

On a Class of Functional Equations over the Real and over the Complex Fields

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Abstract:

In the present review-paper, we start by recalling some of our earlier results on the construction of a nontrivial function ff defined implicitly by the equation (1), without using the implicit function theorem. This is the first aim of the paper. Here the function gg is given, satisfying some conditions. All these considerations work in the real case, for functions and a class of operators. The second aim is to consider the complex case, proving the analyticity of the function ff defined implicitly, under the hypothesis that gg is analytic and verifies natural conditions, related to the real case. Some consequences are deduced. Finally, one illustrates the preceding results by an application to a concrete functional and respectively operatorial equation. Related examples are given, some of them pointing out elementary functions gg for which equation (1) leads to nontrivial solutions ff that can be expressed by means of elementary functions.

Keywords: constructive solutions; function defined implicitly; analyticity; self-adjoint operators; functional calculus.

1. Introduction

The equation

$$gg = gg \circ ff \tag{1}$$

where g is given, while f is the unknown function, always has the trivial solution

$$f(x) = x, \quad \forall x \in D,$$

where D is the domain of definition for f . When (the nonlinear) function g is firstly decreasing and then increasing (hence $-g$ firstly increases and then decreases), there exists exactly one decreasing nontrivial solution f , with the properties stated in Theorem 3.1 below. These equations were studied in [3]-[6]. The present review paper is mainly based on article [4]. For concrete functions g , one obtains special properties of the corresponding solutions f .

The present approach allows the construction of the solutions of such functional and operatorial equations, without using the implicit function theorem. In the operatorial case the solution F is a function of $U \in D \subset X$, where X is the commutative algebra of selfadjoint operators defined by (3). We essentially use the fact that X is also an order-complete vector lattice, with respect to the natural order relation on the real vector space $\mathcal{AA}(H)$ of all selfadjoint operators acting on the Hilbert space H .

The background of this paper is partially covered by some chapters from [1] and [9]. Some other types of functional and operatorial equations are discussed in [2], [7]. The rest of the paper is organized as follows. Section 2 emphasizes the methods applied along this work. In the first part of Section 3, we recall some known results on the subject, especially related to the real case. Then we consider the case of complex analytic functions. The general idea is that analyticity of gg implies the same property for ff . In the end of Section 3, concrete functional and operatorial equations are solved and simple examples when the solution can be written explicitly in terms of elementary functions are given. Section 4 concludes the paper.

2. Methods

The main methods used along this paper are:



- 1) Constructing a real function f defined implicitly by a given real continuous function g of one real variable, under appropriate assumptions on g (see Theorem 3.1, assertion (x)), without using the implicit function theorem. Deducing basic properties of the "unknown" function f .
- 2) Solving operatorial equations involving functions gg, ff of self-adjoint operators, derived from real continuous functions gg, ff by means of functional calculus (see [1] and the references there). For an illustrative example, see Theorem 3.7 from below. In particular, equations involving symmetric matrices with real entries are under attention.
- 3) In the case when gg is a complex analytic function of one complex variable, deducing the analyticity of ff . Here elements of elementary complex analysis [9] are applied.

3. Results and Discussion

1) The real case

Theorem 3.1. (see also [3]-[6]). *Let $t u, v \in \overline{\mathbf{R}}, u < v, \alpha \in]u, v[$ and let $g :]u, v[\rightarrow \mathbf{R}$ be a continuous function. Assume that*

$$(a) \lim_{x \downarrow u} g(x) = \lim_{x \uparrow v} g(x) = w \in \overline{\mathbf{R}},$$

(b) g is strictly decreasing on $]u, \alpha[$ and strictly increasing on $[\alpha, v[$.

Then there exists $f :]u, v[\rightarrow]u, v[$ such that

$$g(x) = g(f(x)), \quad \forall x \in]u, v[$$

and f has the following qualities

(i) f is strictly decreasing on $]u, v[$ and we have

$$\lim_{x \downarrow u} f(x) = v, \quad \lim_{x \uparrow v} f(x) = u;$$

(ii) α is the unique fixed point of f ;

(iii) we have $f^{-1} = f$ on $]u, v[$;

(iv) f is continuous on $]u, v[$;

(v) if $g \in C^n(]u, v[\setminus\{\alpha\}), n \in \mathbb{N} \cup \{\infty\}, n \geq 1$, then $f \in C^n(]u, v[\setminus\{\alpha\})$;

(vi) if g is derivable on $]u, v[\setminus\{\alpha\}$, so is f ;

(vii) if $g \in C^2(]u, v[), g''(\alpha) \neq 0$ and there exists $\rho_1 := \lim_{x \rightarrow \alpha} f'(x) \in \overline{\mathbf{R}}$ then

$$f \in C^1(]u, v[) \cap C^2(]u, v[\setminus\{\alpha\}) \text{ and } f'(\alpha) = -1;$$

(viii) if $g \in C^3(]u, v[), g''(\alpha) \neq 0$ and there exist $\rho_1 := \lim_{x \rightarrow \alpha} f'(x) \in \overline{\mathbf{R}}$ and $\rho_2 := \lim_{x \rightarrow \alpha} f''(x) \in \mathbf{R}$, then

$$f \in C^2(]u, v[) \cap C^3(]u, v[\setminus\{\alpha\})$$

and

$$f''(\alpha) = \rho_2 = -\frac{2}{3} \cdot \frac{g'''(\alpha)}{g''(\alpha)};$$

(ix) if g is analytic at α , then f is derivable at α and $f'(\alpha) = -1$;

(x) let $g_l := g|_{]u, \alpha[}, g_r := g|_{[\alpha, v[}$; then we have

$$f(x_0) = (g_r^{-1} \circ g_l)(x_0) = \sup\{x \in [\alpha, v[; g_r(x) \leq g_l(x_0)\} \quad \forall x_0 \in]u, \alpha[$$



and

$$f(x_0) = (g_l^{-1} \circ g_r)(x_0) = \inf\{x \in]u, \alpha]; g_l(x) \leq g_r(x_0)\} \quad \forall x_0 \in [\alpha, v[$$

We recall the geometric meaning of the construction of f . If $x \in]u, v[\setminus \{\alpha\}$, consider the horizontal passing through the point $(x, g(x))$. Thanks to the qualities of g , this straight line intersect once again the graph of g at exactly one point

$$(x_1, g(x_1)) = (x, g(x)), \quad x_1 \neq x.$$

We define

$$f(x) = x_1.$$

Then we have

$$g(x) = g(x_1) = g(f(x)), \quad x \in]u, v[\setminus \{\alpha\}, \quad f(\alpha) := \alpha.$$

When x runs over the interval $]u, \alpha]$, $f(x)$ runs over the interval $[\alpha, v[$, in the decreasing sense, from v to α . When x runs over the interval $[\alpha, v[$ (in the increasing sense), $f(x)$ runs over the interval $]u, \alpha]$, in the decreasing sense, from α to u .

Let $\alpha \in]0, \infty[$ and denote by G the set of all continuous functions $g :]0, \infty[\rightarrow]0, \infty[$, $g(\alpha) = 0$, which are decreasing on $]0, \alpha]$ and increasing on $[\alpha, \infty[$, such that

$$\lim_{x \downarrow 0} g(x) = \lim_{x \uparrow \infty} g(x) = \infty.$$

For $g, h \in G$, an interesting problem is the following one: find necessary and sufficient conditions on g, h for the equality:

$$f_g = f_h,$$

where f_g, f_h are the corresponding functions attached to g , respectively to h by Theorem 3.1. The following statement is giving the answer. For the detailed proof see the references of one of the papers [3]-[6].

Theorem 3.2. *Let $g, h \in G$, $\lambda \in]0, \infty[$. Then $g + h, \lambda g, g \cdot h$ are also elements of G and the following statements are equivalent*

(a) $f_g = f_h$;

(b) $h_r \circ g_r^{-1} = h_l \circ g_l^{-1}$, $g_l := g|_{]0, \alpha]}$, $g_r := g|_{[\alpha, \infty[}$, $g \in G$;

(c) *there exists $\varphi :]0, \infty[\rightarrow]0, \infty[$ such that $\varphi(0) = 0$, φ is continuous and increasing, verifying the relation*

$$h = \varphi \circ g.$$

Next we consider the abstract operatorial version of Theorem 2.1. In the sequel, X will be an order-complete vector lattice, and $Izom_+(X)$ will be the set of all vector space isomorphisms $T : X \rightarrow X$ which apply X_+ onto itself.

Theorem 3.3. *Let X be an order-complete vector lattice, X_+ its positive cone, $\alpha \in X$, D_l a convex subset such that*

$$\alpha \in D_l \subset \{x \in X; x \leq \alpha\};$$

D_r a convex subset such that

$$\alpha \in D_r \subset \{x \in X; x \geq \alpha\};$$

Let $g_l : D_l \rightarrow X$ be a convex operator such that

$$\partial g_l(x) \cap (-Izom_+(X)) \neq \emptyset \quad \forall x \in D_l \setminus \{\alpha\}$$

Let $g_r : D_r \rightarrow X$ be a convex operator such that

$$\partial g_r(x) \cap (Izom_+(X)) \neq \emptyset \quad \forall x \in D_r \setminus \{\alpha\}$$

We also assume that

$$g_l(\alpha) = g_r(\alpha) \text{ and } R(g_l) = R(g_r),$$

where $R(g)$ is the range of g , while $\partial g(x)$ is the set of all subgradients of g at x . Let

$$g : D := D_l \cup D_r \rightarrow X, \\ g(x) = g_l(x) \quad \forall x \in D_l, g(x) = g_r(x) \quad \forall x \in D_r$$

Then there exists $F : D \rightarrow D$ such that

$$g(x) = g(F(x)), \quad \forall x \in D$$

F is strictly decreasing in D and it has the following properties:

- (i) α is the only fixed point of F ;
- (ii) there exists F^{-1} and $F^{-1} = F$ on D ;
- (iii) we have

$$F(x_0) = g_r^{-1}(g_l(x_0)) = \sup\{x \in D_r; g_r(x) \leq g_l(x_0)\} \quad \forall x_0 \in D_l, \\ F(x_0) = g_l^{-1}(g_r(x_0)) = \inf\{x \in D_l; g_l(x) \leq g_r(x_0)\} \quad \forall x_0 \in D_r.$$

II) On the analyticity of the solution. The complex case

Application of the complex form of the implicit function theorem for a holomorphic function \tilde{g} , that is the extension of the real function g of Theorem 3.1, might be difficult around α . Namely considering the equation

$$H(z, w) = \tilde{g}(z) - \tilde{g}(w) = 0,$$

we have

$$\frac{\partial H}{\partial w}(\alpha, \alpha) = 0, \quad \lim_{z \rightarrow \alpha} \frac{\tilde{g}'(z)}{\tilde{g}'(w(z))} = \frac{0}{0} = \lim_{z \rightarrow \alpha} w'(z).$$

Note that for points from $]u, v[\setminus \{\alpha\}$ we can apply the implicit function theorem (over the complex field), which leads to the analyticity of \tilde{f} at such points. Therefore, we only have to study the analyticity at α .

Theorem 3.4. Let \tilde{g} be the extension of the function g from Theorem 3.1, such that \tilde{g} is holomorphic in a complex neighborhood of $]u, v[$. Then there is a unique holomorphic solution \tilde{f} of the equation $\tilde{g} = \tilde{g} \circ \tilde{f}$ such that $\tilde{f}\tilde{f}$ is the extension of ff from Theorem 3.1 to a complex neighborhood of $]u, v[$.

Proof. From the preceding remarks, we have to prove the analyticity of \tilde{f} only at α . To this end, let us write the following expansions:

$$\tilde{g}(z) - \tilde{g}(\alpha) = (z - \alpha)^{2k} \left[\frac{\tilde{g}^{(2k)}(\alpha)}{(2k)!} + \frac{\tilde{g}^{(2k+1)}(\alpha)}{(2k+1)!} (z - \alpha) + o(z - \alpha) \right],$$

$$\tilde{g}(f(z)) - \tilde{g}(\alpha) = (f(z) - \alpha)^{2k} \left[\frac{\tilde{g}^{(2k)}(\alpha)}{(2k)!} + \frac{\tilde{g}^{(2k+1)}(\alpha)}{(2k+1)!} (f(z) - \alpha) + o(f(z) - \alpha) \right]$$

Here $m = 2k \geq 2$ is the smallest natural number, which for the derivative of order m of \tilde{g} at α is not vanishing. By Taylor formula, it must be an even number, since α is a minimum point for g . On the other hand, since $\tilde{g} = \tilde{g} \circ \tilde{f}$, elementary computations and the preceding expansions lead to

$$\lim_{z \rightarrow \alpha} f(z) = \alpha (= f(\alpha)).$$

Further computations yield

$$\frac{\tilde{f}(z) - \tilde{f}(\alpha)}{z - \alpha} = \left(\frac{\tilde{g}^{(2k)}(\alpha)/(2k)! + \varphi(z)(z - \alpha)}{\tilde{g}^{(2k)}(\alpha)/(2k)! + \varphi(\tilde{f}(z))(\tilde{f}(z) - \alpha)} \right)^{1/2k},$$

where φ is holomorphic around α . It follows that $\tilde{f}'(\alpha)$ is a $2k$ -order root of the unity. Using the fact that \tilde{f} applies intervals of the real line into the real line and that it is decreasing we deduce that $\tilde{f}'(\alpha) = -1$. This concludes the proof. \square

In the sequel, for a complex neighborhood V of α , we denote

$$V_l^+ = \{z \in V; \operatorname{Re} z < \alpha, \operatorname{Im} z > 0\}, V_l^- = \{z \in V; \operatorname{Re} z < \alpha, \operatorname{Im} z < 0\},$$

$$V_r^+ = \{z \in V; \operatorname{Re} z > \alpha, \operatorname{Im} z > 0\}, V_r^- = \{z \in V; \operatorname{Re} z > \alpha, \operatorname{Im} z < 0\}.$$

Corollary 3.1. *There is a neighborhood V of α such that*

$$\tilde{f} : V \rightarrow \tilde{f}(V)$$

is a one to one mapping and we have $\tilde{f} \circ \tilde{f} = id$, and

$$\tilde{f}(V_l^+) = (\tilde{f}(V))_r^-, \tilde{f}(V_l^-) = (\tilde{f}(V))_r^+, \tilde{f}(V_r^+) = (\tilde{f}(V))_l^-, \tilde{f}(V_r^-) = (\tilde{f}(V))_l^+$$

Proof. The first two assertions follow by the local inversion theorem and respectively from the analytic continuation principle (\tilde{f} is a holomorphic extension of f). The last four relations are consequences of the fact that \tilde{f} is conformal at α , also using the qualities of f (see the comments following Theorem 3.1). The conclusions follow. \square

Corollary 3.2. *There is a complex neighborhood W of α such that \tilde{f} is holomorphic on W .*

Corollary 3.3. *The function $\frac{1}{z - \alpha} + (f(z) - \alpha)$ is univalent in the open disc $|z - \alpha| < 1$ if and only if*

$$f(z) = -z + 2\alpha.$$

Proof. For the only if part, assume that

$$\frac{1}{z - \alpha} + (f(z) - \alpha) = \frac{1}{z - \alpha} - (z - \alpha) + \frac{f''(\alpha)}{2}(z - \alpha)^2 + \dots$$

is univalent in the open disc $|z - \alpha| < 1$. Then by the area theorem 14.13 [9], we should have

$$\sum_{n=1}^{\infty} n \cdot |a_n|^2 \leq 1,$$

where $a_n, n \geq 1$ are the coefficients of the holomorphic part of the preceding expansion. Since $a_1 = -1$, it follows that all the others $a_n, n \geq 2$ are vanishing, so that

$$f(z) - \alpha = -z + \alpha, f(z) = -z + 2\alpha.$$

Conversely, if this last relation is verified, then a straightforward computation shows that the function

$$\frac{1}{z - \alpha} + (f(z) - \alpha)$$

is univalent in the unit open disc centered at $\alpha\alpha$. This concludes the proof. \square

Remark 3.1. The function $f(z) = -z + 2\alpha$ is an extreme point of the convex set of all holomorphic functions with real coefficients $\psi(z) = \sum_{n=1}^{\infty} a_n(z - \alpha)^n, \sum_{n=1}^{\infty} n \cdot a_n^2 \leq 1$. It is also an extreme point of the convex subset of all functions

$$h(z) = \sum_{n=1}^{\infty} a_n(z - \alpha)^n, a_1^2 + \sum_{n=2}^{\infty} p_n |a_n|^{m_n} \leq 1, p_n \geq 0, m_n > 1, \forall n \geq 2,$$

where a_n, p_n, m_n are given numbers with the properties from above.

Theorem 3.5. In a small neighborhood of $\alpha\alpha$, we have

$$\tilde{f}(z) \approx z - 2 \frac{\tilde{g}'(z)}{\tilde{g}''(z)}.$$

Proof. The following relations hold true

$$0 = \tilde{g}(f(z)) - \tilde{g}(z) = \tilde{g}'(z)(f(z) - z) + \frac{\tilde{g}''(z)}{2}(f(z) - z)^2 + o((f(z) - z)^2)$$

Dividing by $f(z) - z \neq 0$ and neglecting the remained terms in $f(z) - z$, the conclusion follows. \square

Corollary 3.4. For \tilde{g} as above and any linear bounded operator U acting on a Hilbert space H , with spectrum $\sigma(U)$ in a small neighborhood of $\alpha\alpha$, there is a holomorphic function \tilde{f} such that

$$\begin{aligned} \tilde{g}(\tilde{f}(U)) &= \tilde{g}(U), \tilde{f}(\tilde{f}(U)) = U, \\ \sigma(U) \subset \{\text{Im } z \geq 0\} &\Leftrightarrow \sigma(\tilde{f}(U)) \subset \{\text{Im } z \leq 0\}, \\ \tilde{f}(U) &\approx U - 2 \cdot \tilde{g}'(U) \cdot [\tilde{g}''(U)]^{-1}, \end{aligned}$$

for all such operators UU .

Proof. Part of the relations follows by analytic functional calculus. For the third relation, one applies Corollary 3.1. For the last relation one uses Theorem 3.5. \square

III) Examples and applications

We consider the functional equation

$$x^a \exp(-\beta \cdot x) = (f(x))^a \exp(-\beta \cdot f(x)), x > 0, a > 0, \beta > 0.$$

This equation is equivalent to the following one

$$x \exp(-bx) = f(x) \exp(-bf(x)), x > 0, b := \beta/a > 0 \tag{2}$$

Theorem 3.6. There exists a unique decreasing solution $f:]0, \infty[\rightarrow]0, \infty[$ of the equation (2) and this solution has the following properties:



- (i) $f(0+) = \infty-, f(\infty-) = 0+$;
- (ii) $\alpha = 1/b$ is the unique fixed point of f ;
- (iii) $f^{-1} = f$ on $]0, \infty[$;
- (iv) the following constructive formulae for $f(x)$ hold

$$f(x_0) = \sup\{x \in [1/b, \infty[; x \cdot \exp(-bx) \geq x_0 \cdot \exp(-bx_0)\}, \forall x_0 \in]0, 1/b[,$$

$$f(x_0) = \inf\{x \in]0, 1/b]; x \cdot \exp(-bx) \geq x_0 \cdot \exp(-bx_0)\}, \forall x_0 \in [1/b, \infty[;$$

- (v) f is the restriction of a holomorphic function \tilde{f} on a complex neighborhood of $]0, \infty[$, such that $\tilde{f}'(1/b) = -1, \tilde{f} \circ \tilde{f} = id$, and \tilde{f} has the properties mentioned in Corollary 3.1;
- (vi) in a small neighborhood of $1/b$, we have:

$$\tilde{f}(z) \approx z + \frac{2}{b} \cdot \frac{1-bz}{2-bz}.$$

Proof. The function $g(x) = -x \cdot \exp(-bx)$ decreases from 0 to $g(1/b) = -e^{-1}b^{-1}$ in the interval $]0, b^{-1}[$ and increases from $g(b^{-1})$ to 0 in the interval $]b^{-1}, \infty[$. Hence the conclusions (i)-(iv) follow from Theorem 3.1. The function g is the restriction of a holomorphic function \tilde{g} , with

$$\tilde{g}'(b^{-1}) = 0, \tilde{g}^{(n)}(z) = (-1)^n b^{n-1} \cdot \exp(-bz)(n - bz), n \in \mathbb{N}, n \geq 1, z \in \mathbb{C}.$$

In particular, $\tilde{g}^{(n)}(b^{-1}) \neq 0$, so that one applies Corollary 3.1 that leads to the conclusion (v) of the present statement. The assertion (vi) follows from Theorem 3.5. The proof is complete. \square

Let g be the function from the proof of Theorem 3.6. Then there exists appropriate intervals $]u, v[$ containing b^{-1} such that g is convex on $]u, v[, g(u) = g(v)$. The convexity is required in Theorem 3.3 in order to deduce the existence of a subgradient, which allows the construction of the solution. Next we apply Theorem 3.3 to the operatorial equation corresponding to (2). The case of arbitrary linear bounded operators follows from Corollary 3.4, for

$$\tilde{g}(z) = -z \cdot \exp(-bz), \text{Re } z > 0, \alpha = b^{-1}, \tilde{f}$$

being the holomorphic extension of f from Theorem 3.6 to a complex neighborhood of $]0, \infty[$ (see also Theorem 3.1). Now we consider the case of the associated operator equation in a commutative algebra of self-adjoint operators $XXXX = XX(AA), A \in \mathcal{A}(HH)$ being a fixed self-adjoint operator acting on the Hilbert space HH . We define

$$X_1 = \{U \in \mathcal{A}(H); UA = AU\}, X = X(A) = \{V \in X_1; VU = UV, \forall U \in X_1\} \tag{3}$$

$$X_+ = \{V \in X; \langle Vh, h \rangle \geq 0, \forall h \in H\}$$

It is known that X defined by (3) is an order complete Banach lattice and a commutative algebra (cf. [1]). We denote:

$$D_l = \{V \in X; \sigma(V) \subseteq [u, 1/b] \cup \{b^{-1}I\},$$

$$D_r = \{V \in X; \sigma(V) \subseteq [1/b, v] \cup \{b^{-1}I\}$$

$$D = D_l \cup D_r$$

where $\sigma(V)$ is the spectrum of V .

Theorem 3.7. *There exists a decreasing mapping $F : D \rightarrow D$, such that*

$$U \exp(-bU) = F(U) \exp(-bF(U)), U \in D.$$

This mapping has the following properties:

(i) $b^{-1}I$ is the unique fixed point of F ;

(ii) F is invertible and $F^{-1} = F$ on D ;

(iii) F applies D_l onto D_r and D_r onto D_l ;

(iv) the following constructive formulas for F hold

$$F(U_0) = \sup \{U \in D_r; U \exp(-bU) \geq U_0 \exp(-bU_0)\}, \forall U_0 \in D_l,$$

$$F(U_0) = \inf \{U \in D_l; U \exp(-bU) \geq U_0 \exp(-bU_0)\}, \forall U_0 \in D_r;$$

(v) for $U \in A$, with the spectrum in a small neighborhood of b^{-1} , we have:

$$F(U) \approx U + (2b^{-1})(I - bU)(2I - bU)^{-1}.$$

Proof. We have to verify the conditions from the statement of Theorem 3.3. To prove the convexity of $g(U) = -U \exp(-bU)$ on the convex subsets D_l, D_r, D_l, D_r we use the convexity of the scalar function g on $]uu, vv[$ and the positivity of the spectral measures attached to the elements $U_1, U_2 \in D_l, D_l$ respectively $U_j U_j \in D_r, j = 1, 2$. The fact that U_1, U_2 are commuting operators is essential. For $t \in [0, 1]$, we have (thanks to the properties of functional calculus attached to the tuple (UU_1, UU_2) , which "preserves" inequalities)

$$g((1-t)U_1 + tU_2) \leq (1-t)g(U_1) + tg(U_2).$$

Hence the convexity on the subsets D_l, D_r is proved. On the other hand, for $U \in D_l \setminus \{b^{-1}I\}$ we have:

$$g'_l(U) = \exp(-bU)(bU - I) < 0$$

as a product of two commuting self-adjoint operators, the first one being positive and the second one being negative. In the same way, one shows that

$$g'_r(U) > 0, \forall U \in D_r \setminus \{b^{-1}I\}$$

It remains to prove that the ranges of the operators g_l, g_r, g_r coincide, and also the assertions (iii) and (v) of the present statement. Let

$$g_l(U_1) \in r(g_l), U_2 := F(U_1),$$

where F is associated to f by means of functional calculus. Using the qualities of f , we obtain

$$\sigma(U_2) = \sigma(F(U_1)) = f(\sigma(U_1)) \subset [1/b, v] \Rightarrow U_2 \in D_r.$$

From the equality $g_l(t) = g_r(f(t)), t \in [u, 1/b] \supset \sigma(U_1)$, by integration with respect to the spectral measure attached to UU_1 one obtains

$$g_l(U_1) = \int_{\sigma(U_1)} g_l(t) d_{E_{U_1}} = \int_{\sigma(U_1)} g_r(f(t)) d_{E_{U_1}} = g_r(F(U_1)) = g_r(U_2) \in R(g_r)$$

Similarly, one proves that $R R(g_r) \subset R(g_l)$. Now all the conclusion (except (v)), follow by application of Theorem 3.3. The assertion (v) is a consequence of (vi), Theorem 3.6. This concludes the proof. \square

We next give some examples for which the exact analytic expression of the solution ff can be determined explicitly.

Theorem 3.8. *The nontrivial solution of the equation*

$$e^{-x}(1 - e^{-x}) = e^{-f(x)}(1 - e^{-f(x)}), x > 0,$$

is given by

$$f(x) = -\ln(1 - e^{-x}), x > 0.$$

Its holomorphic extension is

$$\tilde{f}(z) = -\ln(1 - e^{-z}), \operatorname{Re} z > 0.$$

Proof. One can prove easily that the function

$$g(x) = -e^{-x}(1 - e^{-x}), x \in]0, \infty[$$

verifies the conditions of Theorem 3.1, where $\alpha = \ln(2)$. To find the analytic expression of ff , we rewrite our functional equation as

$$e^{-f(x)} - e^{-x} = e^{-2f(x)} - e^{-2x} = (e^{-f(x)} - e^{-x})(e^{-f(x)} + e^{-x})$$

Dividing by

$$e^{-f(x)} - e^{-x} \neq 0, x \neq \ln 2,$$

and doing some straightforward computations, one obtains the result. This concludes the proof. \square

Example 3.1. The unique nontrivial solution \tilde{f} of the equation

$$\tilde{g}(z) = \tilde{g}(\tilde{f}(z)), |z| < 1, \tilde{g}(z) = \left(\frac{z}{2-z}\right)^2$$

is given by $\tilde{f}(z) = \frac{z}{z-1}$.

Example 3.2. The unique nontrivial solution of the functional equation

$$g(x) = g(f(x)), g(x) = x + \frac{1}{x}, x \in]0, \infty[$$

is $f(x) = \frac{1}{x}$. A similar remark holds true for $x \in]-\infty, 0[$. It seems that this example remains valid in complex, for $g(z) = z + \frac{1}{z}, f(z) = \frac{1}{z}, z \in \mathbb{C} \setminus \{0\}$.

4. Conclusions

We have proved that under certain assumptions on the given function gg , there exists a unique nontrivial solution ff of the functional equation (1). In case of real functions gg of one real variable, verifying the condition of Theorem 3.1, the nontrivial strictly decreasing solution ff has as unique fixed point the unique minimum point of gg . The real case for scalar-valued functions leads to generalization to functions of self-adjoint operators, by means of functional calculus (when inequalities are preserved). Along the proofs of these first results, the implicit function theorem is not used, even in the case when gg is smooth. In the case of real valued functions gg , a geometric meaning of the construction of the nontrivial solution ff is pointed out in the comment following Theorem 3.1. This idea is basic for the construction of ff , not only for real valued functions, but also in the context of abstract (and concrete) order complete vector

lattices, according to Theorem 3.3 (and respectively Theorem 3.7). In the case of complex-valued functions gg , if gg is analytic, so is ff . Finally, we emphasize three elementary functions gg for which the solution ff is expressible by means of elementary functions as well. This could be a direction of research for future work on this subject.

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