On Tensor Products and Elementary Operators

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Abstract

In This Paper We Describe Operator Systems And Elementary Operators Via Tensor Products. We Also Discuss Norms Of Elementary Operators.

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Introduction

Spectral Analysis Is One Of The Important Directions Of Functional Analysis. The Development Of Physical Sciences Is Becoming More And More A Challenge To Mathematicians. In Particular, The Resolution Of The Problems Associated With The Physical Processes And, Consequently, The Study Of Partial Differential Equations And Mathematical Physics Equations, Requires A New Approach. The Method Of Separation Of Variables In Many Cases Turns Out To Be The Only Acceptable, Since It Reduces Finding A Solution To A Complex Equation With Many Variables To Find A Solution To A System Of Ordinary Differential Equations, Which Are Much Easier To Study. In This Paper We Use Tensor Products To Characterize Elementary Operators Which We Describe In This Section In Details. The Field Of Analysis Has Been Very Interesting Especially On The Study Of Elementary Operators For Many Decades. Sylvester In 1880s [10], Computed The Eigenvalues Of The Matrix Operators On A Square Matrix. This Work Has Been Of Great Concern Especially In The Applications Of Operator Theory And Functional Analysis. Later, Lumer And Rosenblum [5] Described The Elementary Operator From A Mapping T : A → A If It Can Be Expressed As T : B(H) → B(H) By $T_{ai,Bi}(X) = \sum^{N}_{i=1} A_i X B_i$ ∀ X ∈ B(H) And ∀ A_i , B_i Fixed In B(H) And 1 ≤ I < N. The Study Of Operator Theory Has Been Significant Dating Back Many Decades Ago. Some Research Has Been Done Though Not Exhaustive. Studies About Elementary Operators Have Been Of Much Concern. We Define An Elementary Operator T : B(H) → B(H) [6] By $T_{ai,Bi}(X) = \sum^{N}_{i=1} A_i X B_i$ ∀ X ∈ B(H) And ∀ A_i , B_i Fixed In B(H) Where I = 1, . . . , N. [3] From This Operator, We Can Define The Generalized Adjoint By $T_{ai,Bi}(X) = \sum^{N}_{i=1} A_i^* X B_i^*$ And We Say That T Is Normal If And Only If T T*= T*T. Now AC = CA, BD = DB, Together With AA*= A*A, BB*= B*B, CC*= C*C And DD*= D*D Ensures That The Operator $T_{ai,Bi}(X)$ = AXC + BXD Is Normal. Some Of Our Results Show That; If T ∈ B(H) Be A P-Hyponormal And T = U |T | Be Polar Decomposition Of T Such That U^{n0}= U* For Some Positive Integer N0 Then T Is Normal. Moreover, If T ∈ B(H) Be A P-Hyponormal Ant T = U |T | Be The Polar Decomposition Of T Such That U^*N→1 Or Un→ 1 As N → ∞, Where Limits Are Taken In The Strong Operator Topology. Then T Is Normal. For An Operator A To Be Normal, It Is Also Necessary That A = A*. It Is Sufficient That For An Operator A To Be Normal Then The Condition AA* = A*A Holds. This Knowledge Is Important Especially In Quantum Physics Especially The Formulation Of Heisenberg Uncertainty Principle For Linear Transformations And Non-Zero Scalars Such That AX − XA = Ai. The Study Can Also Be Used In The Solutions Of Schr ¨Ondinger Wave Equations Since The Infimum Of The Hamiltonian Operator Is Always An Eigenvalue And Its Corresponding Eigenvector Are Called The Ground State Energies E Giving Us A Formulation Of E As ($E_{C3, H8}$). Over The Past Years, Several Scholars Have Joined In Research To Describe Several Properties Related To The Structure Of The Elementary Operators. In [11] They Described Sylvester And Lyapunov Operators In Real And Complex Matrices Which Included In Particular Cases Operators Arising From The Theory Of Linear Time Invariant System. Fanqyan [30] Described The Multiplicative Mappings Of Operator Algebras. They Described The Nest Algebra As Being The Natural Analogues Of Upper Triangular Matrix Algebra In The Infinite

Dimensional Hilbert Space. Gheondea [1], Described The Normality Of Elementary Operators Based On The Spectral Theorem For The Normal Operators. This Study Postulated That If N ∈ B(H) Is A Normal Arbitrary Such That AN = NA Then AN*= N*A As Well Is Normal. This Shows That That If A, B ∈ B(H) Are Two Normal Operators That Commute And Each Commutes With Its Adjoint, Then Their Product Is AB Is Normal. The Study Further States That If A And B Are Bounded Operators Such That AB Is Normal And Compact, Then BA Is Normal And Compact As Well And Sk(AB) = Sk(BA) For All K = 1, 2, . . . In This Paper We Describe Operator Systems And Elementary Operators Via Tensor Products. We Also Discuss Norms Of Elementary Operators.

Preliminaries

In This Section We Give The Preliminary Concepts Which Are Useful In The Sequel.

*Definition 1***.** A Norm Is A Non-Negative Real Valued Function That Takes The Elements Of A Vector Space To A Field Of Real Numbers Denoted By ‖.‖: V → R Satisfying The Following Conditions:

(I.) Non-Negativity: ‖X‖ ≥ 0, ∀ X ∈ V.

(Ii.) Zero Property: ‖X‖ = 0, If And Only If X=0, For All X ∈ V.

(Iii.) Homogeneity: ‖Ax‖ ≤ |A|‖X‖, ∀ X ∈ V And A ∈ F

(Iv.) Triangle Inequality: ‖X + Y‖ ≤ ‖X‖ + ‖Y‖, ∀ X And Y ∈ V

The Pair (V, ‖.‖) Is Called A Normed Linear Space.

*Definition 2***.** Let H Be An Infinite Dimensional Complex Hilbert Space And B(H) Be An Algebra Of All Bounded Linear Operators On The H . We Define An Elementary Operator T : B(H) → B(H) By $T_{ai,Bi}(X) = \sum_{i=1}^{N} A_i X B_i$ ∀ X ∈ B(H) And ∀ A_i , B_i Fixed In B(H) Where I = 1, . . . , N. Examples Of Elementary Operators Include:

(I). The Left Multiplication Operator L_A: B(H) By: $L_A(X)$ = AX , ∀X ∈ B(H).

(Ii). The Right Multiplication Operator R_B: B(H) By: R_B (X)=BX , ∀X ∈ B(H).

(Iii). The Basic Elementary Operator (Implemented By A, B) By: $M_{A,B}$ (H) = AXB, ∀X ∈ B(H).

(Iv). The Jordan Elementary Operator (Implemented By A, B) By: $U_{A,B}$ (X)=AXB + BXA, ∀X ∈B(H).

(V). The Generalized Derivation (Implemented By A, B) By: $\Delta_{a,B} = L_A - R_B$.

(Vi).The Inner Derivation (Implemented By A, B) By: Δ_a = AX – XA.

*Definition 3***.** Let H Be An Infinite Dimensional Complex Hilbert Space And B(H) Be The Algebra Of All Bounded Linear Operators On H. We Define An Elementary Operator, T : B(H) → B(H) By $T_{ai,Bi}(X) = \sum_{i=1}^{N} A_i X B_i$ ∀ X ∈ B(H) And ∀ A_i , B_i Fixed In B(H) Where I = 1, . . . , N**.** From This Operator, We Can Define The Generalized Adjoint By $T_{ai,Bi}(X) = \sum_{i=1}^{N} A_i^* X B_i^*$ And We Say That T Is Normal If And Only If T T*= T*T. Now AC = CA, BD = DB, Together With AA*= A*A, BB*= B*B, CC*= C*C And DD*= D*D Ensures That The Operator $T_{ai,Bi}(X)$ = AXC + BXD Is Normal. Therefore, The Elementary Operator Of The Form: $T_{ai,Bi}(X) = \sum_{i=1}^{N} A_i X B_i$ Where A_i And B_i Are Commuting Families Of Normal Operators Are Called Normally Represented Elementary Operator.

Next We Give In Details Some Definitions And Concepts From The Theory Of Multiparameter Operator Systems Necessary For Understanding Of The Further Considerations.

Let The Linear Multiparameter System Be In The Form:

$$B_k(\lambda)x_k = (B_{0,k} + \sum_{i=1}^{n} \lambda_i B_{i,k})x_k = 0, \quad k = 1, 2, \ldots, n \qquad (1)$$

Where Operators $B_{k,i}$ Act In The Hilbert Space H_i

Definition 4. [1,2,11] $\lambda=(\lambda_1,\lambda_2,...,\lambda_n)\in C^n$ Is An Eigenvalue Of The System (1) If There Are Non-Zero Elements $x_i\in H_i$, $i=1,2,...,n$ Such That (1) Is Satisfied, And Decomposable Tensor $x=x_1\otimes x_2\otimes...\otimes x_n$ Is Called The Eigenvector Corresponding To Eigenvalue $\lambda=(\lambda_1,\lambda_2,...,\lambda_n)\in C^n$.

Definition 5. The Operator $B_{s,i}^{+}$ Is Induced By An Operator $B_{s,i}$,Acting In The Space H_i , Into The Tensor Space $H=H_1\otimes...\otimes H_n$, If On Each Decomposable Tensor $x=x_1\otimes...\otimes x_n$ Of Tensor Product Space $H=H_1\otimes...\otimes H_n$ We Have $B_{s,i}^{+}x=x_1\otimes...\otimes x_{i-1}\otimes B_{s,i}x_i\otimes x_{i+1}\otimes...\otimes x_n$ And On All The Other Elements Of $H=H_1\otimes...\otimes H_n$ The Operator $B_{s,i}^{+}$ Is Defined On Linearity And Continuity.

Definition 5 ([5], [6]). Let $x_{0,...,0}=x_1\otimes x_2\otimes...\otimes x_n$ Be An Eigenvector Of The System (1), Corresponding To Its Eigenvalue $\lambda=(\lambda_1,\lambda_2,...,\lambda_n)$; Then $x_{m_1,...,m_n}$ Is $m_1,m_2,...,m_n$ - Th Associated Vector (See[4]) To An Eigenvector $x_{0,0,...,0}$ Of The System (1) If There Is A Set Of Vectors $\{x_{i_1,i_2,...,ii_n}\}\subset H_1\otimes\cdots\otimes H_n$, Satisfying To Conditions

$$B_{0,i}^{+}(\lambda)x_{is,s_2,...,s_n}+B_{1,i}^{+}x_{s_1-1,s_2,...,s_n}+...+B_{n,i}^{+}x_{s_1,...,s_{n-1},s_n-1}=0\,.\,x_{i_{s1},s_2,...,s_n}=0\ ,\quad \text{When}\quad s_i<0 \quad (2)$$

$$0\le s_r\le m_r,\ r=1,2,...,n,\ i=1,...,n$$

For The Indices $s_1,s_2,...,s_n$ In Element $(x_{i_1,i_2,...,i_n})\subset H_1\otimes\cdots\otimes H_n$, There Are Various Arrangements From Set Of Integers On n With $0\le s_r\le m_r,\ r=1,2,...,n,$.

Definition 6. In [1,3, 11] The System (1) Is An Analogue Of The Cramer's Determinants, When The Number Of Equations Is Equal To The Number Of Variables, And Is Defined As Follows: On Decomposable Tensor $x=x_1\otimes...\otimes x_n$ Operators Δ_i Are Defined With Help The Matrices

$$\sum_{i=0}^{n}\alpha_i\Delta_i x==\otimes\begin{pmatrix}\alpha_0 & \alpha_1 & \alpha_2 & .... & \alpha_n\\ B_{0,1}x_1 & B_{1,1}x_1 & B_{2,1}x_1 & ... & B_{n,1}x_1\\ B_{0,2}x_2 & B_{1,2}x_2 & B_{2,2}x_2 & ... & B_{n,2}x_2\\ B_{0,3}x_3 & B_{1,3}x_3 & B_{2,3}x_3 & ... & B_{n,3}x_3\\ ... & ... & ... & ... & ...\\ B_{0,n}x_n & B_{1,n}x_n & B_{2,n} & ... & B_{n,n}\end{pmatrix} \quad (3)$$

Where $\alpha_0,\alpha_1,...,\alpha_n$ Are Arbitrary Complex Numbers, Under The Expansion Of The Determinant Means Its Formal Expansion, When The Element $x=x_1\otimes x_2\otimes...\otimes x_n$ Is The Tensor Products Of Elements $x_1,x_2,...,x_n$ If $\alpha_k=1,\alpha_i=0,\ i\ne k$, ,Then Right Side Of (10) Equal To $\Delta_k x$, Where $x=x_1\otimes x_2\otimes...\otimes x_n$ On All The Other Elements Of The Space H Operators Δ_i Are Defined By Linearity And Continuity. $E_s(s=1,2,...,n)$ Is The Identity Operator Of The Space H_i .Suppose That For All $x\ne 0$, $(\Delta_0 x,x)\ge\delta(x,x),\ \delta>0$, And All $B_{i,k}$ Are Self adjoint Operators In The Space H_i . Inner Product [.,.] Is Defined As Follows; If $x=x_1\otimes x_2\otimes...\otimes x_n$ And $y=y_1\otimes y_2\otimes...\otimes y_n$ Are Decomposable Tensors, Then $[x,y]=(\Delta_0 x,y)$ Where (x_i,y_i) Is The Inner Product In The Space. H_i .On All The Other Elements Of The Space H The Inner Product Is Defined On Linearity And Continuity. In Space H With Such A Metric All Operators $\Gamma_i=\Delta_0^{-1}\Delta_i$ Are Self adjoin

Definition 7.([7],[8] [10]). Let Two Operator Pencils Depending On The Same Parameter And acting In, Generally Speaking, In Various Hilbert Spaces Be As Follows

$$A(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2 + ... + \lambda^n A_n,$$
$$B(\lambda) = B_0 + \lambda B_1 + \lambda^2 B_2 + ... + \lambda^m B_m$$

Operator $\operatorname{Res}(A(\lambda), B(\lambda))$ Is Presented By The Matrix

$$\begin{pmatrix} A_0 \otimes E_2 & A_1 \otimes E_2 & ... & A_n \otimes E_2 & ... & 0 \\ . & . & ... & . & ... & . \\ 0 & 0 & ...A_0 \otimes E_2 & A_1 \otimes E_2 & ... & A_n \otimes E_2 \\ E_1 \otimes B_0 & E_1 \otimes B_1 & ... & E_1 \otimes B_m & .. & 0 \\ . & . & ... & . & ... & . \\ . & . & ...E_1 \otimes B_0 & E_1 \otimes B_1 & .... & E_1 \otimes B_m \end{pmatrix}$$

Which acts In The $(H_1 \otimes H_2)^{n+m}$ - Direct Sum Of $n+m$ Copies of The Space $H_1 \otimes H_2$ In A Matrix (4),The Number Of Rows With Operators A_i Is Equal To Leading Degree Of The Parameter λ In Pencils $B(\lambda)$ And The Number Of Rows With B_i Is Equal To The Leading Degree Of Parameter λ In $A(\lambda)$. The Notion Of Abstract Analog Of Resultant Of Two Operator Pencils Is Considered In The[7] For The Case Of The Same Leading Degree Of The Parameter In Both Pencils And In The [2]For, Generally Speaking, Different Degree Of The Parameters In The Operator Pencils.

Theorem1 [7, 8]. Let For All Operators Bounded In Corresponding Hilbert Spaces, One Of Operators A_n Or B_m Has Bounded Inverse. Then Operator Pencils $A(\lambda)$ And $B(\lambda)$ Have A Common Point Of Spectra If And Only If

$$Ker \operatorname{Res}(A(\lambda), B(\lambda)) \neq \{\vartheta\}$$

Remark1. If The Hilbert Spaces H_1 And H_2 Are The Finite Dimensional Spaces Then A Common Points Of Spectra Of Operator Pencils $A(\lambda)$ And $B(\lambda)$ Are Their Common Eigenvalues.(See [6], [7].)

$$\{B_i(\lambda) = B_{0,i} + \lambda B_{1,i} + ... + \lambda^{k_i} B_{k_i,i}, \quad i = 1, 2, ..., n$$

$B_i(\lambda)$ - Operator Bundles Acting In A Finite Dimensional Hilbert Space H_i Correspondingly. Suppose That $k_1 \geq k_2 \geq ... \geq k_n$. In The Space $H^{k_1+k_2}$ (The Direct Sum Of $k_1 + k_2$ Tensor Product $H = H_1 \otimes ... \otimes H_n$ Of Spaces $H_1, H_2, ..., H_n$) Are Introduced The Operators R_i $(i = 1, ..., n-1)$ With The Help Of Operational Matrices (3.12) Let $B_i(\lambda)$ Be The Operational Bundles Acting In A Finite Dimensional Hilbert Space H_i, Correspondingly. Without Loss Of copies With

$$R_{i-1} = \begin{pmatrix} B^+_{0,1} & B^+_{1,1} & \cdots & B^+_{k_1,1} & \cdots & 0 \\ 0 & B^+_{0,1} & B^+_{1,1} \cdot\cdot & B^+_{k_1-1.1} & B^+_{k_1,1} \cdot\cdot & 0 \\ . & . & \cdot\cdot & . & \cdots & . \\ 0 & 0 & \cdot\cdot B^+_{0,1} & B^+_{1,1} & \cdots & B^+_{k_1,1} \\ B^+_{0,i} & B^+_{1,i} & \cdots & B^+_{k_i,i} & 0.\cdot\cdot & 0 \\ 0 & B^+_{0,i} & B^+_{1,i} \cdots & \cdot & B^+_{k_i,i} \cdot\cdot & 0 \\ . & . & \cdots & . & \cdots & . \\ 0 & 0 & \cdots B^+_{0,i} & B^+_{1,i} & \cdots & B^+_{k_i,i} \end{pmatrix},$$

$$i = 2, 3, ..., n$$

The Number Of Rows With Operators $B_{s,1}, s=0,1,...,k_1$ In The Matrix R_{i-1} Is Equal To k_2 And The Number Of Rows With Operators $B_{s,i}, s=0,1,...,k_i$ Is Equal To k_1. We Designate $\sigma_p(B_i(\lambda))$ The Set Of Eigenvalues Of An Operator $B_i(\lambda)$.From [5] We Have The Result:

Theorem 2. [9] $\bigcap_{i=1}^{n} \sigma_p(B_i(\lambda)) \neq \{\theta\}$ If And Only If $\bigcap_{i=1}^{n-1} KerR_i \neq \{\theta\}$, $(KerB_{k_1} = \{\theta\})$.

Tensor Product And Operator Systems

Consider The System

$$A_{i,j,s}(\lambda)x_s = (A_{0,s} + \sum_{r=1}^{k_{1,s}} \lambda_1^r A_{1,r,s} + ... + \sum_{r=1}^{k_{n,s}} \lambda_n^r A_{n,r,s} + \\ + \sum \lambda_1^{i_1} \lambda_2^{i_2} ... \lambda_n^{i_n} A_{i_1,...,i_n} \\ s = 1,2,...,n \tag{4}$$

The Parameters $\lambda_1, \lambda_2, ..., \lambda_n$ Enter The System Nonlinearly, And The System (4) Contains Also The Products Of These Parameters. Divide The System Of Equations (4) Into Groups Of n In Each Group. If Some Equations Remains Outside, These Equations We Add By Other Operators From The System (4). Each Group Contains n Operators And Will Be Considered Separately.

In (4) The Coefficients Of The Parameter λ_m^r, $r \le k_m$, $m = 1,2,...,n$ Are The Operators $A_{i,m,j}$, Which Act In The Space H_j, Index i Indicate On The Parameter λ_i, Index k - On The Degree Of The Parameter λ_i.

We Introduce The Notations:

$$\lambda_m^r = \lambda_{k_1+k_2+...+k_{m-1}+r}, \quad r \le k_m, \quad m = 1,2,...,n \tag{5}$$

Further , We Numerate The Different Products Of Variables $\lambda_1, \lambda_2, ..., \lambda_n$ In The System (4) On Increasing Of The Degrees Of The Parameter λ_1. Let The Numbers Of Term With The Products Of The Parameters $\lambda_1, \lambda_2, ..., \lambda_n$ Are Equal To r Put Further

$$\lambda_1^{i_1} \lambda_2^{i_2} ... \lambda_n^{i_n} = (\lambda_1^{i_1} \lambda_2^{i_2} ... \lambda_n^{i_n})_t = \tilde{\lambda}_{k_1+k_2+...+k_n+t}, \quad t \le r,$$

Where $t \le s$ Is The Number Which Correspond The Multiplier At $\lambda_1^{i_1} \lambda_2^{i_2} ... \lambda_n^{i_n}$ The Ordering Of Multiplies Of Parameters In The System (4). So In New Notations To The Product $\lambda_1^{i_1} \lambda_2^{i_2} ... \lambda_n^{i_n}$ Correspond The Parameter $\tilde{\lambda}_{k_1+k_2+...+k_n+t}$, $t \le r$ ($\lambda_1^{i_1} \lambda_2^{i_2} ... \lambda_n^{i_n} \to \tilde{\lambda}_{k_1+k_2+...+k_n+t}$, $t \le r$), Accordingly, Operators

$$A_{r,s,i} = D_{k_1+k_2+...+k_{s-1}+s,i}, r = 1,2,...,n; s = 1,2,...,k_r; \\ i = 1,2,...,n$$

$$k_r = \max k_{r,i}, i = 1,2,...,k, \tag{6}$$

$$A_{k_1,k_2...,k_n;i} = D_{k_1+k_2+...+k_m+t,i}, t = 1,2,...,s;; i = 1,2,...,n$$

When s Is The Number Of Different Products Of Parameters, Entering The System(4). In New Notations The System (4) In The Tensor Product Of Spaces $H_1 \otimes H_2 \otimes ... \otimes H_n$ Contains $k_1 + k_2 + ... + k_n + s$ Parameters And n

Equations. Let $k_1+k_2+...+k_n=k$ Then

$$\sum_{r=0}^{n}\sum_{k=1}^{k_r}[\tilde{\lambda}_{k_1+k_2+...+k_{r-1}+k}D_{k_1+k_2+...+k_{r-1}+k,i}]x_i+[\sum_{k=1}^{r}\tilde{\lambda}_{k+t}D_{k+t,i}]x_{i=0}=0 \quad (7)$$
$$k_0=0;\ k_{-i}=0;\ i=1,2,...,n$$

Adding The System (7) With Help Of New Equations So Manner That The Connections Between The Parameters, Following From The Equations Of The System (4), Satisfy. Introduce The Operators $T_0,T_1,T_2,\overline{T_0},\overline{\overline{T}}_0$ Acting In The Finite Dimensional Space R^2 And Defining With Help Of The Matrices

$$T_0=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad T_1=\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad T_2=\begin{pmatrix}0&0\\0&1\end{pmatrix},\quad \overline{\overline{T}}_0=\begin{pmatrix}0&1\\0&0\end{pmatrix},$$

$$T_{1,s_1,r}=\begin{pmatrix}1&0&0&...&0&0&0\\0&1&0&...&0&0&0\\0&0&1&...&0&0&0\\0&0&0&...&0&0&0\\.&.&.&...&.&.&.\\0&0&0&...&0&0&0\\0&0&0&...&0&0&0\end{pmatrix},...T_{k_n+1,s_n,r}=\begin{pmatrix}0&0&0&...&0&0&0\\0&0&0&...&0&0&0\\0&0&0&...&0&0&0\\0&0&0&...&0&0&0\\.&.&.&...&.&.&.\\0&0&0&...&0&1&0\\0&0&0&...&0&0&1\end{pmatrix}\ T_{(s_1,s_2,...,s_n)_r}=\begin{pmatrix}0&0&0&...&0&0&1\\1&0&0&...&0&0&0\\0&1&0&...&0&0&0\\0&0&1&...&0&0&0\\.&.&.&...&.&.&.\\0&0&0&...&1&0&0\\0&0&0&...&0&1&0\end{pmatrix} \quad (8)$$

The Number1 Stands On The Diagonal Elements Of The First s_1 Rows Of The Matrix $T_{1,s_1,r}$; Diagonal Elements Of The Rows $s_1+s_2+...+s_{i-1}+1,...,s_1+s_2+...+s_i$ Of The Matrix $T_{k_i+1,s_{i+1},r}$ Is Equal Also To 1 And So On. Besides, All Matrices $T_{1,s_1,r},...,T_{k_i+1,s_{i+1},r},...,T_{(s_1,s_2,...,s_n)_r}$ Have The Order $s_1+s_2+...+s_n$.

Adding The System (7) By The Following Equations

$$(T_{2,n+1}+\tilde{\lambda}_1T_{0,n+1}+\tilde{\lambda}_2T_{1,n+1})x_{n+1}=0$$
$$..$$
$$(\tilde{\lambda}_{k_1+k_2-2}T_{2,n+k_1+k_2-2}+\tilde{\lambda}_{k_1+k_2-1}T_{0,n+k_1+k_2-2}+$$
$$+\tilde{\lambda}_{k_1+k_2}T_{1,n+k_1+k_2-2})x_{n+k_1+k_2-2}=0$$
$$..$$

$$(\tilde{\lambda}_{k_1+...+k_{n-1}-2}T_{2,1+\sum_{i=1}^{n-1}k_i}+\tilde{\lambda}_{k_1+...+k_{n-1}-1,0}T_{0,1+\sum_{i=1}^{n-1}k_i})x_{n+k_1+...+k_{n-1}-2}=0$$

$$(\tilde{\lambda}_{k_1+...+k_n-2}T_2+\tilde{\lambda}_{k_1+...+k_n-1}T_0+\tilde{\lambda}_kT_1)x_k=0$$
$$x_s\in R^2,\ s>n \quad (9)$$

$$(T_{0,t}+\tilde{\lambda}_1T_{i_1,t}+\tilde{\lambda}_{k_1+1}T_{i_2,t}+...+\tilde{\lambda}_{k_1+...+k_{n-1}+1}T_{i_n,t}-$$
$$-\tilde{\lambda}_{k+(i_1,i_2,...,i_n)_t}T_{(i_1,i_2,...,i_n)})x_t=0$$
$$t=1,2,...,r$$

Denote $\tilde{\lambda}_{k+(i_{11},i_2,...,i_n)_r}$ The Multiplier $\lambda_1^{i_1}\lambda_2^{i_2}...\lambda_n^{i_n}$ Of The Parameters, Entering The System (4) Having The Coefficient $A_{(i_{11},i_2,...,i_n)_r}$. System ((4),(9)) Form The Linear Multiparameter System, Containing $k_1+k_2+...+k_n+r$ Equations And $k_1+k_2+...+k_n+s$ Parameters. To This System We May Apply All Results, Given In The Beginning Of This Paper.

Theorem3. [4]. Let The Following Conditions:

A) Operators $A_{k,t}, A_{k_1,k_2,...,k_n;t}$ In The Space H_i Are Bounded At The All Meanings i And k.

B) Operator Δ_0^{-1} Exists And Bounded satisfy:

Then The System Of Eigen And Associated Vectors Of (4) Coincides With The System Of Eigen And Associated Vectors Of Each Operators $\Gamma_i (i = 1,2,...,n)$ Given Two Equations From (9). Let The Equations Be:

$$\begin{aligned}(T_2 + \lambda_1 T_0 + \lambda_2 T_1)x_{n+1} = 0 \\ (\lambda_1 T_2 + \lambda_2 T_0 + \lambda_3 T_1)x_{n+2} = 0\end{aligned} \quad (10)$$

Let $\lambda_1 \neq 0$ И $x_{n+1} = (\alpha_1, \beta_1) \neq 0$ Is The Component Of The Eigenvector Of The System ((4),(9)). We Have

$$\left(\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} + \lambda_1 \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \lambda_2 \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right)(\alpha_1, \beta_1) = 0,$$

$$\lambda_1 \beta_1 + \lambda_2 \alpha_1 = 0, \ \beta_1 + \lambda_1 \alpha_1 = 0, \ \lambda_2 \neq 0; \lambda_2 = \lambda_1^2.$$

Further From The Condition $\lambda_1 \neq 0, \lambda_2 \neq 0, x_{n+2} = (\alpha_2, \beta_2) \neq 0$ It Follows $\lambda_2 \beta_2 + \lambda_3 \alpha_2 = 0, \ \lambda_1 \beta_2 + \lambda_2 \alpha_1 = 0$ And Consequently, $\tilde{\lambda}_1 \tilde{\lambda}_3 = \tilde{\lambda}_2^2$. Earlier We Proved That $\tilde{\lambda}_2 = \lambda_1^2$, Consequently, $\tilde{\lambda}_3 = \lambda_1^3$.

On Analogy For Other Parameters Of ((4),(9)): If $(\tilde{\lambda}_1, \tilde{\lambda}_2, ..., \tilde{\lambda}_{k_1+k_2+...+k_n+s})$ Is The Eigenvalue Of The System -((4), (9)), Then $\lambda_4 = \lambda_1^4$, ..., $\lambda_{k_1} = \lambda_1^{k_1}$, ... , $\tilde{\lambda}_{k_1+k_2+...+k_r+s} = \lambda_{r+1}^s$, $r = 1,2,...,n-1; s = 1,2,...,k_n$.

To Each Multiplier Of Parameters $(\tilde{\lambda}_{j_1}^{r_{j_1}} \tilde{\lambda}_{j_2}^{r_{j_2}} \cdots \tilde{\lambda}_{j_k}^{r_{j_k}})_t = \tilde{\lambda}_{k+t}$; $t \leq r$ It Is Corresponded The Equation

$$\begin{aligned}&(T_{0,t+k} + \tilde{\lambda}_1 T_{1,i_1,k+t} + \tilde{\lambda}_{k_1+1} T_{2,i_2,k+t} + ... + \tilde{\lambda}_{k_1+k_2+...+k_{n-1}+1} T_{n,i_n,k+t} - \\ &-\tilde{\lambda}_{k+(i_1,i_2,...,i_n)_t} T_{(i_1,...,i_n)_t,t+k})x_{k+t} = 0\end{aligned}$$

Consider The Last Equation, In Which

$$T_{1,s_1,r} = \begin{pmatrix}1 & 0 & 0 & ... & 0 & 0 & 0\\ 0 & 1 & 0 & ... & 0 & 0 & 0\\ 0 & 0 & 1 & ... & 0 & 0 & 0\\ 0 & 0 & 0 & ... & 0 & 0 & 0\\ . & . & . & ... & . & . & .\\ 0 & 0 & 0 & ... & 0 & 0 & 0\\ 0 & 0 & 0 & ... & 0 & 0 & 0\end{pmatrix}, ...,$$

$$T_{k_1+...+k_{n-1}+1,s_n,r} = \begin{pmatrix} 0 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & 0 \\ . & . & . & ... & . & . & . \\ 0 & 0 & 0 & ... & 0 & 1 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & 1 \end{pmatrix}$$

$$\tilde{T}_{(s_1,...,s_n)_t,r} = \begin{pmatrix} 0 & 0 & 0 & ... & 0 & 0 & 1 \\ 0 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & 0 \\ . & . & . & ... & . & . & . \\ 0 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{T}_{0,r} = \begin{pmatrix} 0 & 0 & 0 & ... & 0 & 0 & 0 \\ 1 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 1 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & ... & 1 & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 1 & 0 \end{pmatrix}$$

For Operators, Defining With Help The Matrices $T_{1,s_1,k+t}$, $T_{2,s_2,k+t}, ..., T_{n,s_n,k+t}, T_{0.k+t}$ Act In Space $R^{s_1+...+s_n}$ On Eigenvector $(\alpha_1,...,\alpha_{s_1+s_2+...+s_n}) \in R^{s_1+...+s_n}$.

$$(-\tilde{T}_{0,r} + \tilde{\lambda}_1 T_{1,s_1,r} + ... + \tilde{\lambda}_{1+\sum_{i=1}^{n-1} k} T_{1+k_1+...+k_{n-1},s_n,r})\tilde{\alpha} =$$

$$= \tilde{\lambda}_{k+t} T_{(s_1...s_{nt}} r)\tilde{\alpha}$$

Consequently,

$$\tilde{\lambda}_1 \alpha_1 = \tilde{\lambda}_{k+t} \alpha_{s_1+...+s_n}$$

....

$$\tilde{\lambda}_1 \alpha_{s_1} = \alpha_{s_1-1}$$

$$\tilde{\lambda}_{k_1+1} \alpha_{s_1+1} = \alpha_{s_1}$$

....

$$\tilde{\lambda}_{k_1+1} \alpha_{s_1+s_2} = \alpha_{s_1+s_2-1}$$

....

$$\tilde{\lambda}_{1+\sum_{i=1}^{n-1} k_i} \alpha_{s_1+s_2+...+s_{1+\sum_{i=1}^{n-1} k_i}} = \alpha_{s_1+s_2+...+s_{1+\sum_{i=1}^{n-1} r_i}-1}$$

....

Hence, $\lambda_1^{s_1} \lambda_2^{s_2} \cdots \lambda_n^{s_n} = \lambda_{k+s}$; $s \le r$.

For The Obtained Linear Multiparameter System We Construct Operator Δ_0 On Rule (3). The Condition $Ker\Delta_0^{-1} = \{\vartheta\}$ Means That Operators $\Gamma_i = \Delta_0^{-1}\Delta_i$ Are Pair Commute[2]. So Operators Γ_i Act In Finite Dimensional Space H And Operators $\Gamma_{k_1+k_2+...+k_{r-1}+1}$ Have Not The Zero Eigenvalues Then For The Any Eigenvalue $(\lambda_1, \lambda_2, ..., \lambda_{k_1+k_2+...+k_n})$ Of The System((4),(9)) And From Equation 2.47 And Equation 2.48 In [7] It Follows That There Is Such Eigen Element z That The Equalities, $\Gamma_{i,s} z = \lambda_{i,s} z$, $i = 1, 2, ..., k_1 + k_2 + ... + k_n$ Satisfy. For Analogy Conditions We Obtain The Analogy Results For All Groups. We Have The Several Systems Of Operator Polynomials In One Parameter. We Apply The Results Of [9]. The System Has The Form

$$\Delta_{i,} z = \lambda_{i,s}\Delta_{o,i} z$$

....

$$\Delta_{k_1+k_2+...+k_{i-1}+1,i} z_i = \lambda_{i,s}\Delta_{o,i} z_i$$

Theorem 4. Let The Conditions Of The Theorem1 Be Fulfilled Then Operators $\Delta_{o,i}$ Have Inverses. Moreover, The System(4) Has The Common Eigenvalue If And Only If

$$Ker \bigcap (\Delta_{k_1+k_2+...+k_{i-1}+1,i} - \lambda_{i,s}\Delta_{o,i}) \neq 0\,.$$

Elementary Operators

We Use Tensor Products To Determine The Norms Of Elementary Operators. Details Can Be Found In[11] And The References Therein. Below Is The First Result.

Theorem 5. Let A, B ∈ B(H) And $U_{A,B}$ =A ⊗H B + B ⊗H A Be Normally Represented Then ‖ $U_{A,B}$ $‖_{Inj}$ ≥ 2(√2 − 1)‖A‖‖B‖.

Proof. Let ‖A‖ = ‖B‖ = 1 And A, B Be Functions On D := (B(H)*), And $U_{A,B}$ As A Function On D × D. Taking Dot Products Of A And B Using A Suitable Scalars Of Modulus 1, We Let $A(X_0)$ = 1 And $B(Y_0)$ ∀ X_0, Y_0 ∈ D. Putting A_1 = $A(X_0)$, And B_1 = $B(Y_0)$. Then It Gives $U_{A,B}$ (X_0, Y_0) = $2B_1$, $U_{A,B}$ (Y_0, Y_0) = 2A, $U_{A,B}$(X_0, Y_0) = 1 + A_1B_1 If |A_1| Or|B_1| ≥ √(2 − 1). This Completes The Proof. On The Other Hand, If Suppose That |A_1| < √(2 − 1) And |B_1| <√(2 − 1) Then,

|1+ A_1B_1| > |− (√(2 − 1))^2|=2(√(2 −1)) ‖A‖‖B‖.

Corollary 1. Let R =$\sum^N_{i=1}$Ai ⊗ Bi ∈ B(H) ⊗ B(H). Then We Have ‖R‖$_{Inj}$= Sup{‖Ai ⊗ Bi‖ : X ∈ B(H)}, ‖X ‖ = 1 If And Only If X Is Rank One Operator.

Proof. Let R(X) =$\sum^N_{i=0}$$A_i$ ×B_i And ‖R‖$_B$ = Sup{‖Rx‖ : X ∈ B(H), ‖X ‖ =1, And, Rank (X) = 1}. It Is Known [70] That Every Rank One Operator X ∈ B(H) Is Of The Form X = V ⊗ ¯Σ For All V, Σ ∈ H Then This Gives

‖R‖$_B$= Sup{| $\sum^N_{I=1}$⟨A_i × B_i⟩ Ξη| : ‖X ‖ = 1, Rank (X) = 1, ‖Ξ ‖ = ‖H‖ = 1}

= Sup{|N ∑I=1⟨A_i, V, H ⟩⟨B_1ξ, Z ⟩|: ‖Z ‖ = ‖V‖ = ‖Ξ ‖ = ‖H‖ = 1}

= Sup{|$\sum^N_{I=1}$f (A_i) G(B_i)|}.

Taking The Last Supremum All Over All Functionals Of The Form F = V ⊗¯H, G = Ξ ⊗¯Z. For All Elements In The Product U(H) Of B(H) Is A Norm Limit Of Convex Combinations Of Elements Of The Form V ⊗ ¯ H And The Unit Ball U(H) Is A Weak Dense In The Unit Ball Of Dual Of B(H). This Implies That ‖R‖$_B$ = ‖R‖$_{Inj}$.

Conclusions

Tensor Product [10] Is A Very Important Technique Used In Solving Problems Of Norms In Hilbert Spaces. Norms Are Very Important Properties Of Operators And Interesting Studies Have Been Directed On Them. The Field Of Elementary Operators Has Been So Interesting Over The Past Decades And Much Have Been Done. The Norm Property In Particular Has Attracted Many Scholars But A Lot Can Be Done Further. In Our Study, We Considered The Normally Represented Elementary Operators. We Recommend That Other Properties Of The Normally Represented Elementary Operators Can Be Studied Like Numerical Ranges, Positivity And Spectrum. The Norm Property Is Also Not Exhausted.

Conflicts Of Interest

Authors Declare That There Is No Conflicts Of Interest.

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