Numerical Simulation of Robust Recursive Least-Squares Wiener Estimators for Observations with Random Delays and Packet Dropouts in Systems with Uncertainties

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Abstract

This paper investigates the numerical estimation characteristics of the robust recursive least-squares (RLS) Wiener estimators by using the observed values with random delays, packet dropouts, and out-of-order packets for the systems with or without the uncertain parameters in the system matrix and the observation vector. The estimation characteristics are compared with the existing estimators.

(1) The estimation accuracy of the robust RLS Wiener filter is superior to the RLS Wiener filter and fixed-point smoother.

(2) The estimation accuracy of the robust RLS Wiener filter is superior to the RLS Wiener filter and fixed-point smoother, which are designed for the delayed and uncertain observations, except for the observation noise $N(0, 0.5^2)$, provided that the signal exists in the observed values.

(3) In the case of the observations with random delays and without including the uncertain parameters in the system matrix and the observation vector, the estimation accuracies of the robust RLS Wiener filter and fixed-point smoother are superior to the RLS Wiener filter and fixed-point smoother, which are designed for the delayed and uncertain observations.

It should be noted that the robust RLS Wiener estimators do not assume any knowledges of the probabilities of the random delays, and the uncertain parameters.

Keywords: Robust filter; robust fixed-point smoother; delayed observation; autoregressive model; packet dropout.

1. Introduction

Usually, the recursive least-squares (RLS) Wiener estimators (Nakamori, 1995) assume the knowledge of the state-space model except the input matrix and the variance of the white-noise input. If the observed values are randomly delayed or the parameters in the state-space model contain uncertainties, the estimation accuracies of the RLS Wiener estimators become degraded in comparison with the case of using the complete state-space model. For this reason, there have been investigated on the RLS Wiener estimators with the randomly delayed and uncertain observations (Nakamori, 2018), and the robust RLS Wiener estimators (Nakamori, 2019a, 2019b) with the observations, generated by the system with the uncertain parameters in the system and observation matrices. The robust RLS Wiener estimators (Nakamori, 2019a, 2019b) do not use any information on the uncertain parameters in the system. In Nakamori (2018), the uncertain observation means that the observation does not contain the signal with a certain probability. In Dong & You (2006), the robust finite horizon Kalman-type filter is designed for a class of discrete-time uncertain systems, including random delays, packet dropouts, and out-of-order packets. In addition to the random delays in the observed values, there might occur the case, where some parameters in the state-space model include uncertainties. In Chen & Zhang (2011) the robust Kalman filter is designed in uncertain stochastic systems with time-invariant state delayed; bounded random observation delays and missing measurements. The robust Kalman filter (Chen & Zhang, 2011) uses the information on the probabilities of the delays of the observations. This paper investigates numerically on the estimation characteristics of the robust RLS Wiener filter and fixed-point smoother (Nakamori, 2019a, 2019b) for the randomly delayed, packet dropouts and out-of-order packets observed values in the state-space model with or without the uncertain parameters in the system matrix and the observation vector. The estimation characteristics are compared with the existing estimators. The main simulation results for the scalar signal are summarized as follows.

(1) The estimation accuracy of the robust RLS Wiener filter (Nakamori, 2019a, 2019b) is superior to the RLS Wiener filter and fixed-point smoother (Nakamori, 1995).

(2) The estimation accuracy of the robust RLS Wiener filter (Nakamori, 2019a, 2019b) is superior to the RLS Wiener filter and fixed-point smoother (Nakamori, 2018), which are designed for the delayed and uncertain observations, except for the observation noise $N(0, 0.5^2)$.

(3) In the case of the observations with random delays, packet dropouts, out-of-order packets and without including the uncertain parameters in the system matrix and the observation vector, the estimation accuracies of the robust RLS Wiener filter and fixed-point smoother (Nakamori, 2019a, 2019b) are superior to the RLS Wiener filter and fixed-point smoother (Nakamori, 2018), which are designed for the delayed and uncertain observations. Here, the probability containing the signal in the observed value is set to one.

In the simulation, it should be noted that the robust RLS Wiener estimators do not assume any knowledges on the probabilities of random delays of the observations, and the uncertain parameters.

In section 2, we consider an example of sequence with random delays, packet dropouts and out-of-order packets by referring to Liu, Yang, Zhou, Naeem, Wang & Wang (2018). In section 3, Theorem 1 presents the robust RLS Wiener filtering and fixed-point smoothing algorithms (Nakamori, 2019a, 2019b). In section 4, the estimation characteristics of the filter and the fixed-point smoother are shown numerically for the RLS Wiener filter and fixed-point smoother (Nakamori, 1995), the robust RLS Wiener estimators (Nakamori, 2019a, 2019b) and the RLS Wiener estimators (Nakamori, 2018), which are designed for the randomly delayed and uncertain observations.

2. Observations with random delays, packet dropouts and out-of-order packets

Table 1 Samples of degraded observations with one-step or two-steps delays. "*": Observation with one-step delay; "**": Observation with two-steps delay. Packet dropouts: y(22), y(24), y(28), y(30), y(85), y(86).

Time k	1	2	3	4	5	6	7	8	9	10
Observations	y(1)	y(2)	y(3)	y(4)	y(5)	y(6)	y(7)	y(8)	y(9)	y(10)
Degraded delayed observations	$y_d(1)$	$y_d(2)$	$y_d(3)$	$y_d(4)$	$y_d(5)$	$y_d(6)$	$y_d(7)$	$y_d(8)$	$y_d(9)$	$y_d(10)$

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Time k	21	22	23	24	25	26	27	28	29	30
Observations	y(21)	y(22)	y(23)	y(24)	y(25)	y(26)	y(27)	y(28)	y(29)	y(30)
Degraded delayed observations	$y_d(21)$	$y_d(21)$ *	$y_d(23)$	$y_d(23)$ *	$y_d(25)$	$y_d(26)$	$y_d(27)$	$y_d(27)$ *	$y_d(29)$	$y_d(29)$ *

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Time k	81	82	83	84	85	86	87	88	89	90
Observations	y(81)	y(82)	y(83)	y(84)	y(85)	y(86)	y(87)	y(88)	y(89)	y(90)
Degraded delayed observations	$y_d(81)$	$y_d(82)$	$y_d(83)$	$y_d(84)$	$y_d(84)$ *	$y_d(84)$ **	$y_d(87)$	$y_d(88)$	$y_d(89)$	$y_d(90)$

Referring to Fig.1 in Liu et al. (2018), Table 1 shows the sequence of the observations with random one-step or two-steps delays. Here, the probabilities of one step and two-steps delays are 0.1 and 0.05 respectively. Packet dropout occurs when the observed value is delayed. This phenomena are shown in the observation sequence of Table 1. $y_d(k)$ in Table 1 represents the degraded observed value of the observed value y(k) by the random delays and the uncertain parameters in the system matrix and the observation vector. Section 3 describes on the robust estimation technique based on the state-space model including the uncertain parameters, and the robust RLS Wiener filtering and fixed-point smoothing algorithms with the degraded observations (Nakamori, 2019a, 2019b).

3. **Robust RLS Wiener filtering and fixed-point smoothing algorithms** (Nakamori, 2019a, 2019b)

Let the signal $z(k)$ be generated by the state-space model in linear discrete-time wide-sense stationary stochastic

$$\begin{aligned} &y(k) = z(k) + v(k), z(k) = Hx(k),\\ &x(k+1) = \Phi x(k) + \Gamma w(k),\\ &E[v(k)v(s)] = R\delta_K(k-s),\\ &E[w(k)w^T(s)] = Q\delta_K(k-s). \end{aligned} \qquad (1)$$

systems. Here, $z(k)$: signal to be estimated; H: m × n observation matrix; $x(k)$: $n \times 1$ state vector; $v(k)$: zero-mean white observation noise with variance R; Φ: state transition matrix; $w(k)$: white-noise input with variance Q; Γ: $n \times l$ input matrix. Let the signal process be independent of the observation noise process. Let the signal process be fit to the autoregressive (AR) model of the finite order M.

$$\begin{aligned} &z(k) = -\underline{a}_1 z(k-1) - \underline{a}_2 z(k-2) \cdots\\ &-\underline{a}_M z(k-M) + \underline{e}(k),\\ &E[\underline{e}(k)\underline{e}(s)] = \underline{Q}\delta_K(k-s) \end{aligned} \qquad (2)$$

The system matrix Φ in (1) is not necessarily restricted to the controllable canonical form. Let us express the signal $z(k)$ by the state vector $\underline{x}(k) as$

$$\begin{aligned} &z(k) = \underline{H}\underline{x}(k), \underline{H} = [I_{m\times m} \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0],\\ &\underline{x}(k) = \begin{bmatrix} \underline{x}_1(k) \\ \underline{x}_2(k) \\ \vdots \\ \underline{x}_{M-1}(k) \\ \underline{x}_M(k) \end{bmatrix} = \begin{bmatrix} z(k) \\ z(k+1) \\ \vdots \\ z(k+M-2) \\ z(k+M-1) \end{bmatrix}. \end{aligned} \qquad (3)$$

From (2), the state equation for $\underline{x}(k)$ is given by

$$\begin{aligned} &\underline{x}(k+1) = \underline{\Phi}\underline{x}(k) + \underline{\Gamma}\underline{w}(k),\\ &E[\underline{w}(k)\underline{w}^T(s)] = \underline{Q}\delta_K(k-s),\\ &\underline{\Phi} = \begin{bmatrix} 0 & I_{m\times m} & 0 & \cdots & 0 \\ 0 & 0 & I_{m\times m} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_{m\times m} \\ -\underline{a}_M & -\underline{a}_{M-1} & -\underline{a}_{M-2} & \cdots & -\underline{a}_1 \end{bmatrix},\\ &\underline{w}(k) = \underline{e}(k+N), \end{aligned} \qquad (4)$$

with the system matrix $\underline{\Phi}$ in the controllable canonical form. The AR parameters $\underline{a}_i$, $1 \le i \le M$, are calculated by the Yule-Walker equation (Nakamori, 2019b), which uses the auto-covariance function of the signal $z(k)$, $K_z(k,s) = E[z(k)z^T(s)] = K_z(i)$, $i = k - s$, $0 \le i \le N$.

$$K(k,k)\begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_{M-1}^T \\ a_M^T \end{bmatrix} = -\begin{bmatrix} K_z^T(1) \\ K_z^T(2) \\ \vdots \\ K_z^T(M-1) \\ K_z^T(M) \end{bmatrix},$$

$$K(k,k) = \begin{bmatrix} K_z(0) & K_z(1) & \cdots & K_z(M-2) & K_z(M-1) \\ K_z^T(1) & K_z(0) & \cdots & K_z(M-3) & K_z(M-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_z^T(M-2) & K_z^T(M-3) & \cdots & K_z(0) & K_z(1) \\ K_z^T(M-1) & K_z^T(M-2) & \cdots & K_z^T(1) & K_z(0) \end{bmatrix}. \quad (5)$$

In Nakamori (2019a, 2019b), the degraded observations $\breve{y}(k)$ are generated by the state-space model with the uncertain quantities $\Delta\Phi(k)$ in the system matrix and ΔH(k) in the observation matrix. $\breve{y}(k)$ is given as the sum of the degraded signal $\breve{z}(k)$ and the white observation noise $v(k)$. The robust estimators (Nakamori, 2019a, 2019b) do not require any information on $\Delta\Phi(k)$ and ΔH(k).

$$\begin{aligned} &\breve{y}(k) = \breve{z}(k) + v(k), \breve{z}(k) = \breve{H}(k)\breve{x}(k), \\ &\breve{x}(k+1) = \breve{\Phi}(k)\breve{x}(k) + \breve{\Gamma}\zeta(k), \\ &\breve{\Phi}(k) = \Phi + \Delta\Phi(k), \breve{H}(k) = H(k) + \Delta H(k) \end{aligned} \quad (6)$$

In Nakamori (2019a, 2019b), the degraded signal $\breve{z}(k)$ is also fitted to the AR model of the N-th order.

$$\begin{aligned} &\breve{z}(k) = -\breve{a}_1\breve{z}(k-1) - \breve{a}_2\breve{z}(k-2)\cdots \\ &-\breve{a}_N\breve{z}(k-N) + \breve{e}(k), \\ &E[\breve{e}(k)\breve{e}(s)] = \breve{Q}\delta_K(k-s) \end{aligned} \quad (7)$$

In (6), it is considered that the degraded signal $\breve{z}(k)$ is influenced by the uncertainties in the system matrix $\breve{\Phi}(k)$, the observation matrix $\breve{H}(k)$, the input matrix $\breve{\Gamma}$, the white-noise input $\zeta(k)$. In Nakamori (2019a, 2019b), the uncertainties of the system and observation matrices are supposed in particular. Since the degraded signal $\breve{z}(k)$ is modelled in terms of the AR model, it results in that the robust RLS Wiener estimators (Nakamori, 2019a, 2019b) do not use the information on $\Delta\Phi(k)$ and $\Delta H(k)$ at all. If the observation $\breve{y}(k)$ is delayed, the degraded signal $\breve{z}(k)$ is also delayed accordingly. The additional observation noise v(k) is statistically independent of $\breve{z}(k)$. From this fact, the robust RLS Wiener estimators might be suggested to estimate the signal $z(k)$ with the randomly delayed observations regarding the systems having the uncertain parameters in the system and observation matrices. This paper examines to estimate the signal with the delayed and degraded observations from numerical aspect. In terms of the state vector $\breve{x}(k)$, $\breve{z}(k)$ is expressed as

$$\begin{aligned} &\breve{z}(k) = \breve{H}\breve{x}(k), \breve{H} = [I_{m\times m} \quad 0 \quad 0 \quad \cdots \quad 0], \\ &\breve{x}(k) = \begin{bmatrix} \breve{x}_1(k) \\ \breve{x}_2(k) \\ \vdots \\ \breve{x}_{N-1}(k) \\ \breve{x}_N(k) \end{bmatrix} = \begin{bmatrix} \breve{z}(k) \\ \breve{z}(k+1) \\ \vdots \\ \breve{z}(k+N-2) \\ \breve{z}(k+N-1) \end{bmatrix}. \end{aligned} \quad (8)$$

From (7), the state equation for the state vector $\breve{x}(k)$ is given by

$$\begin{aligned}
&\breve{x}(k+1) = \breve{\Phi}\breve{x}(k) + \breve{\Gamma}\zeta(k),\\
&E[\zeta(k)\zeta^T(s)] = \breve{Q}\delta_K(k-s),\\
&\breve{\Phi} = \begin{bmatrix} 0 & I_{m\times m} & 0 & \cdots & 0\\ 0 & 0 & I_{m\times m} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & I_{m\times m}\\ -\breve{a}_N & -\breve{a}_{N-1} & -\breve{a}_{N-2} & \cdots & -\breve{a}_1 \end{bmatrix},\\
&\Gamma = \begin{bmatrix} 0\\ 0\\ 0\\ \vdots\\ I_{m\times m} \end{bmatrix},\\
&\zeta(k) = \breve{e}(k+N).
\end{aligned} \quad (9)$$

The auto-covariance function $\breve{K}(k,s)$ of the state vector $\breve{x}(k)$ is represented as

$$\begin{aligned}
&\breve{K}(k,s) = \begin{cases} A(k)B^T(s), 0 \le s \le k,\\ B(k)A^T(s), 0 \le k \le s, \end{cases}\\
&A(k) = \breve{\Phi}^k, B^T(s) = \breve{\Phi}^{-s}\breve{K}(s,s).
\end{aligned} \quad (10)$$

Here, $\breve{K}(k,k)$ is expressed, with the auto-variance function $K_{\breve{z}}(k-s) = E[\breve{z}(k)\breve{z}^T(s)]$ of the degraded signal $\breve{z}(k)$, as

$$\begin{aligned}
\breve{K}(k,k) &= E\left[\begin{bmatrix} \breve{z}(k)\\ \breve{z}(k+1)\\ \vdots\\ \breve{z}(k+N-2)\\ \breve{z}(k+N-1) \end{bmatrix}\right.\\
&\times [\breve{z}^T(k) \quad \breve{z}^T(k+1) \quad \cdots\\
&\left.\breve{z}^T(k+N-2) \quad \breve{z}^T(k+N-1)]\right]\\
&= \begin{bmatrix} K_{\breve{z}}(0) & K_{\breve{z}}(-1) & \cdots & K_{\breve{z}}(-N+2) & K_{\breve{z}}(-N+1)\\ K_{\breve{z}}(1) & K_{\breve{z}}(0) & \cdots & K_{\breve{z}}(-N+3) & K_{\breve{z}}(-N+2)\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ K_{\breve{z}}(N-2) & K_{\breve{z}}(N-3) & \cdots & K_{\breve{z}}(0) & K_{\breve{z}}(-1)\\ K_{\breve{z}}(N-1) & K_{\breve{z}}(N-2) & \cdots & K_{\breve{z}}(1) & K_{\breve{z}}(0) \end{bmatrix}.
\end{aligned} \quad (11)$$

By using $K_{\breve{z}}(i)$, $0 \le i \le N$, the Yule-Walker equation for the AR parameters $\breve{a}_i$, $1 \le i \le N$, is given by (9) in Nakamori (2019a, 2019b). Also, the cross-covariance function $K_{\underline{x}\breve{x}}(k,s) = E[\underline{x}(k)\breve{x}^T(s)]$ of the state vector $\underline{x}(k)$ with $\breve{x}(s)$ is represented as

$$\begin{aligned}
&K_{\underline{x}\breve{x}}(k,s) = \alpha(k)\beta^T(s), 0 \le s \le k,\\
&\alpha(k) = \underline{\Phi}^k, \beta^T(s) = \underline{\Phi}^{-s}K_{\underline{x}\breve{x}}(s,s).
\end{aligned} \quad (12)$$

Based on the prerequisites on the linear robust estimation problem in the above, Theorem 1 presents the robust RLS Wiener filtering and fixed-point smoothing algorithms (Nakamori, 2019a, 2019b) for estimating the signal $z(k)$.

Theorem 1 (Nakamori, 2019a, 2019b) Let the state-space model containing the uncertain quantities ΔΦ(k) and $H(k)$ be given by (6) in linear discrete-time stochastic systems. Let the state-space model for the signal $z(k)$ be given by (1). Let the signal $z(k)$ and the degraded signal $\breve{z}(k)$ be fitted to the AR models of the orders M and N respectively. Let the variance $\breve{K}(k,k)$ of the state vector $\breve{x}(k)$ for the degraded signal $\breve{z}(k)$ and the cross-variance $K_{\underline{x}\breve{x}}(k,k)$ of the state vector $\underline{x}(k)$ for the signal $z(k)$ with the state vector $\breve{x}(k)$ for the degraded signal

$\breve{z}(k)$ be given by (12). Let the variance of the white observation noise $v(k)$ be R. Then, the robust RLS Wiener estimation algorithms for the filtering estimate $\hat{z}(k,k)$ and the fixed-point smoothing estimate $\hat{z}(k,L)$ of the signal $z(k)$ consist of (13)-(23).

Fixed-point smoothing estimate of the signal $z(k)$ at the fixed point k: $\hat{z}(k,L)$

$$\hat{z}(k,L) = \underline{H}\hat{\underline{x}}(k,L) \quad (13)$$

Fixed-point smoothing estimate of the state vector $\underline{x}(k)$ at the fixed point k: $\hat{\underline{x}}(k,L)$

$$\begin{aligned} &\hat{\underline{x}}(k,L) = \hat{\underline{x}}(k,L-1) + h(k,L,L)(\breve{y}(L) - \breve{H}\breve{\Phi}\hat{\breve{x}}(L-1,L-1)), \\ &\hat{\underline{x}}(k,L)|_{L=k} = \hat{\underline{x}}(k,k) \end{aligned} \quad (14)$$

Smoother gain for $\hat{\underline{x}}(k,L)$ in (14): $h(k,L,L)$

$$\begin{aligned} &h(k,L,L) = [K_{\underline{x}\breve{x}}(k,k)(\breve{\Phi}^T)^{L-k}\breve{H}^T - q(k,L-1)\breve{\Phi}^T\breve{H}^T] \\ &\times \{R + \breve{H}[\breve{K}(L,L) - \breve{\Phi}S_0(L-1)\breve{\Phi}^T]\breve{H}^T\}^{-1} \end{aligned} \quad (15)$$

$$\begin{aligned} &q(k,L) = q(k,L-1)\breve{\Phi}^T + h(k,L,L)\breve{H}[\breve{K}(L,L) - \breve{\Phi}S_0(L-1)\breve{\Phi}^T], \\ &q(k,k) = S_0(k) \end{aligned} \quad (16)$$

Filtering estimate of the signal $z(k)$: $\hat{z}(k,k)$

$$\hat{z}(k,k) = \underline{H}\hat{\underline{x}}(k,k) \quad (17)$$

Filtering estimate of $\underline{x}(k)$: $\hat{\underline{x}}(k,k)$

$$\begin{aligned} &\hat{\underline{x}}(k,k) = \underline{\Phi}\hat{\underline{x}}(k-1,k-1) + G(k)(\breve{y}(k) - \breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1)), \\ &\hat{\underline{x}}(0,0) = 0 \end{aligned} \quad (18)$$

Filter gain for $\hat{\underline{x}}(k,k)$ in (18): $G(k)$

$$\begin{aligned} &G(k) = [K_{\underline{x}\breve{z}}(k,k) - \underline{\Phi}S(k-1)\breve{\Phi}^T\breve{H}^T] \\ &\times \{R + \breve{H}[\breve{K}(k,k) - \breve{\Phi}S_0(L-1)\breve{\Phi}^T]\breve{H}^T\}^{-1}, \\ &K_{\underline{x}\breve{z}}(k,k) = K_{\underline{x}\breve{x}}(k,k)\breve{H}^T \end{aligned} \quad (19)$$

Filtering estimate of $\breve{x}(k)$: $\hat{\breve{x}}(k,k)$

$$\begin{aligned} &\hat{\breve{x}}(k,k) = \breve{\Phi}\hat{\breve{x}}(k-1,k-1) + g(k)(\breve{y}(k) - \breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1)), \\ &\hat{\breve{x}}(0,0) = 0 \end{aligned} \quad (20)$$

Filter gain for $\hat{\breve{x}}(k,k)$ in (20): $g(k)$

$$\begin{aligned} &g(k) = [\breve{K}(k,k)\breve{H}^T - \breve{\Phi}S_0(k-1)\breve{\Phi}^T\breve{H}^T] \\ &\times \{R + \breve{H}[\breve{K}(k,k) - \breve{\Phi}S_0(L-1)\breve{\Phi}^T]\breve{H}^T\}^{-1} \end{aligned} \quad (21)$$

Auto-variance function of $\hat{\breve{x}}(k,k)$: $S_0(k) = E[\hat{\breve{x}}(k,k)\hat{\breve{x}}^T(k,k)]$

$$\begin{aligned} &S_0(k) = \breve{\Phi}S_0(k-1)\breve{\Phi}^T + g(k)\breve{H}[\breve{K}(k,k) - \breve{\Phi}S_0(k-1)\breve{\Phi}^T], \\ &S_0(0) = 0 \end{aligned} \quad (22)$$

Cross-variance function of $\hat{x}(k,k)$ with $\hat{\breve{x}}(k,k)$: $S(k) = E[\hat{x}(k,k)\hat{\breve{x}}^T(k,k)]$

$$\begin{aligned} &S(k) = \underline{\Phi}S(k-1)\breve{\Phi}^T + G(k)\breve{H}[\breve{K}(k,k) - \breve{\Phi}S_0(k-1)\breve{\Phi}^T], \\ &S(0) = 0 \end{aligned} \quad (23)$$

Section 4 shows the estimation characteristics numerically for the filter and the fixed-point smoother by the RLS Wiener filter and fixed-point smoother (Nakamori, 1995), the robust RLS Wiener estimators (Nakamori, 2019a, 2019b), and the RLS Wiener estimators (Nakamori, 2018), which are designed for the randomly delayed and uncertain observations.

4. A numerical simulation example

Let a scalar observation equation and a state equation for $x(k)$ be given by

$$\begin{aligned} &y(k) = z(k) + v(k), z(k) = Hx(k), H = [0.95 \quad -0.4], x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \\ &x(k+1) = \Phi x(k) + \Gamma w(k), \Phi = \begin{bmatrix} 0.05 & 0.95 \\ -0.98 & 0.2 \end{bmatrix}, \Gamma = \begin{bmatrix} 0.952 \\ 0.2 \end{bmatrix}, \\ &E[v(k)v(s)] = R\delta_K(k-s), E[w(k)w(s)] = Q\delta_K(k-s), Q = 0.5^2. \end{aligned} \tag{24}$$

As shown in (2), the signal $z(k)$ is fitted to the M-th order AR model, M = 5. The state-space model contains the uncertain quantities $\Delta H(k)$ and $\Delta\Phi(k)$ as

$$\begin{aligned} &\breve{y}(k) = \breve{z}(k) + v(k), \breve{z}(k) = \bar{H}(k)\breve{x}(k), \breve{x}(k) = \begin{bmatrix} \breve{x}_1(k) \\ \breve{x}_2(k) \end{bmatrix}, \\ &\bar{H}(k) = H + \Delta H(k) = [1 + \Delta_3(k) \quad 0], \Delta H(k) = [\Delta_3(k) \quad 0], \Delta_3(k) = 0.3\zeta_3(k), \\ &\breve{x}(k+1) = \bar{\Phi}(k)\breve{x}(k) + \Gamma w(k), \bar{\Phi}(k) = \Phi + \Delta\Phi(k), \Delta\Phi(k) = \begin{bmatrix} \Delta_1(k) & 0 \\ 0 & \Delta_2(k) \end{bmatrix}, \\ &\Delta_1(k) = 0.1\zeta_1(k), \Delta_2(k) = 0.2\zeta_2(k). \end{aligned} \tag{25}$$

The observed values $\breve{y}(k)$ in Theorem 1 are generated as the random delays of the observed values $\breve{y}(k)$ in (25) by one step or two steps. The probabilities of one-step and two-steps delays are 0.1 and 0.05 respectively. $\zeta_1(k)$, $\zeta_2(k)$ and $\zeta_3(k)$ in (25) represent the uniformly distributed random variables taking values in the range 0 to 1. $\Delta_1(k)$, $\Delta_2(k)$ and $\Delta_3(k)$ consist of the deterministic mean values and the zero-mean random variables, which are mutually independent. The degraded signal $\breve{z}(k)$ is fitted to the N-th order AR model, N = 5. The system matrix $\breve{\Phi}$ for the state vector $\breve{x}(k)$ is given in (9). In terms of the auto-covariance function $K_{\breve{z}}(k-s) = K_{\breve{z}}(s-k) = E[\breve{z}(k)\breve{z}(s)]$ of the degraded signal $\breve{z}(k)$, the auto-variance function $\breve{K}(k,k)$ of the state vector $\breve{x}(k)$ is given by

$$\begin{aligned} \breve{K}(k,k) &= E\left[\begin{bmatrix} \breve{z}(k) \\ \breve{z}(k+1) \\ \vdots \\ \breve{z}(k+N-2) \\ \breve{z}(k+N-1) \end{bmatrix} [\breve{z}(k) \quad \breve{z}(k+1) \quad \cdots \quad \breve{z}(k+N-2) \quad \breve{z}(k+N-1)]\right] \\ &= \begin{bmatrix} K_{\breve{z}}(0) & K_{\breve{z}}(1) & \cdots & K_{\breve{z}}(N-2) & K_{\breve{z}}(N-1) \\ K_{\breve{z}}(1) & K_{\breve{z}}(0) & \cdots & K_{\breve{z}}(N-3) & K_{\breve{z}}(N-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_{\breve{z}}(N-2) & K_{\breve{z}}(N-3) & \cdots & K_{\breve{z}}(0) & K_{\breve{z}}(1) \\ K_{\breve{z}}(N-1) & K_{\breve{z}}(N-2) & \cdots & K_{\breve{z}}(1) & K_{\breve{z}}(0) \end{bmatrix}. \end{aligned} \tag{26}$$

Let $K_{z\breve{z}}(k,s) = E[z(k)\breve{z}(s)]$ represent the cross-covariance function of the signal $z(k)$ with the degraded signal $\breve{z}(s)$. From (3) and (12), the cross-covariance function $K_{\underline{x}\breve{x}}(k,s)$ is given by

$$K_{\underline{x}\check{x}}(k,s) = \underline{\Phi}^{k-s} K_{\underline{x}\check{x}}(s,s), 0 \le s \le k,$$

$$K_{\underline{x}\check{x}}(k,k) = E\left[\begin{bmatrix} \underline{x}_1(k) \\ \underline{x}_2(k) \\ \vdots \\ \underline{x}_{M-1}(k) \\ \underline{x}_M(k) \end{bmatrix} [\check{z}(k) \quad \check{z}(k+1) \quad \cdots \quad \check{z}(k+N-2) \quad \check{z}(k+N-1)]\right]$$

$$= \begin{bmatrix} K_{z\check{z}}(k,k)] & K_{z\check{z}}(k,k+1) & \cdots & K_{z\check{z}}(k,k+N-2) & K_{z\check{z}}(k,k+N-1) \\ K_{z\check{z}}(k+1,k) & K_{z\check{z}}(k+1,k+1) & \cdots & K_{z\check{z}}(k+1,k+N-2) & K_{z\check{z}}(k+1,k+N-1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_{z\check{z}}(k+M-2,k) & K_{z\check{z}}(k+M-2,k+1) & \cdots & K_{z\check{z}}(k+M-2,k+N-2) & K_{z\check{z}}(k+M-2,k+N-1) \\ K_{z\check{z}}(k+M-1,k) & K_{z\check{z}}(k+M-1,k+1) & \cdots & K_{z\check{z}}(k+M-1,k+N-2) & K_{z\check{z}}(k+M-1,k+N-1) \end{bmatrix}. \quad (27)$$

Fig. 1 illustrates the signal z(k), the filtering estimate $\hat{z}$(k, k) and the fixed-point smoothing estimate $\hat{z}(k, k+1)$ by the robust RLS Wiener filter and fixed-point smoother (Nakamori, 2019a, 2019b) vs. time k with the observations randomly delayed and degraded by the uncertain parameters in the system matrix and the observation vector for the white Gaussian observation noise N(0,0.3^2). Fig. 2 illustrates the mean-square values (MSVs) of the filtering errors z(k) $-\hat{z}(k,k)$ and the fixed-point smoothing errors z(k) $-\hat{z}(k,k+Lag)$, 1 ≤ Lag ≤ 10, 1 ≤ k ≤ 2000, vs. Lag by the RLS Wiener filter and the fixed-point smoother (Nakamori, 1995) with the observations randomly delayed and degraded by the uncertain parameters in the system matrix and the observation vector. Fig. 2 indicates, in comparison with the MSV of the filtering errors, that the MSV of the fixed-point smoothing errors decreases for each observation noise. Fig. 3 Illustrates the MSVs of the filtering errors z(k) $-\hat{z}(k,k)$ and the fixed-point smoothing errors z(k) $-\hat{z}(k,k+Lag)$, 1 ≤ Lag ≤ 10, 1 ≤ k ≤ 2000, vs. Lag by the robust RLS Wiener filter and fixed-point smoother (Nakamori, 2019a, 2019b) with the observations randomly delayed and degraded by the uncertain parameters in the system matrix and the observation vector. Fig. 3 indicates that the MSV of the fixed-point smoothing errors is larger than the MSV of the filtering errors for each observation noise. From Fig. 2 and Fig. 3, the estimation accuracy of the robust RLS Wiener filter is superior to the RLS Wiener filter and fixed-point smoother for each observation noise. Fig. 4 illustrates the MSVs of the filtering errors z(k) $-\hat{z}(k,k)$ and the fixed-point smoothing errors z(k) $-\hat{z}(k,k+Lag)$, 1 ≤ Lag ≤ 10, 1 ≤ k ≤ 2000, vs. Lag by the RLS Wiener filter and fixed-point smoother (Nakamori, 2018), which are designed for the delayed and uncertain observations, with the observations randomly delayed for the system with the uncertain parameters in the system matrix and the observation vector. From Fig. 3 and Fig. 4, the estimation accuracies of the robust RLS Wiener estimators (Nakamori, 2019a, 2019b) are superior in estimation accuracy to the RLS Wiener estimators (Nakamori, 2018) for the observation noises N(0, 0.1^2), N(0, 0.3^2) and N(0, 0.7^2), except for N(0, 0.5^2). Fig. 5 illustrates the MSVs of the filtering errors z(k) $-\hat{z}(k,k)$ and the fixed-point smoothing errors z(k) $-\hat{z}(k,k+Lag)$, 1 ≤ Lag ≤ 10, 1 ≤ k ≤ 2000, vs. Lag by the robust RLS Wiener filter and fixed-point smoother (Nakamori, 2019a, 2019b) with the randomly delayed observations, provided that the uncertain parameters are $\Delta_1(k) = 0$, $\Delta_2(k) = 0$ and $\Delta_3(k) = 0$. Fig. 6. Illustrates the MSVs of the filtering errors z(k) $-\hat{z}(k,k)$ and the fixed-point smoothing errors z(k) $-\hat{z}(k,k+Lag)$, 1 ≤ Lag ≤ 10, 1 ≤ k ≤ 2000, vs. Lag by the RLS Wiener filter and fixed-point smoother (Nakamori, 2018) with the randomly delayed observations. From Fig. 5 and Fig. 6, it is shown, for the randomly delayed observed values, that the estimation accuracies of the robust RLS Wiener filter and fixed-point smoother (Nakamori, 2019a, 2019b) are superior to the RLS Wiener estimators (Nakamori, 2018).

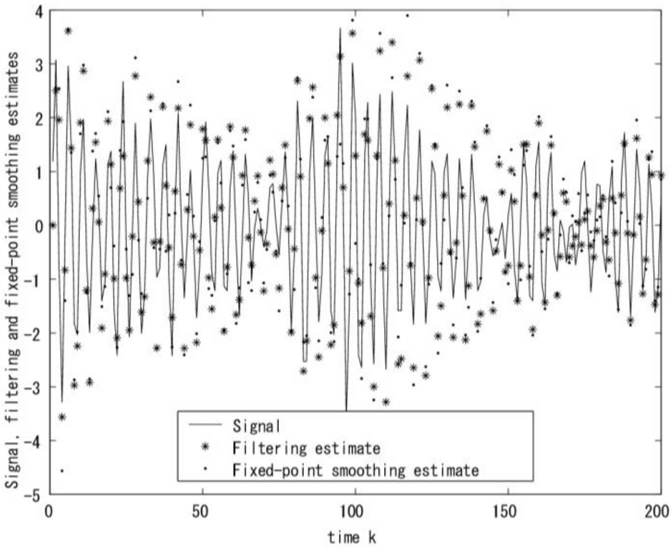

Fig. 1 Signal z(k), filtering estimate $\hat{z}(k,k)$ and fixed-point smoothing estimate $\hat{z}(k,k+1)$ by robust RLS Wiener filter and fixed-point smoother (Nakamori, 2019a, 2019b) vs. time k with observations randomly delayed and degraded by uncertain parameters in system matrix and observation vector for white Gaussian observation noise $N(0,0.3^2)$.

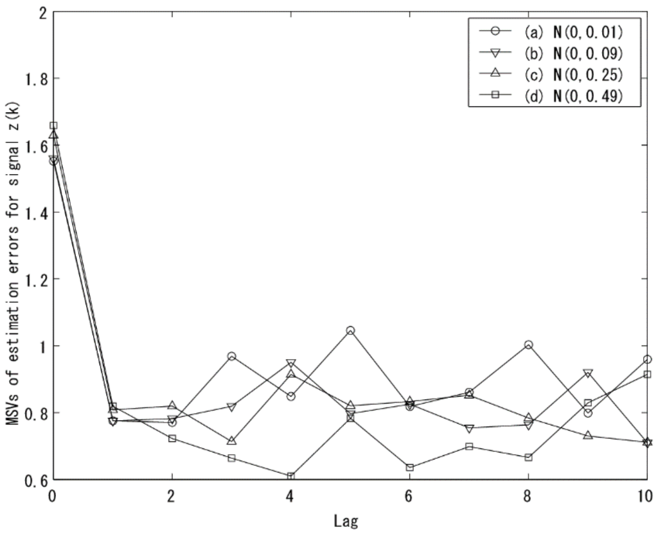

Fig. 2 Mean-square values of filtering errors $z(k) - \hat{z}(k,k)$ and fixed-point smoothing errors $z(k) - \hat{z}(k,k+Lag)$, $1 \le Lag \le 10$, $1 \le k \le 2000$, vs. Lag by RLS Wiener filter and fixed-point smoother (Nakamori, 1995) with observations randomly delayed and degraded by uncertain parameters in system matrix and observation vector.

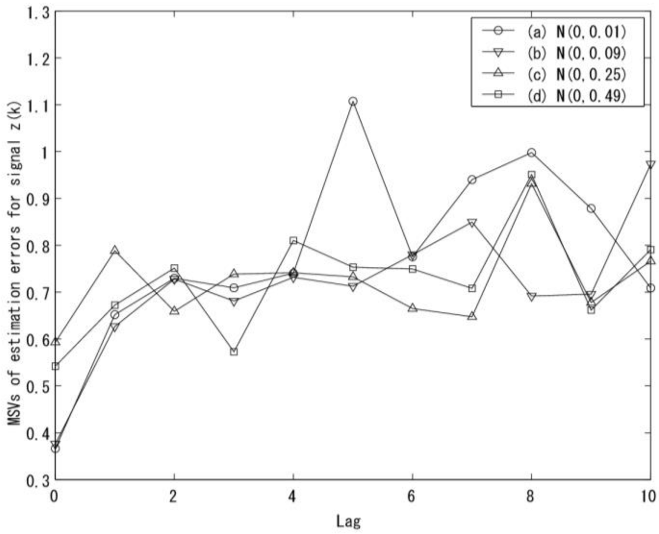

Fig. 3 Mean-square values of filtering errors z(k) $-\hat{z}(k,k)$ and fixed-point smoothing errors z(k) $-\hat{z}(k,k+Lag)$, 1 ≤ Lag ≤ 10, 1 ≤ k ≤ 2000, vs. Lag by robust RLS Wiener filter and fixed-point smoother (Nakamori, 2019a, 2019b) with observations randomly delayed and degraded by uncertain parameters in system matrix and observation vector.

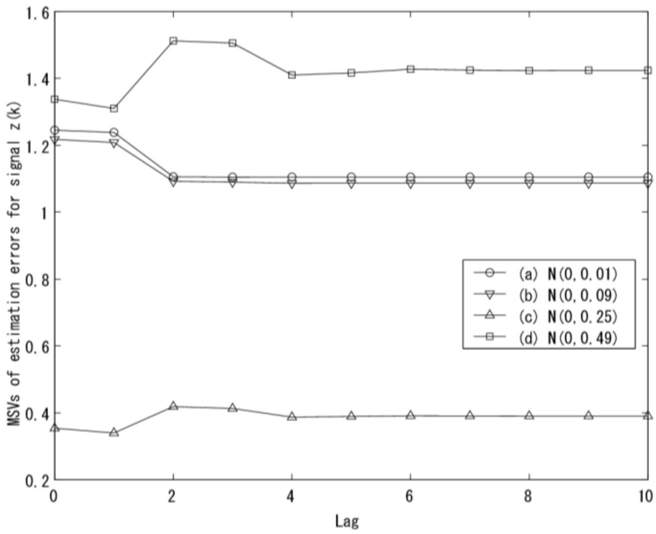

Fig. 4 Mean-square values of filtering errors z(k) $-\hat{z}(k,k)$ and fixed-point smoothing errors z(k) $-\hat{z}(k,k+Lag)$, 1 ≤ Lag ≤ 10, 1 ≤ k ≤ 2000, vs. Lag by RLS Wiener filter and fixed-point smoother (Nakamori, 2018), which are designed for delayed and uncertain observations, with observations randomly delayed and degraded by uncertain parameters in system matrix and observation vector.

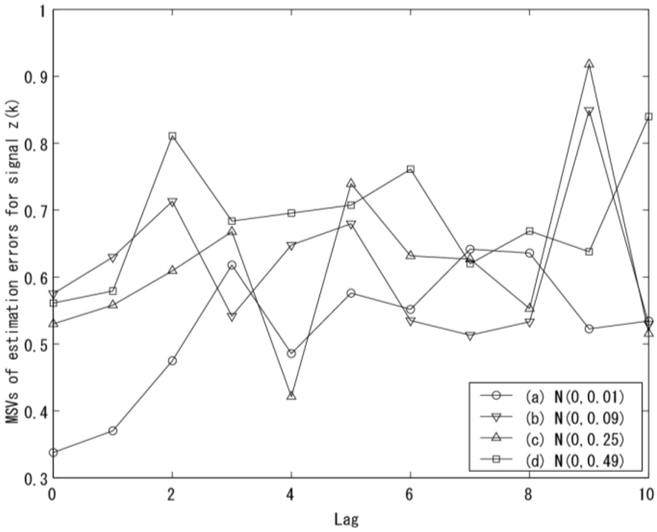

Fig. 5 Mean-square values of filtering errors z(k) $-\hat{z}(k,k)$ and fixed-point smoothing errors z(k) $-\hat{z}(k,k+Lag)$, $1 \le$ Lag ≤ 10, $1 \le$ k ≤ 2000, vs. Lag by robust RLS Wiener filter and fixed-point smoother (Nakamori, 2019a, 2019b) with randomly delayed observations.

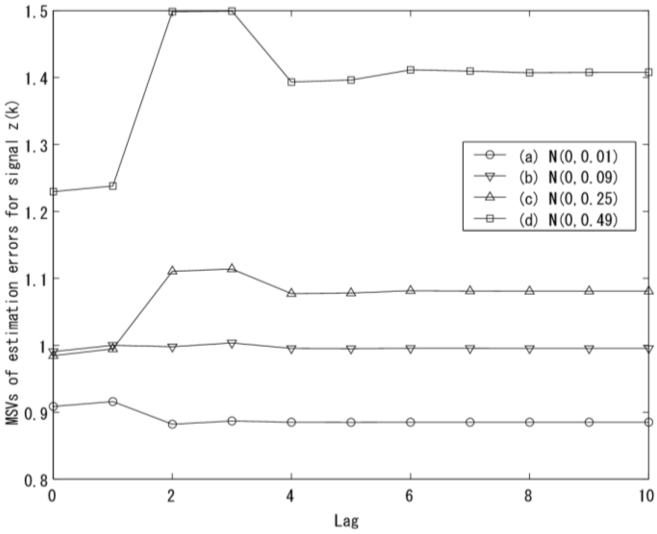

Fig. 6 Mean-square values of filtering errors z(k) $-\hat{z}(k,k)$ and fixed-point smoothing errors z(k) $-\hat{z}(k,k+Lag)$, $1 \le$ Lag ≤ 10, $1 \le$ k ≤ 2000, vs. Lag by RLS Wiener filter and fixed-point smoother (Nakamori, 2018), which are designed for delayed and uncertain observations, with randomly delayed observations.

5. Conclusions

This paper has investigated numerically on the estimation characteristics of the robust RLS Wiener filter and fixed-point smoother with the delayed, packet dropouts and out-of-order packets observed values for the systems with or without the uncertain parameters in the system matrix and the observation vector. The estimation characteristics are compared for the scalar signal with the existing estimators.

(1) The estimation accuracy of the robust RLS Wiener filter is superior to the RLS Wiener filter and fixed-point smoother.

(2) The estimation accuracy of the robust RLS Wiener filter is superior to the RLS Wiener filter and fixed-point smoother, which are designed for the delayed and uncertain observations, for the observation noises N(0, 0.1^2), N(0, 0.3^2) and N(0, 0.7^2), except for N(0, 0.5^2).

(3) In the case of the observations with random delays and without including the uncertain parameters in the system matrix and the observation vector, the estimation accuracies of the robust RLS Wiener filter and fixed-point smoother are superior to the RLS Wiener filter and fixed-point smoother, which are designed for the delayed and uncertain observations. Here, the probability containing the signal in the observed value is set to one.

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Author Biography

Seiichi Nakamori is a Specially Appointed Professor in the Faculty of Education, Kagoshima University, from April, 2017 to date. He was appointed Professor Emeritus, Kagoshima University, in April 2016. He was a Professor, Department of Technology, Graduate School of Education, Kagoshima University from April, 1994 to March 2016, Associate Professor, in the Department from April, 1987 to March, 1994 and a lecturer, Interdisciplinary Chair (Applied Mathematics), Faculty of Engineering, Oita University, from April, 1985 to March, 1987.

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