On Newton's Method for Convex Functions and Operators and its Relationship with Contraction Principle

Octav Olteanu

Department of Mathematics-Informatics, University Politehnica of Bucharest, Romania

octav.olteanu50@gmail.com

Abstract

In this review paper, generally known results on a version of global Newton's method for convex increasing or decreasing functions and operators, as well as afferent examples and applications, are recalled. Connection with the contraction principle is discussed in detail and applied to approximate $AA^{1/pp}$, Where AA is a positive invertible symmetric operator acting on a finite-dimensional Hilbert space, and $pp > 1$ is a real number. Two numerical examples for 2×2 symmetric matrices with real coefficients are given. Some other nonlinear matrix or scalar equations are solved approximately.

Keywords: global Newton method; convex operators; symmetric operators; contraction principle; successive approximations.

1. Introduction

The aim of this review paper is to emphasize a global Newton like method, which works only for increasing (as well as for decreasing) convex functions and operators. The theoretical results are recalled and numerous examples are illustrating the way of applying them. The connection wit contraction principle plays a central role. Namely, we approximate the positive root $A^{1/p}$ of an arbitrary symmetric operator A acting on the finite dimensional Hilbert space H, with the spectrum $\sigma(A) \subset (0, \infty)$, when $pp \in (1, \infty)$ (see [4]). The approximation of the solution under attention is done by means of contraction principle, applied to a contraction operator (defined by means of Newton's method), having the contraction constant equal to $(pp - 1)/pp$. If U is the positive solution under attention, then clearly U verifies the equation $P(U) := U^p - A = \mathbf{00}$. Some related numerical examples are given as well. Recall that Newton's method can be used to approximate iteratively the solution of the equation

$$PP(x) = 0$$

where P is a function or operator satisfying certain conditions (see the following results and examples). In some cases, the iteration $xx_{nn+1} = \varphi\varphi(x_n x_n), nn \in \mathbb{NN}$ of Newton's method can be done by means of a contraction mapping $\varphi\varphi$. In such cases, the evaluations of the norms of the errors given by contraction principle could be better than those ensured by Newton's method. For example, if pp from the above example is close to 1, then the second approximation of the solution furnished by contraction principle is good enough (see the last result and example in the end of the paper). For other types of equations see [7]. Completions and modern approaches are outlined in [1], [2], [3], [6], [8]. Being given a Hilbert space HH, and a symmetric linear operator AA from H H to HH, the construction of a commutative algebra $YY = Y(A)$, (which is also an order complete Banach lattice) of symmetric operators acting on H, is Hstudied in detail (see Section 2 and the monograph [5]). This space of symmetric operators (in particular of symmetric matrices) plays a central role in the present work. The rest of this article is organized as follows. Section 2 mentions briefly the methods applied in the sequel. Section 3 is devoted to the main known results on the subject and is divided in subsections. Section 4 concludes the paper.

2. Methods

The following methods are used along this paper

1) Newton method for convex increasing (or decreasing) functions and operators.
2) Elements of theory of symmetric operators acting on a Hilbert space. Namely, let HH be an arbitrary Hilbert space and AA a symmetric (linear) operator acting on HH. Define

$$Y_1 = \{V \in \mathcal{S}; AV = VA\}, Y = Y(A) = \{U \in Y_1; UV = VU \ \ \forall V \in Y_1\}$$
$$Y_+ = \{U \in Y; < Uh, h > \geq 0 \ \forall h \in H\}$$

Then Y is clearly a commutative real algebra of symmetric operators. It is also an order complete real Banach lattice (for details, see [5]). Here $\mathcal{S}$ is the real ordered space of all symmetric operators acting on H.

3) Contraction principle and related successive approximation method.

3. Main Text

3.1. General type results

Let X be a $\sigma-$ order complete vector lattice, endowed with a solid $(|x| \leq |y| \Rightarrow \|x\| \leq \|y\|)$ and $o-$ continuous norm $(x_n \rightarrow_{in\ order} x \Rightarrow \|x_n - x\| \rightarrow 0)$. Let Y be a normed vector space, endowed with an order relation defined by a closed convex cone. For $a, b \in X, a < b,$ we denote $[a, b] = \{x \in X, a \leq x \leq b\}$. Pet $P \in C^1([a, b], Y)$. In most of our applications, we have $X = Y$, where Y is an order complete Banach lattice of selfadjoint operators, that is also a commutative algebra (see [5], p. 303-305) and Section 2 above).

Theorem 3.1 *Additionally assume that for each* $x\ x \in [a, b], \exists\ [P'(x)]^{-1} \in L_+(Y, X)$ *and that* $a \leq x \leq b \Rightarrow P'(a) \leq P'(x) \leq P'(b)$. *If* $P(a) < 0, P(b) > 0,$ *then there exists a unique solution* x^* *of the equation* $P(x) = 0,$ *where*

$$x^* := \inf x_k = \lim x_k, \ x_0 := b, \ x_{k+1} = x_k - [P'(x_k)]^{-1}[P(x_k)], \ k \in N. \tag{1}$$

Moreover, we have

$$a < x^* < b, \quad \|x_k - x^*\| \leq \left\|[P'(a)]^{-1}\right\| \cdot \|P(x_k)\| \rightarrow 0. \tag{2}$$

Proof. Using induction upon $k,$ we prove that

$$P(x_k) \geq 0, \ x_{k+1} \leq x_k, \quad k \in N. \tag{3}$$

The last relations (1) and the convexity of $P,$ yield

$$P(x_0) = P(b) > 0, \ P(x_{k+1}) \geq P(x_k) + [P'(x_k)](x_{k+1} - x_k) = P(x_k) - P(x_k) = 0$$

Hence $P(x_k) \geq 0 \ \forall k \in N.$ These relations lead to

$$x_{k+1} - x_k = -[P'(x_k)]^{-1}[P(x_k)] \leq 0 \Rightarrow x_{k+1} \leq x_k \quad \forall k \in N.$$

We derive the following useful relations

$$P(a) < 0, -P(x_k) \leq 0 \Rightarrow 0 \geq [P'(x_k)]^{-1}(P(a) - P(x_k)) \geq$$
$$[P'(x_k)]^{-1}([P'(x_k)](a - x_k)) = a - x_k \Rightarrow x_k \geq a, \quad \forall k \in N.$$

Using the hypothesis on the space $X,$ there exists x^* defined by the first relations (1), and from (3) we infer that the sequence $(x_k)_k$ is decreasing. Passing through the limit in the recurrence relations (1) one obtains

$$[P'(x^*)]^{-1}[P(x^*)] = 0 \Rightarrow P(x^*) = 0.$$

From the assumptions on the positivity of $P(b), -P(a),$ and from the definition of x^*, we infer that $a < x^* < b$. In order to prove (2), one uses the convexity once more

$$P(x_k) = P(x_k) - P(x^*) \geq P'(x^*)(x_k - x^*) \geq P'(a)(x_k - x^*) \Rightarrow$$
$$[P'(a)]^{-1}(P(x_k)) \geq x_k - x^* \geq 0 \Rightarrow \|x_k - x^*\| \leq \left\|[P'(a)]^{-1}\right\| \cdot \|(P(x_k))\|, \; k \in N.$$

The uniqueness of the solution follows quite easily:

$$0 = P(x_1^*) - P(x_2^*) \geq P'(x_2^*)(x_1^* - x_2^*) \Rightarrow 0 \geq [P'(x_2^*)]^{-1}(P'(x_2^*)(x_1^* - x_2^*)) = x_1^* - x_2^*.$$

Similarly, we can write: $0 \geq x_2^* - x_1^*$, hence $x_1^* = x_2^*$. □

The corresponding statement for convex decreasing operators holds.

Theorem 3.2 *Assume that for any* $x \in [a,b]$ *there exists* $[P'(x)]^{-1}$ *such that* $-[P'(x)]^{-1}(Y_+) \subset X_+,\, a \leq x \leq b \Rightarrow [P'(x)]^{-1} \geq [P'(b)]^{-1}$. *If* $P(a) > 0,\;\; P(b) < 0,$ *then there exists an unique solution* $x^* \in]a,b[$ *of the equation* $P(x) = 0,\;\; x^*$ *being given by*

$$x^* = \sup x_k = \lim x_k,\, x_0 = a,\, x_{k+1} = x_k - [P'(x_k)]^{-1}(P(x_k)), \quad k \in N.$$

Moreover, the sequence $(x_k)_k$ *is increasing and the convergence rate is given by the inequalities*

$$\|x_k - x^*\| \leq \left\|[P'(b)]^{-1}\right\| \cdot \|P(x_k)\|, \; k \in N.$$

3.2. Direct consequences

During this Section we mention some applications of the general theorems of Section 2. The difficulties consist only in technical details concerning verifying conditions from general theorems. That is why we do not prove all the statements.

Theorem 3.3 (see also [7]). *Let* H *be a finite dimensional Hilbert space and* $X = Y$ *the commutative algebra defined in* [5], *p. 303-305 and in Section 2 of the present paper. Let*

$$B_j \in X_+,\; j \in \{0,1,\ldots,n\},\; B_0 > 0,\; B_n > 0$$

be such that

$$B_0 < \sum_{j=1}^{n} B_j$$

and $nB_nU^{n-1} + \cdots + 2B_2U + B_1$ *is invertible for all* $UU \in [0, II]$. *Then there exists a unique* $\overline{U}, 0 < \overline{U} < I,$ *such that*

$$B_nB_n\overline{U}^n + \cdots + B_1\overline{U} - B_0 = 0,$$

and this solution verifies in particular the relation

$$\|I - \overline{U}\| \leq \left\|\sum_{j=1}^{n} B_j - B_0\right\|$$

Proof. The space X is an order-complete Banach lattice and a commutative algebra of symmetric operators. Let

$$P:[0,I]\subset X_+\to X$$

be defined by

$$P(U)=B_nU^n+\cdots+B_1U-B_0$$

One can show that the operator $P_n(U)=U^n$ is convex on X_+ (see [7] for details). Now it follows easily that P is also convex on X_+, since all the coefficients $B_k\in X_+$ and all the operators in X are permutable. On the other hand, we have

$$P'(U)(V)=(nB_nU^{n-1}+\cdots+2B_2U+B_1)V\Rightarrow$$

$$[P'(U)]^{-1}(\tilde{V})=(nB_nU^{n-1}+\cdots+2B_2U+B_1)^{-1}\tilde{V},\forall\tilde{V},\ U\in[\mathbf{0},\boldsymbol{I}]$$

We also have

$$[P'(U)]^{-1}\geq 0, 0\leq U\leq I\Rightarrow P'(0)V=B_1V\leq$$

$$(nB_nU^{n-1}+\cdots+2B_2U+B_1)V\leq(nB_n+\cdots+2B_2+B_1)V\Rightarrow$$

$$P'(0)\leq P'(U)\leq P'(I)\ \ \forall U\in[0,I]$$

Due to the hypothesis, we infer that

$$P(0)=-B_0<0,\quad P(I)=\sum_{k=1}^{n}B_k-B_0>0,$$

so that all requirements of Theorem 3.1 are accomplished. It follows that there exists a unique solution $\overline{U}\in]0,I[$ of the equation $P(U)=0,$ that verifies the following relation (for $kk=0$ in Theorem 3.1)

$$\|I-U\|\leq\|[P'(0)]^{-1}\|\cdot\|P(I)\|=\|B_1^{-1}\|\cdot\|B_n+\cdots+B_1+B_0\|$$

Now the proof is complete. □

Theorem 3.4. *Let H, X be as in the preceding theorem, $\alpha>1,\ B\in X$ such that the spectrum $S(B)\subset]\ln(\alpha),\infty[$. There is a unique solution $\overline{U}\in]0,I[\subset X$ of the equation*

$$\exp(BU)-\alpha I=0$$

and this solution verifies in particular the relation

$$\|I-\overline{U}\|\leq\|B^{-1}\|\cdot\|\exp B-\alpha I\|.$$

The next result is an application of the scalar version of Theorem 3.2, when $X=Y=R.$

Proposition 3.1 *Let* $\alpha, \beta, \gamma > 0$ *be such that*

$$1-\alpha \cdot \exp(-\alpha)-\beta<\gamma<1.$$

Then there exists a unique solution $x^* \in]0,1[$ *of the equation*

$$\exp(-\alpha x)-\beta x-\gamma=0$$

and we have

$$0<x^* \leq \frac{1-\gamma}{\alpha \exp(-\alpha)+\beta} \rightarrow 0, \quad \gamma \uparrow 1.$$

Theorem 3.5. *Let* H *be a finite dimensional Hilbert space, and* X *the space defined in* [5], *p. 303-305. Let* $\widetilde{A}, B, C \in X$ *be such the spectrums of* $\widetilde{A}$ *and* B *are contained in* $]0,\infty[$. *Assume also that*

$$\exp(-\widetilde{A})-B<C<I.$$

Then there exists a unique solution $\overline{U} \in]0, I[$ *of the equation*

$$\exp(-\widetilde{A}U)-BU-C=0$$

and the following estimation holds true

$$\|\widetilde{U}\| \leq \left\|\left[\widetilde{A}\exp(-\widetilde{A})+B\right]^{-1}\right\| \cdot \|I-C\| \rightarrow 0, C \rightarrow I$$

Proposition 3.2. *There is a unique solution* $x^* \in]3/2, 2[$ *of the equation*

$$2x^3-4x^2+1=0,$$

and this solution verifies

$$2-\frac{27}{152}<x^*<2.$$

Theorem 3.6. *Let* A *be a symmetric operator acting on a finite dimensional Hilbert space, with the spectrum* $S(A) \subset [3/2, 2]$. *Let* $X=Y=X(A)$ *be the space defined in Section 2. Then there exists a unique operator* $\overline{U} \in X$ *such that*

$$2\overline{U}^3-4\overline{U}^2+I=0$$

and the spectrum of this operator verifies the following relation

$$S(\overline{U}) \subset]2-\frac{27}{152}, 2[$$

3.3. Approximating $A^{1/p}$, $p>1$; connection to contraction principle

Let *H* be a finite dimensional Hilbert space, A a symmetric operator acting on H, with the spectrum

$S(A) \subset]0,\infty[$, $X=X(A)$ is defined in Section 2, as being the associated commutative algebra and order complete Banach lattice dicussed in [5], p. 303-305. We denote

$$\omega_A=\inf_{\|h\|=1}\langle Ah, h\rangle, \quad \Omega_A=\sup_{\|h\|=1}\langle Ah, h\rangle$$

Theorem 3.7. *Let* A *be as above,* $A \notin Sp\{I\}$, $p > 1$, $p \in R$. *There exists a unique operator* $U_p \in]\omega_A^{1/p} I, \Omega_A^{1/p} I[$ *such that*

$$U_p^p - A = 0,$$

and this solution verifies the relations

$$\left\|\Omega_A^{1/p} I - U_p\right\| \le \frac{1}{p\omega_A^{(p-1)/p}} \left\|\Omega_A I - A\right\|, \left\|U_p - \omega_A^{1/p} I\right\| \le \frac{\Omega_A^{(p+1)/p}}{p} \left\|\omega_A^{-1} I - A^{-1}\right\|.$$

Corollary 3.1 *With the above notations and assumptions, we have*

$$\ln \Omega_A - \ln \omega_A \le \frac{1}{\omega_A} \left\|\Omega_A I - A\right\| + \Omega_A \left\|\omega_A^{-1} I - A^{-1}\right\|.$$

Remark 3.1 If in the recurrence relation of Newton's method

$$x_{k+1} = \varphi(x_k),\ \varphi(x) := x - [P'(x)]^{-1}(P(x))$$

the mapping φ is a contraction, the rate of convergence of the sequence $(x_k)_k$ is given by contraction principle. Next, we show that this is the case of the operator $P(U) = U^p - A$, which leads to the positive solution $U_p = A^{1/p}$.

Theorem 3.8. (see also [4]). *Let* p, A, X *be as above. Then the Newton recurrence for the equation*

$$P(U) = U^p - A = 0$$

is

$$U_0 = \Omega_A^{1/p} I, \quad U_{k+1} = \varphi(U_k) = \frac{p-1}{p} U_k + \frac{1}{p} U_k^{-p+1} A, \quad k \in N.$$

The convergence rate for $U_k \to A^{1/p}$ *is given by*

$$\left\|U_k - A^{1/p}\right\| \le \left(\frac{p-1}{p}\right)^k \Omega_A^{1/p} \left\|I - \Omega_A^{-1} A\right\|, \quad k \in N.$$

Proof. Newton's sequence for the convex operator P is

$$U_0 = b = \Omega_A^{1/p} I, U_{k+1} = U_k - \left(pU_k^{p-1}\right)^{-1}\left(U_k^p - A\right) =$$
$$U_k - \frac{1}{p} U_k^{-p+1+p} + \frac{1}{p} U_k^{-p+1} A = \frac{p-1}{p} U_k + \frac{1}{p} U_k^{-p+1} A = \varphi(U_k), k \in N.$$

Let

$$M := \{U \in X; U \ge A^{1/p}\},$$

here the root is obtained by the aid of functional calculus for A. Clearly, M is closed in X, hence it is complete. Let $\varphi : M \to X$ be defined by

$$\varphi(U)=\frac{p-1}{p}U+\frac{1}{p}U^{-p+1}A, \quad U\in M.$$

A straightforward computation shows that $\varphi\left(A^{1/p}\right)=A^{1/p}$. First we show that

$$S(U)\subset]0,\infty[\Rightarrow \varphi(U)\in M;$$

(in particular this proves that $\varphi(M)\subset M$). One can show that φ is convex on the subset of all operators in X having the spectrum contained in the positive semiaxis. In particular, φ is convex on M. Direct computations yield

$$\varphi(U)\geq \varphi\left(A^{1/p}\right)+\varphi'\left(A^{1/p}\right)\left(U-A^{1/p}\right)=A^{1/p}.$$

Thus $\varphi(U)\in M$ for all U with spectrum $S(U)\subset]0,\infty[$. Now we prove that $\varphi: M\rightarrow M$ is a contraction, with contraction constant $q=\frac{p-1}{p}$. Precisely we prove that

$$\|\varphi'(U)\|\leq \frac{p-1}{p}, \quad \forall U\in M.$$

Indeed, we have

$$\|\varphi'(U)\|=\frac{p-1}{p}\left\|I-U^{-p}A\right\|; U\in M\Rightarrow U^{p}\geq A\Rightarrow I-U^{-p}A\geq 0\Rightarrow$$

$$\left\|I-U^{-p}A\right\|\leq \|I\|=1\Rightarrow \|\varphi'(U)\|\leq \frac{p-1}{p}, \quad U\in M.$$

Now the conclusion on φ being a contraction follows by a standard differential calculus argument. Application of contraction theorem and an elementary computation shows that

$$\left\|U_k-A^{1/p}\right\|\leq \frac{q^k}{1-q}\|U_0-\varphi(U_0)\|=\left(\frac{p-1}{p}\right)^k \Omega_A^{1/p}\left\|I-\Omega_A^{-1}A\right\|, \quad k\in N.$$

This concludes the proof. □

Numerical examples

1) We approximate $\begin{pmatrix}2 & 2\\ 2 & 5\end{pmatrix}^{1/(\ln 6)}$, evaluating the norm of the error. Notice that $\ln(6)$ is not an integer. Therefore, simple recurring approximating sequences of the matrix from above might be difficult to find. We apply the previous Theorem 3.8. Consider the linear symmetric operator *A* defined by the matrix $\begin{pmatrix}2 & 2\\ 2 & 5\end{pmatrix}$, applying $\mathbf{R}^2$ onto itself. The *spectrum* of *A* is $\sigma(A)=\{1,6\}\subset]0,\infty[$, and *A* is not contained in $\operatorname{Span}\{I\}$, so that all conditions of theorem 3.8 are accomplished. Applying the latter theorem to $p=ln(6)$, one obtains the following two approximations of $A^{1/(\ln 6)}$:

$$U_0=\Omega_A^{1/(\ln 6)}=6^{1/(\ln 6)}I=eI,$$

$$U_1=\frac{\ln 6-1}{\ln 6}U_0+\frac{1}{\ln 6}U_0^{1-\ln 6}A=\frac{e}{6\cdot(\ln 6)}\begin{pmatrix}6\ln 6-4 & 2\\ 2 & 6\ln 6-1\end{pmatrix}.$$

For the evaluation of the norm of the error corresponding to the second approximation U_1, the last relation in the statement of Theorem 3.8 is applied, where $k=1,\ p=\ln 6$. One deduces

$$\| U_1 - A^{1/(\ln 6)} \| \le \frac{\ln 6 - 1}{\ln 6} 6^{1/(\ln 6)} \| I-(1/6)A\| =$$
$$= \left(1-\frac{1}{\ln 6}\right) e \| B\|,\ B := I - \left(\frac{1}{6}\right) A = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 1/6 \end{pmatrix}.$$

The spectrum of the matrix *B* is $\sigma_B = \{0, 5/6\} \subset [0,\infty[$ and *B* is symmetric, so that $\| B\| = \Omega_B = \frac{5}{6}$. Thus

$$\| U_1 - A^{1/(\ln 6)} \| \le \left(1-\frac{1}{\ln 6}\right) e \frac{5}{6} \le \left(1-\frac{1}{1.8}\right) \cdot 2.72 \cdot \frac{5}{6} = 1+\frac{1}{135}.$$

The conclusion is

$$A^{1/(\ln 6)} = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}^{1/(\ln 6)} \approx \frac{e}{6\cdot(\ln 6)} \begin{pmatrix} 6\ln 6 - 4 & 2 \\ 2 & 6\ln 6 - 1 \end{pmatrix} = U_1$$

and the norm of the error in the latter approximation is smaller than $1+\frac{1}{135}$.

2) Next we approximate $\begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}^{1/(n^{1/n})}$, $n \in \mathrm{N},\ n \ge 2,$ evaluating the norm of the error. Using the notations and some of the results for preceding Example 1), and applying Theorem 3.8 to the $X = Y = Y(A)$ defined in Section 2, associated to the operator (or matrix) *A*, we find:

$$U_0(n) = 6^{1/(n^{1/n})} I,\quad U_1(n) = \frac{n^{1/n}-1}{n^{1/n}} 6^{1/(n^{1/n})} I + \frac{1}{n^{1/n}} A \left(6^{1/(n^{1/n})} I\right)^{1-n^{1/n}} =$$
$$= \frac{1}{n^{1/n}} \begin{pmatrix} (n^{1/n}-1)6^{1/(n^{1/n})} + 2\cdot 6^{\frac{1-n^{1/n}}{n^{1/n}}} & 2\cdot 6^{\frac{1-n^{1/n}}{n^{1/n}}} \\ 2\cdot 6^{\frac{1-n^{1/n}}{n^{1/n}}} & (n^{1/n}-1)6^{1/(n^{1/n})} + 5\cdot 6^{\frac{1-n^{1/n}}{n^{1/n}}} \end{pmatrix}.$$

Thus $\begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}^{1/(n^{1/n})} \approx U_1(n)$ and the norm of the error follows from the last evaluation in the statement of Theorem 3.8

$$\| A^{1/(n^{1/n})} - U_1(n) \| \le \frac{n^{1/n}-1}{n^{1/n}} 6^{1/(n^{1/n})} \cdot \left\| I - \frac{1}{6} A \right\|.$$

Applying the result from Example 1) for $\left\| I - \frac{1}{6} A \right\|$, the latter relation further yields

$$\| A^{1/(n^{1/n})} - U_1(n) \| \le \frac{5}{6^{1-1/(n^{1/n})}} \cdot \frac{n^{1/n}-1}{n^{1/n}} \le 5(n^{1/n}-1) \to 0,\quad n \to \infty$$

The conclusion is that for large n, the second approximation $U_1(n)$ is good enough.

4. Conclusions

We have reviewed and applied some of our earlier results on Newton like method for convex increasing (or decreasing) operators (and, in particular, for convex monotone functions), having continuous first derivatives. The method is illustrated by means of examples involving concrete equations. The strength of the method consists in its global character, while the weakness is that is applicable only for convex functions and operators. This is a review paper, completed by two new numerical examples.

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