$y_1(w_8) = y_1(v_0) \simeq y_1(w_7) + \frac{1}{6}\left(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}\right),$$

$$y_2(w_8) = y_2(v_0) \simeq y_2(w_7) + \frac{1}{6}\left(k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2}\right).$$

Where,

$$
\begin{aligned}
k_{1,1}(w_i, y(w_i)) &= min\{(0.0125 + 0.025r \ , \ 0.0625 - 0.025r)f(w_i, u(w_i))| \\
&\quad u(w_i) \in [y_1\left(t,r\right), y_2\left(w_i, r\right)]\}, \\
k_{1,2}(w_i, y(w_i)) &= max\{(0.0125 + 0.025r \ , \ 0.0625 - 0.025r)f(w_i, u(w_i))| \\
&\quad u(w_i) \in [y_1\left(t,r\right), y_2\left(w_i, r\right)]\}, \\
k_{2,1}(w_i, y(w_i)) &= min\{(0.0125 + 0.025r \ , \ 0.0625 - 0.025r)f(w_i + \frac{(0.0125 + 0.025r)}{2}, u(w_i))| \\
&\quad u(w_i) \in [z_{1,1}\left(w_i, y\left(w_i, r\right)\right), z_{1,2}\left(w_i, y\left(w_i, r\right)\right)]\}, \\
k_{2,2}(w_i, y(w_i)) &= max\{(0.0125 + 0.025r \ , \ 0.0625 - 0.025r)f(w_i + \frac{0.0625 - 0.025r}{2}, u(w_i))| \\
&\quad u(w_i) \in [z_{1,1}\left(w_i, y\left(w_i, r\right)\right), z_{1,2}\left(w_i, y\left(w_i, r\right)\right)]\}, \\
k_{3,1}(w_i, y(w_i)) &= min\{(0.0125 + 0.025r \ , \ 0.0625 - 0.025r)f(w_i + \frac{0.0125 + 0.025r}{2}, u(w_i))| \\
&\quad u(w_i) \in [z_{2,1}\left(w_i, y\left(w_i, r\right)\right), z_{2,2}\left(w_i, y\left(w_i, r\right)\right)]\}, \\
k_{3,2}(w_i, y(w_i)) &= max\{(0.0125 + 0.025r \ , \ 0.0625 - 0.025r)f(w_i + \frac{0.0625 - 0.025r}{2}, u(w_i))| \\
&\quad u(w_i) \in [z_{2,1}\left(w_i, y\left(w_i, r\right)\right), z_{2,2}\left(w_i, y\left(w_i, r\right)\right)]\}, \\
k_{4,1}(w_i, y(w_i)) &= min\{(0.0125 + 0.025r \ , \ 0.0625 - 0.025r)f(w_i + 0.0125 + 0.025r, u(w_i))| \\
&\quad u(w_i) \in [z_{3,1}\left(w_i, y\left(w_i, r\right)\right), z_{3,2}\left(w_i, y\left(w_i, r\right)\right)]\}, \\
k_{4,2}(w_i, y(w_i)) &= max\{(0.0125 + 0.025r \ , \ 0.0625 - 0.025r)f(w_i + 0.0625 - 0.025r, u(w_i))| \\
&\quad u(w_i) \in [z_{3,1}\left(w_i, y\left(w_i, r\right)\right), z_{3,2}\left(w_i, y\left(w_i, r\right)\right)]\},
\end{aligned}
$$

$$z_{1,1}\left(w_i, y\left(w_i, r\right)\right) = y_1(w_i, r) + \frac{1}{2}k_{1,1}\left(w_i, y\left(w_i, r\right)\right),$$

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$$
\begin{aligned}
z_{1,2}\left(w_i, y\left(w_i, r\right)\right) &= y_2(w_i, r) + \frac{1}{2}k_{1,2}\left(w_i, y\left(w_i, r\right)\right), \\
z_{2,1}\left(w_i, y\left(w_i, r\right)\right) &= y_1(w_i, r) + \frac{1}{2}k_{2,1}\left(w_i, y\left(w_i, r\right)\right), \\
z_{2,2}\left(w_i, y\left(w_i, r\right)\right) &= y_2(w_i, r) + \frac{1}{2}k_{2,2}\left(w_i, y\left(w_i, r\right)\right), \\
z_{3,1}\left(w_i, y\left(w_i, r\right)\right) &= y_1(w_i, r) + k_{3,1}\left(w_i, y\left(w_i, r\right)\right), \\
z_{3,2}\left(w_i, y\left(w_i, r\right)\right) &= y_2(w_i, r) + k_{3,2}\left(w_i, y\left(w_i, r\right)\right).
\end{aligned}
$$

The exact solution of (11) is $Y = e^{\frac{x^2}{2}+1}$, therefore

$$
Y(v_0) = e^{\frac{v_0{}^2}{2}+1} = \left(e^{0.005r^2+0.03r+1.045}, e^{0.005r^2-0.05r+1.125}\right)
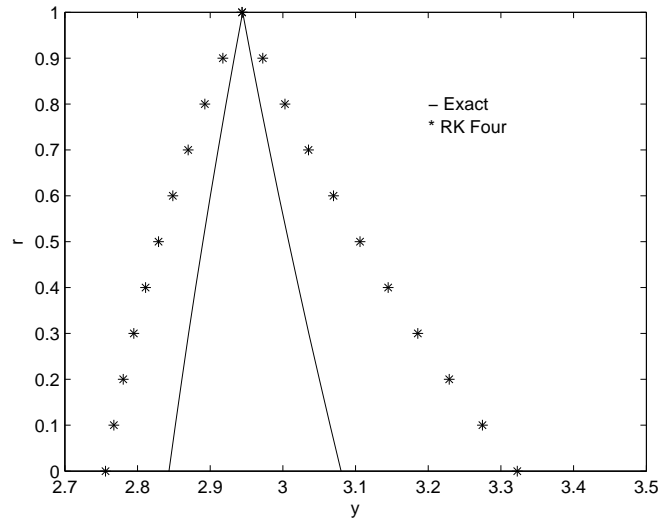$$

Figure 2: (for n=8)

Example 5.2
Consider the following differential equation with fully fuzzy initial value.

$$
\begin{cases}
y'(t) = y(t) \\
y(u_0) = U_0 = \left(e^{0.1r}, e^{0.2-0.1r}\right)
\end{cases}
\tag{15}
$$

where $u_0 = (0.1r, 0.2 - 0.1r)$ and $f(x, y(t)) = y(t)$. We want to approximate $y(v_0)$ by $v_0 = (0.3 + 0.1r, 0.5 - 0.1r)$.
Again we used of n=4, we define w_0, w_1, w_2, w_3, w_4 similarly as that of the Example 3.1.
Where

$$
y_1(w_{i+1}) \simeq y_1(w_i) + \frac{1}{6}\left(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}\right),
$$

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$$y_2(w_{i+1}) \quad \simeq \quad y_2(w_i) + \frac{1}{6}\left(k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2}\right),$$

where $i = 0, 1, 2, 3$

By
$$\tfrac{1}{4}v_0 + (-\tfrac{1}{4})u_0 = (0.025 + 0.05r, 0.125 - 0.05r),$$

$$y_1(w_{i+1}) \simeq y_1(w_i) + \frac{1}{6}\left(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}\right), i = 0, 1, 2, 3,$$

$$y_2(w_{i+1}) \simeq y_2(w_i) + \frac{1}{6}\left(k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2}\right), i = 0, 1, 2, 3,$$

$$y_1(w_4) = y_1(v_0) \simeq y_1(w_3) + \frac{1}{6}\left(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}\right),$$

$$y_2(w_4) = y_2(v_0) \simeq y_2(w_3) + \frac{1}{6}\left(k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2}\right).$$

Where,

$$
\begin{aligned}
k_{1,1}(w_i, y(w_i)) &= min\{(0.025 + 0.05r \ , \ 0.125 - 0.05r)f(w_i, u(w_i))| \\
&\qquad u(w_i) \in [y_1(t, r), y_2(w_i, r)]\}, \\
k_{1,2}(w_i, y(w_i)) &= max\{(0.025 + 0.05r \ , \ 0.125 - 0.05r)f(w_i, u(w_i))| \\
&\qquad u(w_i) \in [y_1(t, r), y_2(w_i, r)]\}, \\
k_{2,1}(w_i, y(w_i)) &= min\{(0.025 + 0.05r \ , \ 0.125 - 0.05r)f(w_i, u(w_i))| \\
&\qquad u(w_i) \in [z_{1,1}(w_i, y(w_i, r)), z_{1,2}(w_i, y(w_i, r))]\}, \\
k_{2,2}(w_i, y(w_i)) &= max\{(0.025 + 0.05r \ , \ 0.125 - 0.05r)f(w_i, u(w_i))| \\
&\qquad u(w_i) \in [z_{1,1}(w_i, y(w_i, r)), z_{1,2}(w_i, y(w_i, r))]\}, \\
k_{3,1}(w_i, y(w_i)) &= min\{(0.025 + 0.05r \ , \ 0.125 - 0.05r)f(w_i, u(w_i))| \\
&\qquad u(w_i) \in [z_{2,1}(w_i, y(w_i, r)), z_{2,2}(w_i, y(w_i, r))]\}, \\
k_{3,2}(w_i, y(w_i)) &= max\{(0.025 + 0.05r \ , \ 0.125 - 0.05r)f(w_i, u(w_i))| \\
&\qquad u(w_i) \in [z_{2,1}(w_i, y(w_i, r)), z_{2,2}(w_i, y(w_i, r))]\}, \\
k_{4,1}(w_i, y(w_i)) &= min\{(0.025 + 0.05r \ , \ 0.125 - 0.05r)f(w_i, u(w_i))| \\
&\qquad u(w_i) \in [z_{3,1}(w_i, y(w_i, r)), z_{3,2}(w_i, y(w_i, r))]\}, \\
k_{4,2}(w_i, y(w_i)) &= max\{(0.025 + 0.05r \ , \ 0.125 - 0.05r)f(w_i, u(w_i))| \\
&\qquad u(w_i) \in [z_{3,1}(w_i, y(w_i, r)), z_{3,2}(w_i, y(w_i, r))]\},
\end{aligned}
$$

$$
\begin{aligned}
z_{1,1}(w_i, y(w_i, r)) &= y_1(w_i, r) + \frac{1}{2}k_{1,1}(w_i, y(w_i, r)), \\
z_{1,2}(w_i, y(w_i, r)) &= y_2(w_i, r) + \frac{1}{2}k_{1,2}(w_i, y(w_i, r)), \\
z_{2,1}(w_i, y(w_i, r)) &= y_1(w_i, r) + \frac{1}{2}k_{2,1}(w_i, y(w_i, r)), \\
z_{2,2}(w_i, y(w_i, r)) &= y_2(w_i, r) + \frac{1}{2}k_{2,2}(w_i, y(w_i, r)), \\
z_{3,1}(w_i, y(w_i, r)) &= y_1(w_i, r) + k_{3,1}(w_i, y(w_i, r)), \\
z_{3,2}(w_i, y(w_i, r)) &= y_2(w_i, r) + k_{3,2}(w_i, y(w_i, r)).
\end{aligned}
$$

The exact solution of (15) is $Y = e^x$ therefore $Y(v_0) = e^{v_0} = (e^{0.3+0.1r}, e^{0.5-0.1r})$.
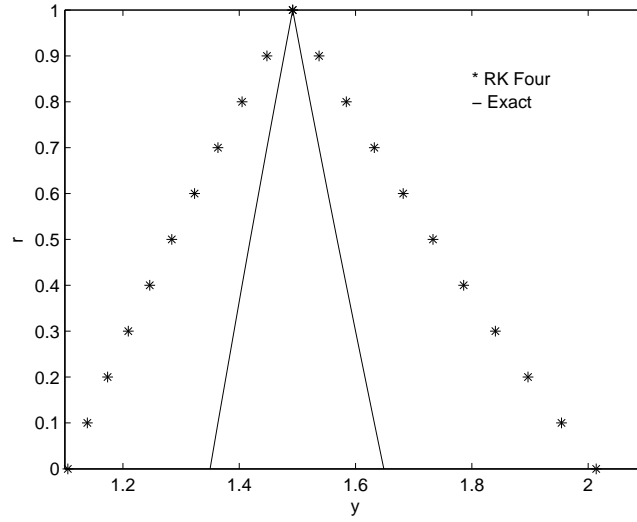
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Figure 3: (for n=4)

Again if we used of n=8, we define $w_0, w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8$ similarly as that of the Example 3.1.
Where

$$
\begin{aligned}
y_1(w_{i+1}) &\simeq y_1(w_i) + \frac{1}{6}\left(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}\right), \\
y_2(w_{i+1}) &\simeq y_2(w_i) + \frac{1}{6}\left(k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2}\right),
\end{aligned}
$$

where $i = 0, 1, ..., 7$
By $\qquad \frac{1}{8}v_0 + (-\frac{1}{8})u_0 = (0.0125 + 0.025r, 0.0625 - 0.025r)$

$$
y_1(w_{i+1}) \simeq y_1(w_i) + \frac{1}{6}\left(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}\right), i = 0, 1, 2, ..., 7,
$$

$$
y_2(w_{i+1}) \simeq y_2(w_i) + \frac{1}{6}\left(k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2}\right), i = 0, 1, 2, ..., 7,
$$

$$
y_1(w_8) = y_1(v_0) \simeq y_1(w_7) + \frac{1}{6}\left(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}\right),
$$

$$
y_2(w_8) = y_2(v_0) \simeq y_2(w_7) + \frac{1}{6}\left(k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2}\right),
$$

Where,

$$
\begin{aligned}
k_{1,1}(w_i, y(w_i)) &= min\{(0.0125 + 0.025r \ , \ 0.0625 - 0.025r)f(w_i, u(w_i))| \\
&\qquad u(w_i) \in [y_1\left(t, r\right), y_2\left(w_i, r\right)]\}, \\
k_{1,2}(w_i, y(w_i)) &= max\{(0.0125 + 0.025r \ , \ 0.0625 - 0.025r)f(w_i, u(w_i))| \\
&\qquad u(w_i) \in [y_1\left(t, r\right), y_2\left(w_i, r\right)]\}, \\
k_{2,1}(w_i, y(w_i)) &= min\{(0.0125 + 0.025r \ , \ 0.0625 - 0.025r)f(w_i, u(w_i))| \\
&\qquad u(w_i) \in [z_{1,1}\left(w_i, y\left(w_i, r\right)\right), z_{1,2}\left(w_i, y\left(w_i, r\right)\right)]\}, \\
k_{2,2}(w_i, y(w_i)) &= max\{(0.0125 + 0.025r \ , \ 0.0625 - 0.025r)f(w_i, u(w_i))| \\
&\qquad u(w_i) \in [z_{1,1}\left(w_i, y\left(w_i, r\right)\right), z_{1,2}\left(w_i, y\left(w_i, r\right)\right)]\},
\end{aligned}
$$

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$$
\begin{aligned}
k_{3,1}(w_i, y(w_i)) &= min\{(0.0125 + 0.025r \;,\; 0.0625 - 0.025r)f(w_i, u(w_i))| \\
&\quad u(w_i) \in [z_{2,1}(w_i, y(w_i, r)), z_{2,2}(w_i, y(w_i, r))]\}, \\
k_{3,2}(w_i, y(w_i)) &= max\{(0.0125 + 0.025r \;,\; 0.0625 - 0.025r)f(w_i, u(w_i))| \\
&\quad u(w_i) \in [z_{2,1}(w_i, y(w_i, r)), z_{2,2}(w_i, y(w_i, r))]\}, \\
k_{4,1}(w_i, y(w_i)) &= min\{(0.0125 + 0.025r \;,\; 0.0625 - 0.025r)f(w_i, u(w_i))| \\
&\quad u(w_i) \in [z_{3,1}(w_i, y(w_i, r)), z_{3,2}(w_i, y(w_i, r))]\}, \\
k_{4,2}(w_i, y(w_i)) &= max\{(0.0125 + 0.025r \;,\; 0.0625 - 0.025r)f(w_i, u(w_i))| \\
&\quad u(w_i) \in [z_{3,1}(w_i, y(w_i, r)), z_{3,2}(w_i, y(w_i, r))]\}, \\
z_{1,1}(w_i, y(w_i, r)) &= y_1(w_i, r) + \frac{1}{2}k_{1,1}(w_i, y(w_i, r)), \\
z_{1,2}(w_i, y(w_i, r)) &= y_2(w_i, r) + \frac{1}{2}k_{1,2}(w_i, y(w_i, r)), \\
z_{2,1}(w_i, y(w_i, r)) &= y_1(w_i, r) + \frac{1}{2}k_{2,1}(w_i, y(w_i, r)), \\
z_{2,2}(w_i, y(w_i, r)) &= y_2(w_i, r) + \frac{1}{2}k_{2,2}(w_i, y(w_i, r)), \\
z_{3,1}(w_i, y(w_i, r)) &= y_1(w_i, r) + k_{3,1}(w_i, y(w_i, r)), \\
z_{3,2}(w_i, y(w_i, r)) &= y_2(w_i, r) + k_{3,2}(w_i, y(w_i, r)).
\end{aligned}
$$

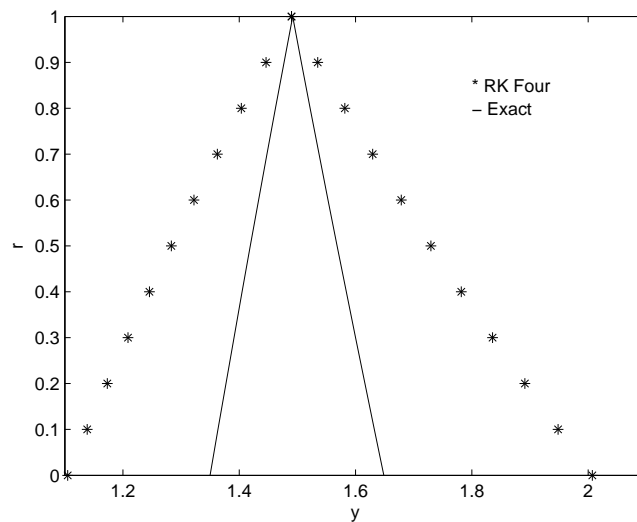The exact solution of(15) is $Y = e^x$ therefore $Y(v_0) = e^{v_0} = (e^{0.3+0.1r}, e^{0.5-0.1r})$.

Figure 4: (for n=8)

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