On Markov moment problem and related inverse problems

Octav Olteanu

Department of Mathematics-Informatics, Politehnica University of Bucharest, Splaiul Independenței 313, 060042 Bucharest, Romania

olteanuoctav@yahoo.ie

Abstract

Existence and construction of the solutions of some Markov moment problems are discussed. Starting from the moments of a solution, one recalls a method of recovering this solution, also solving approximately related systems with infinite many nonlinear equations and infinite unknowns. This is the first aim of this review paper. Extension of linear forms with two constraints is applied. Measure theory arguments play a central role. Other results in analysis and functional analysis are used tacitly, sending the reader to the references for unproved stated theorems. Secondly, in the end, existence of solutions of special Markov moment problems is studied.

Keywords: Markov Moment Problem; Truncated Moment Problem; Full Moment Problem; Inverse Problems; Constrained Extension

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1 Introduction

We start by recalling the following abstract version of the Markov moment problem, motivated by the classical moment problem [1], [2], [3]. The point (a) of the next theorem shows clearly that it is an interpolation problem with two constraints.

Theorem 1.1. ([4]). *Let X be an ordered vector space, Y an order complete vector lattice, $\{x_j\}_{j\in J} \subset X$, $\{y_j\}_{j\in J} \subset Y$ given families and $F_1, F_2 \in L(X,Y)$ two linear operators. The following statements are equivalent*

(a)*there is a linear operator $F \in L(X,Y)$ such that*

$$F_1(x) \le F(x) \le F_2(x)\ \forall x \in X_+,\ F(x_j) = y_j\ \forall j \in J;$$

(b)*for any finite subset $J_0 \subset J$ and any $\{\lambda_j\}_{j\in J_0} \subset R$, we have*

$$\left(\sum_{j\in J_0} \lambda_j x_j = \psi_2 - \psi_1, \psi_1, \psi_2 \in X_+\right) \Rightarrow \sum_{j\in J_0} \lambda_j y_j \le F_2(\psi_2) - F_1(\psi_1).$$

If X is a space (usually a Banach space) of functions defined on a closed subset A of $\mathbb{R}^n, (n \in \mathbb{N}, n \ge 1)$, containing polynomials and continuous compactly supported functions (with their supports contained in A), and

$$x_j(t) = t^j := t_1^{j_1} \cdots t_n^{j_n}, t = (t_1, \dots, t_n) \in A, j = (j_1, \dots, j_n) \in \mathbb{N}^n, \quad (1)$$

then we have a classical Markov moment problem. If $n = 1$, we have a one-dimensional problem, while in case of $n \geq 2$, we have a multidimensional (real) moment problem. In most of cases, $F_1 = 0$, so the solution F is positive. Consequently, it can be represented by a positive scalar or vector measure. The upper constraint $F \leq F_2$ on the positive cone X_+ of X might ensure the continuity and controls the norm of the solution (under the assumption that F_2 is continuous). The elements $y_j, j \in J$ are called the moments, and the equalities $F(x_j) = y_j$, $j \in J$ are the interpolation moment-conditions. Three aspects are studied in solving such problems: the existence, the uniqueness and the construction of the solution (if it is unique). The first aspect is solved by Theorem 1.1 (in the case of the abstract version). For some other related results and/or completions see [5]- [12]. For discussion on the uniqueness of some solutions, see [13], [14], [15]. Considering the classical Markov moment problem, under the hypothesis that the subspace of polynomials is dense in X and the solution F is continuous, then its uniqueness follows too. For the construction of the solution, see [6], [7], [8], [9], [10], [12]. The background of this work is contained in some chapters from [16], [17], [18]. The rest of the paper is organized as follows. Section 2 is devoted mainly to inverse problems related to some Markov moment problems. The one-dimensional case as well as the multidimensional case is illustrated. In the end, a Markov moment problem not necessarily involving polynomials is discussed in a very general setting. An example of a two-dimensional classical moment problem is given as a consequence. Section 3 contains the proofs and related methods used before it. Section 4 concludes the paper.

2 The results

We start by recalling a truncated and a full one-dimensional Markov moment problem, as well as a related inverse problem (see [8], [9] and the references therein). The first result is a truncated moment problem, because only the first n moments are given. Although, the solution represented by $h \in L^\infty([0,b])$ (with respect to Lebesgue measure) is defined on the whole space $L^1([0,b])$. Let $b \in (0,\infty), \psi_j(t) := jt^{j-1}, t \in [0,b], j \in \mathbb{N}\setminus\{0\}$.

Theorem 2.1. *For a given family of numbers* $(m_j)_{j=1}^n$, *consider the following statements*

(a) *there exists* $h \in L^\infty([0,b])$ *such that*

$$0 \leq h(t) \leq 1 \ a.e., \quad m_j = j\int_0^b t^{j-1} h(t)dt, \quad j = 1,2,\dots,n;$$

(b) *for any family of scalars* $(\lambda_j)_{j=1}^n$,*one has*

$$\sum_{j=1}^n \lambda_j m_j \leq \sum_{j=1}^n \lambda_j b^j \ ;$$

(c) *there exists a Borel subset* B *such that*

$$\int_B j \cdot t^{j-1} dt = m_j, \quad j = 1,\dots,n.$$

Then (b) $\Rightarrow$ (a) $\Leftrightarrow$ (c)

Corollary 2.1. *Under the equivalent conditions (a), (c) of Theorem 2.1, there exist sequences*

$$y_{1,n} < x_{1,n} < y_{2,n} < x_{2,n} < \ldots < y_{l,n} < x_{l,n} < \ldots, \quad n \in \mathbb{N},$$

such that the following relations hold

$$m_k = \inf_{n\in \mathbb{N}} \left(\sum_{j=1}^{\infty} \left(x_{j,n}^k - y_{j,n}^k \right) \right), \quad k = 1, \ldots, n$$

Remark 2.1. To approximate the numbers, $x_{j,n}^k, y_{j,n}^k$ one can make use of Fourier approximate expansion of h with respect to the orthonormal sequence attached to the functions kt^{k-1} via Gram-Schmidt algorithm, also using the values of the moments m_k . Thus, one obtains a smooth approximation-function $\tilde{h}$ of h, and the intervals of ends $y_{l,n}, x_{l,n}$ are connected components of the open sets approximating from above subsets of the following form, in the sense of the measures of these sets:

$$\left\{ t; \frac{j_l}{2^{p(n)}} \le \tilde{h}(t) < \frac{j_l + 1}{2^{p(n)}} \right\}.$$

Remark 2.2. A similar result to that of Theorem 2.1 in several dimensions holds, with the same proof.

Next, we go on with the full moment problems, when all the moments have prescribed values $m_k, k \in \mathbb{N}\setminus\{0\}$. Here the solution h is the weak limit of the sequence of solutions of truncated moment problems (see [8], [9] for the proof). The weak topology on $L^\infty([0,b])$ is considered with respect to the dual pair $(L^1([0,b]), L^\infty([0,b]))$.

Theorem 2.2. *With the notations from Theorem 2.1, let $(m_k)_{k\ge 1}$ be a sequence of real numbers. Consider the following statements*

(a) *there exists a Borel function h such that*

$$0 \le h(t) \le 1 \quad a.e., \; m_k = k\int_a^b t^{k-1} h(t)dt, \quad k \in \mathbb{N}\setminus\{0\};$$

(b) *for any natural number $n \ge 1$, and any $\varepsilon > 0$, there exist nonnegative scalars β_j, $j = 1, \ldots, n$, and sequences: $y_{j,1} < x_{j,1} < \ldots < y_{j,l} < x_{j,l} < \ldots, \quad j = 1, \ldots, n$ such that*

$$1 - \varepsilon \le \sum_{j=1}^{n} \beta_j \le 1, \quad m_k = \lim_n \left(\sum_{j=1}^{n} \beta_j \left(\sum_l \left(x_{j,l}^k - y_{j,l}^k \right) \right) \right), \; k \in \mathbb{N}\setminus\{0\};$$

(c) *for any $n \in \mathbb{N}\setminus\{0\}$, there exists a Borel subset B_n such that*

$$m_k = k\int_{B_n} t^{k-1}dt, \quad k = 1,\ldots, n.$$

(d) *for any natural* $n \geq 1$ *and any* $\{\lambda_1,\ldots,\lambda_n\} \subset \mathbb{R}$ *the following relation holds true:*

$$\sum_{k=1}^{n} \lambda_k m_k \leq \sum_{k=1}^{n} \lambda_k (b^k - a^k).$$

Then (d) $\Rightarrow$ *(a)* $\Leftrightarrow$ *(b)* $\Leftrightarrow$ *(c)*

Remark 2.3. For the full moment problem, the following algorithm holds in determining (approximating) $y_{j,l}, x_{j,l}$.

Step 1. (Approximating the function *h*). Let $(e_n)_{n\geq 1}$ be a Hilbert base constructed by the aid of Gram Schmidt procedure, applied to the system of linearly independent functions

$$\varphi_n(t) = n \cdot t^{n-1}, \quad n \in \mathbb{N} \setminus \{0\}.$$

Then for each fixed natural number $n \geq 1$, one has:

$$e_n = \sum_{j=1}^{n} a_j^{(n)} \varphi_j \Rightarrow \langle h, e_n \rangle = \sum_{j=1}^{n} a_j^{(n)} \langle h, \varphi_j \rangle = \sum_{j=1}^{n} a_j^{(n)} m_j,$$

where the coefficients $a_j^{(n)}$ are known from Gram Schmidt procedure. Hence, we can determine each Fourier coefficient of *h*, that is we can approximate h in L^2-norm by a sequence of polynomial functions h_n, $n \geq 1$. Then there exists a subsequence

$$h_{k_n} \to h$$

pointwise convergent almost everywhere in $[a,b]$.

Step 2. For each $n \in \mathbb{N} \setminus \{0\}$, the subsets

$$\left\{ t; \frac{m_l}{2^p} \leq h_{k_n}(t) < \frac{m_l + 1}{2^p} \right\}, \quad p, m_l \in \mathbb{N},$$

can be approximated (in measure) by open subsets. The connected components of these open sets have as end points approximations of the unknowns. $y_{j,l}, x_{j.l}$. Using a weakly compactness standard argument, we can obtain h as the limit of a subsequences of $\aleph_{B_n}$, where B_n are as in assertion (c) of Theorem 2.2. Considering a suitable open set D_{k_n}, with $B_{k_n} \subset D_{k_n}$, from (a) one obtains

$$m_k \approx \int_{D_{k_n}} k \cdot t^{k-1} dt = \sum_l (x_{n,l}^k - y_{n,l}^k), \quad D_{k_n} = \mathrm{Y}_l \;]y_{n,l}, x_{n,l}[\; .$$

Thus, the unknowns are approximated by the ends of the connected components of open subsets D_{k_n}, while the latter subsets can be determined starting from the given moments and the previous arguments of this algorithm. Note that all results of this section can be adapted to the multidimensional moment problem, with similar proofs. In the sequel, we propose such a method.

Let $A = (0,1)^n$, $d\nu = (-\ln t_1)dt_1 \Lambda \, (-\ln t_n)dt_n$. Assume that there exists a $h \in L_\nu^\infty(A), 0 \le h \le 1 \, a.e.$, such that

$$m_j = \int_A t_1^{j_1} (-\ln(t_1))) \Lambda \; t_n^{j_n} (-\ln(t_n)) h(t_1, \ldots, t_n) dt_1 \Lambda \; dt_n,$$
$$j = (j_1, \ldots, j_n), \; j_k \in N, \; k = 1, \ldots, n.$$

Denote $\varphi_j(t_1, \ldots t_n) = \cdot t_1^{j_1} \Lambda \; t_n^{j_n}$, $j_k \in \{0,1,2\ldots\}, k = 1, \ldots, n, (t_1, \ldots t_n) \in A.$ Consider the system of equations

$$m_j = \int_A \varphi_j \cdot h \, d\nu \approx \int_A \varphi_j \cdot \tilde{h} \, d\nu \approx$$
$$\sum_{p,q \le M} c_{p,q} \left(\sum_{m \in \mathrm{N}} \prod_{k=1}^n \left(\frac{x_{k,m,p,q}^{j_k+1} \ln(x_{k,m,p,q}) - y_{k,m,p,q}^{j_k+1} \ln(y_{k,m,p,q})}{j_k + 1} + \frac{y_{k,m,p,q}^{j_k+1} - x_{k,m,p,q}^{j_k+1}}{(j_k+1)^2} \right) \right), \; j_k \ge 0, \; k = 1, \ldots, n \quad (2)$$

Theorem 2.3. *An approximation for the solution of* (2) *is given by the coordinates* $x_{k,m,pq,}\, y_{k,m,pq,}\; k \in \{1, \ldots, n\}$ *of the vertices of the cells from the cell – decomposition of the open subsets* $D_{p,q}$ *associated to the known polynomials* $\tilde{h}$.

For some other methods applied to similar types of problems see [12].

In the sequel, we give sufficient conditions for the existence of a solution, in a very general setting. The method works in an arbitrary measurable space, which may not involve polynomials. In the particular case of subsets of $\mathbb{R}^n$, the form of positive polynomials is not necessarily used. Along this short Section we follow some results from [7].

Theorem 2.4. *Let* (T, ν) *be a measurable space, where* ν *is a positive* σ *- finite measure on T. Let* $X = L_\nu^1(T)$, $\{x_j\}_{j \in J} \subset X$, $\{y_j\}_{j \in J} \subset R$ *. Consider the following statements*:

(a) *there exists* $h \in L_\nu^\infty(T)$ *such that*

$$\int_T x_j(t) \cdot h(t) d\nu = y_j, \quad j \in J, \; -1 \le h(t) \le 1 \; \nu - a.e.;$$

(b) *for any finite subset* $J_0 \subset J$ *and any* $\{\lambda_j\}_{j \in J_0} \subset R$ *we have*

$$\sum_{i,j\in J_0}\lambda_i\lambda_j y_i y_j \le \sum_{i,j\in J_0}\lambda_i\lambda_j\int_T x_i(t)dv\cdot\int_T x_j(t)dv\,;$$

(c) *for any finite subset* $J_0 \subset J$ *and any* $\{\lambda_j\}_{j\in J_0}\subset R$, *we have*

$$\sum_{i,j\in J_0}\lambda_i\lambda_j y_i y_j \le \sum_{i,j\in J_0}|\lambda_i|\cdot|\lambda_j|\cdot\int_T |x_i(t)|\,dv\cdot\int_T |x_j(t)|\,dv$$

Then (b) $\Rightarrow$ (a) $\Rightarrow$ (c)

Corollary 2.2. *Let*

$$T=\{(t_1,t_2)\in R^2;\quad 0\le t_1<\infty, 0\le t_2\le \exp(-t_1)\},\ \{y_{(j_1,j_2)}\}_{(j_1,j_2)\in Z_+^2}\subset R.$$

Consider the following statements

(a) *there exists a Lebesgue measurable "function" h on T such that*

$$\int_T t_1^{j_1}t_2^{j_2}h(t_1,t_2)dt_1dt_2 = y_{(j_1,j_2)},\quad \forall (j_1,j_2)\in Z_+^2,$$
$$-1\le h(t_1,t_2)\le 1\quad a.e.;$$

(b) *for any finite subset* $J_0\subset Z_+^2$ *and any* $\{\lambda_{(t_1,t_2)};(j_1,j_2)\in J_0\}\subset R$ *we have*

$$\sum_{(i_1,i_2),(j_1,j_2)\in J_0}\lambda_{(i_1,i_2)}\lambda_{(j_1,j_2)}y_{(i_1,i_2)}y_{(j_1,j_2)}\le$$
$$\sum_{(i_1,i_2),(j_1,j_2)\in J_0}\lambda_{(i_1,i_2)}\lambda_{(j_1,j_2)}\frac{i_1!}{(i_2+1)^{i_1+2}}\cdot\frac{j_1!}{(j_2+1)^{j_1+2}}$$

Then (b) $\Rightarrow$ (a)

Note that T is a closed unbounded and non-semi-algebraic subset. Details can be found in [7]. In the same paper [7], the solution of a moment problem on the ellipse is constructed.

3 Proofs and related methods

Proof of Theorem 2.3. We propose the following algorithm for approximating the solutions of the system of equations (2), and an explanation for the last relations (2) (accompanied by explicit significance of notations).

Step 1. Find an approximation $\tilde{h}$ of the solution h in terms of the moments $m_j,\ j\in N^n$. To this end, since $h\in L_\nu^\infty(A)\subset L_\nu^2(A)$, h has a Fourier expansion with respect to the Hilbert base $(\psi_j)_{j_k\ge 0}$ associated following

Gram-Schmidt procedure to the complete system of linearly independent polynomials $(\varphi_j)_{j_k \geq 0}$. The Fourier coefficients $\langle h, \psi_j \rangle$ are given by

$$\langle h, \psi_j \rangle = \sum_{\substack{l_k \leq j_k, \\ k=1,\dots,n}} \alpha_l \langle h, \varphi_l \rangle = \sum_{\substack{l_k \leq j_k, \\ k=1,\dots,n}} \alpha_l m_l,$$

where α_l are given by the Gram-Schmidt procedure, so that we know h in terms of the moments. Recall that there exists a subsequence of the sequence of Fourier partial sums, which converges pointwise to h. This fact is a consequence of the remark that the partial sums of the Fourier series converge in an L^2- space. In the sequel we can write: $h \approx \tilde{h}$, where $\tilde{h}$ is a partial sum of the Fourier series of h. Note that all these partial sums are polynomials, so that they are continuous.

Step 2. Let $\tilde{h}$ be a partial sum of the Fourier series with respect to the orthogonal polynomials $(\psi_j)_{j_k \geq 0}$. Using Schwarz inequality, and approximation of continuous functions $\tilde{h}$ by simple functions

$$\tilde{h}(t_1, \dots, t_n) \approx \sum_{p,q \leq M} c_{p,q} \chi_{D_{p,q}}(t_1, \dots, t_n).$$

The numbers $c_{p,q}$ are the values of $\tilde{h}$ at some points in

$$\left\{ (t_1, \dots, t_n); \frac{m_q}{2^p} \leq \tilde{h}(t_1, \dots, t_n) < \frac{m_q + 1}{2^p} \right\},$$

where p is large and m_q is suitable chosen for approximating $\tilde{h}$.

One deduces

$$m_j \approx \int_A \varphi_j \tilde{h} \, d\nu \approx \int_A \varphi_j \left(\sum_{p,q} c_{p,q} \cdot \chi_{D_{p,q}}(t_1, \dots, t_n) \right) d\nu =$$

$$\sum_{p,q} c_{p,q} \cdot \left(\sum_{m \in N} \int_A \varphi_j \cdot \chi_{[x_{1,m,p,q}, y_{1,m,p,q})}(t_1) \Lambda \, \chi_{[x_{n,m,p,q}, y_{n,m,p,q})}(t_n) \, d\nu \right),$$

where $D_{p,q}$ are open subsets approximating in measure the subsets $\left\{ (t_1, \dots, t_n); \frac{m_q}{2^p} \leq \tilde{h}(t_1, \dots, t_n) < \frac{m_q + 1}{2^p} \right\}$, and whose cell decompositions may be written as below

$$(0,1)^n \supset D_{p,q} = \Upsilon_{m \in \mathrm{N}} [x_{1,m,p,q}, y_{1,m,p,q}) \times \Lambda \times [x_{n,m,p,q}, y_{n,m,p,q}) \supset$$
$$\left\{ (t_1, \dots, t_n); \frac{m_q}{2^p} \leq \tilde{h}(t_1, \dots, t_n) < \frac{m_q + 1}{2^p} \right\}.$$

The above arguments yield

$$m_j \approx \sum_{p,q} c_{p,q}\left(\sum_{m\in N}\int_{x_{1,m,p,q}}^{y_{1,m,p,q}} t_1^{j_1}(-\ln t_1)dt_1 \Lambda \int_{x_{n,m,p,q}}^{y_{n,m,p,q}} t_n^{j_n}(-\ln t_n)dt_n\right) =$$

$$\sum_{p,q} c_{p,q}\left(\sum_{m\in N}\prod_{k=1}^{n}\left(-\frac{t_k^{j_k+1}\ln t_k}{j_k+1}|_{x_{k,m,p,q}}^{y_{k,m,p,q}} + \frac{t_k^{j_k+1}}{(j_k+1)^2}|_{x_{k,m,p,q}}^{y_{k,m,p,q}}\right)\right) =$$

$$\sum_{p,q} c_{p,q}\left(\sum_{m\in N}\prod_{k=1}^{n}\left(\frac{x_{k,m,p,q}^{j_k+1}\ln(x_{k,m,p,q}) - y_{k,m,p,q}^{j_k+1}\ln(y_{k,m,p,q})}{j_k+1} + \frac{y_{k,m,p,q}^{j_k+1} - x_{k,m,p,q}^{j_k+1}}{(j_k+1)^2}\right)\right).$$

The conclusion is that we can determinate (approximate) the "unknowns" $y_{k,m,p,q}, x_{k,m,p,q}, k=1,\dots,n$ by means of the cell decomposition of the open subsets $D_{p,q}$ associated to the known polynomial $\tilde{h}$ (cf. [18, section 2.19]). The basic relations can be summarized as the system of equations (2), where m_j are given, $c_{p,q}$ are known from Step 1, and the unknowns $x_{k,m,p,q}, y_{k,m,p,q}$ are determined in terms of the cell - decomposition of the suitable chosen open subsets D_{pq}, deduced from the known polynomial $\tilde{h}$ (the measure ν is outer regular [18]). The unknowns are the coordinates of the vertices of the cells (see. [18, section 2.19]). Clearly, the solution is not unique. □

Proof of Theorem 2.4. (b) $\Rightarrow$ (a). The following implications hold:

$$\sum_{j\in J_0}\lambda_j x_j = \varphi_2 - \varphi_1, \varphi_k \in X_+, k=1,2 \Rightarrow$$

$$-\int_T \varphi_1 dv - \int_T \varphi_2 dv \le \int_T \varphi_2 dv - \int_T \varphi_1 dv = \sum_{j\in J_0}\lambda_j \int_T x_j dv \le \int_T \varphi_2 dv + \int_T \varphi_1 dv =$$

$$\int_T \varphi_2 dv - \left(-\int_T \varphi_1 dv\right) := F_2(\varphi_2) - F_1(\varphi_1) \Rightarrow \left|\sum_{j\in J_0}\lambda_j \int_T x_j dv\right| \le F_2(\varphi_2) - F_1(\varphi_1).$$

On the other hand, the condition (b) can be rewritten as

$$\left(\sum_{j\in J_0}\lambda_j y_j\right)^2 \le \left(\sum_{j\in J_0}\lambda_j \int_T x_j dv\right)^2 \Leftrightarrow \left|\sum_{j\in J_0}\lambda_j y_j\right| \le \left|\sum_{j\in J_0}\lambda_j \int_T x_j dv\right|.$$

From the preceding relation, we conclude

$$\sum_{j\in J_0}\lambda_j y_j \le \left|\sum_{j\in J_0}\lambda_j y_j\right| \le \left|\sum_{j\in J_0}\lambda_j \int_T x_j(t) dv\right| \le F_2(\varphi_2) - F_1(\varphi_1).$$

Whence the condition from the statement of Theorem 1.1 are accomplished, so that there exists a linear form *F* on *X* such that

$$-\int_T \varphi dv = F_1(\varphi) \le F(\varphi) \le F_2(\varphi) = \int_T \varphi dv \Rightarrow |F(\varphi)| \le \int_T \varphi dv \quad \forall \varphi \in X_+ \Rightarrow$$
$$|F(\varphi)| \le |F(\varphi^+)| + |F(\varphi^-)| \le \int_T (\varphi^+ + \varphi^-)dv = \int_T |\varphi|\, dv = \|\varphi\|_1, \varphi \in X \Rightarrow \|F\| \le 1;$$
$$F(x_j) = y_j, j \in J$$

According to representation of continuous linear functionals on L^1 spaces, there exists

$$h \in L_\nu^\infty(T), \quad F(\varphi) = \int_T \varphi(t)h(t)dv \ \ \forall \varphi \in L_\nu^1(T),$$
$$\|h\|_\infty = \|F\| \le 1 \Rightarrow -1 \le h(t) \le 1 \quad v - a.e.$$

We also infer that

$$y_j = F(x_j) = \int_T x_j(t)h(t)dv, \ \ \forall j \in J.$$

Thus, the proof of the implication (b) $\Rightarrow$ (a) is finished. The implication (a) $\Rightarrow$ (c) is almost obvious

$$\left|\sum_{j\in J_0} \lambda_j y_j\right| = \left|\sum_{j\in J_0} \lambda_j \int_T x_j \cdot h \cdot dv\right| \le \sum_{j\in J_0} |\lambda_j| \cdot \int_T |x_j(t)| \cdot |h(t)| \cdot dv \le \sum_{j\in J_0} |\lambda_j| \cdot \int_T |x_j(t)| \cdot dv$$

Taking the squares of the two members, the inequality is preserved. This concludes the proof. □

Conclusions

Section 2 refers to some earlier results on truncated and on full Markov moment problem [6], [8], recently recalled in [9]. These results suggest the algorithms for the two inverse problems solved in section 2. The proofs which are missing in this section make use of various notions and results, such as extreme points, bang-bang principle, Krein-Milman theorem, Alaoglu's theorem, theorem 1.1 recalled above, measure theory [18]. Theorem 2.4 gives a sufficient condition for the existence of a solution $h \in L_\nu^\infty(T), \|h\|_\infty \le 1$ for the general moment problem, not necessarily involving polynomials. Condition (b) is formulated in terms of quadratic forms. Here only theorem 1.1 and measure theory arguments are applied. Corollary 2.2 follows, also using Gamma Euler's function. This corollary gives a sufficient condition for the existence of a solution for a classical moment problem on an unbounded closed subset of finite two-dimensional Lebesgue measure. Moreover, the classical moments of arbitrary natural orders on this subset, with respect to Lebesgue measure, are finite.

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