On the Derivation and Analysis of a Highly Efficient Method for the Approximation of Quadratic Riccati Equations

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Abstract
A highly efficient method is derived and analyzed in this paper for the approximation of Quadratic Riccati Equations (QREs) using interpolation and collocation procedure. The derivation is carried out within a two-step integration interval $[x_n, x_{n+2}]$. We are motivated to derive a method that approximates QREs (which are nonlinear differential equations that have a great deal of applications in science and engineering). Furthermore, the basic properties of the newly derived method, which include the order of accuracy, convergence, zero-stability, consistence and region of absolute stability were analyzed. The method derived was also applied to solve some QREs and from the results generated, it was clear that the new method performed better than the ones with which we juxtaposed our results with.
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1 INTRODUCTION

The QDEs are nonlinear differential equations that find applications in financial mathematics [1], robust stabilization, stochastic realization theory, network synthesis and optimal control [2], random processes, optimal control and diffusion problem [3] . The QREs is also an essential tool used in modeling many physical situations such as spring mass systems, resistor-capacitor-induction circuits, bending of beams, chemical reactions, pendulum, the motion of rotating mass around body and so on, [4].The QRE was named after an Italian, Riccati Francesco Jacopo $(1676 - 1754)$.

In this paper, we shall derive and analyze a highly efficient method for the approximation of QREs of the form;

$$y' = a(t) + b(t)y(t) + c(t)y^2(t), \quad y(t_0) = y_0,\ 0 \le t \le T \tag{1.1}$$

We assume that equation (1.1) satisfies the hypotheses of the existence theorem below.

Theorem 1.1 [5]

Let $f(t,y)$, where $f:\mathbb{R}\times\mathbb{R}\rightarrow\mathbb{R}$, be defined and continuous for all (t,y) in the region D defined by $a\leq t\leq b,\ -\infty<y<\infty$, where a and b are finite and let there exist a constant L such that,

$$\|f(t,y)-f(t,y^{*})\|\leq L\|y-y^{*}\| \tag{1.2}$$

holds for every $(t,y),(t,y^{*})\in D$. Then for $\eta\in\mathbb{R}$,there exists a unique solution $y(t)$ of the problem (1.1), where $y(t)$ is continuous and differentiable for all $(t,y^{*})\in D$. The requirement (1.2) is known as Lipchitz condition and the constant L as a Lipchitz constant.

Definition 1.2 [6]

A numerical method is called $A(\alpha)$-stable for some $\alpha\in\left[0,\frac{\pi}{2}\right]$ if the wedge

$$S_{\alpha}=\{z:|Arg(-z)|<\alpha,\ z\neq 0\} \tag{1.3}$$

is contained in its stability region. The largest $\alpha\,(i.e.\ \alpha_{\max})$ is called the angle of absolute stability.

Definition 1.3 [6]

A numerical method is called $A(0)$-stable if it is $A(\alpha)$-stable for some for some $\alpha\in\left(0,\frac{\pi}{2}\right)$ .Note that $A\left(\frac{\pi}{2}\right)$-stability $\equiv$ A-stability

Over the years, some scholars have developed different methods for approximating QREs of the form (1.1). These methods range from predictor-corretor to hybrid methods. Inspite of the successes the predictor-corrector methods recorded, they have some shortcoming. This is because the predictors usually occur in reducing order of accuracy, another disadvantage is that there is high cost involved in developing separate predictor for the corrector, high cost of computer time and human efforts are also involved, [7].This led to the development of block methods to carter for some of these shortcomings. The first set of block methods were developed in 1953 by Milne basically to serve as predictors for predictor-corrector algorithms. Later, the block methods were adopted as full methods. One of the advantages of block methods is that they generate simultaneous numerical approximations at different grid points within an integration interval, [8]. The block methods are less expensive in terms of the number of function evaluations compared to the linear multistep and the Runge-Kutta methods. We must hoever state that despite all these advantage, the block method also have a major setback. The setback is that the order of interpolation points must not exceed that of the differential equations. This led to the development of highly effcient methods called the hybrid methods. These methods allow for the incorporation of function evaluation at off-step points, thus; affording the opportunity to circumvent the "Dahlquist zero-stabilty barrier". Thus, with hybrid methods, it is possible to obtain convergent $k-$step methods with order $2k+1$ up to $k=7$.The hybrid method helps in reducing the step number of a method and still maintain its zero-stability property, [9].

The Adomian Decomposition Method (ADM) could be cumbersome at time because the Adomian polynomials may be very complicated to construct. The Variational Iteration Method (VIM) also has a major disadvantage because identifying the Lagrange multipliers usually yield an underlying accuracy. The formation of linear functional equations (which is needed in each iteration) could be very difficult when applying the Homotopy Perturbation Method (HPM). The performance of Homotopy Analysis Method (HAM) largely depends on the choice of the auxiliary parameter of the zero-order deformation equation. Furthermore, the implemnetation and convergence region method is very small.

Different methods have been adopted by researchers in approximating QREs. These methods include the Non-Standard Finite Difference Method (NSFDM) [2], ADM [10, 11, 12, 13], VIM [14, 15, 16, 17, 18, 19, 20], Runge-Kutta method [21], Chebyshev wavelets [22], hybrid function and Tau method [23], Differential Transformation Method (DTM) [24, 25], HAM [26, 27], HPM [28, 29], among others.

It is in view of the short-comings of these methods that we were motivated to develop a highly efficient method for the approximation of QREs. It is expected that this method will perform better than the existing ones.

2 DERIVATION OF THE METHOD

We shall employ a basis function given by,

$$y(t) = \sum_{j=0}^{r+s-1} a_j t^j \tag{2.4}$$

in the derivation of a method of the form

$$A^{(0)}\mathbf{Y}_m = E\mathbf{y}_n + hd\mathbf{f}(\mathbf{y}_n) + hb\mathbf{F}(\mathbf{Y}_m) \tag{2.5}$$

for the solution of QREs of the form (1.1), where $A^{(0)}$, E, d and b $are\, r \times r$ matrices (r is the number of collocation points), s is also the number of interpolation points. Also note that $\mathbf{Y}_m$, $\mathbf{y}_n$, $\mathbf{F}(\mathbf{Y}_m)$ and $\mathbf{f}(\mathbf{y}_n)$ are vector matrices with r entries.

Differentiating equation (2.4), we obtain

$$y'(t) = \sum_{j=0}^{r+s-1} j a_j t^{j-1} \tag{2.6}$$

Equation (2.4) is interpolated at point x_{n+s}, $s = \frac{5}{3}$ and equation (2.6) is collocated at points x_{n+r}, $r = 0\left(\frac{1}{3}\right)2$. This leads to the system of equatons,

$$TA = U \tag{2.7}$$

where

$$A = [a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7]^T$$

$$U = \left[y_n \quad f_n \quad f_{n+\frac{1}{3}} \quad f_{n+\frac{2}{3}} \quad f_{n+1} \quad f_{n+\frac{4}{3}} \quad f_{n+\frac{5}{3}} \quad f_{n+1} \right]^T$$

and

$$T = \begin{bmatrix} 1 & t_{n+\frac{5}{3}} & t^2_{n+\frac{5}{3}} & t^3_{n+\frac{5}{3}} & t^4_{n+\frac{5}{3}} & t^5_{n+\frac{5}{3}} & t^6_{n+\frac{5}{3}} & t^7_{n+\frac{5}{3}} \\ 0 & 1 & 2t_n & 3t^2_n & 4t^3_n & 5t^4_n & 6t^5_n & 7t^6_n \\ 0 & 1 & 2t_{n+\frac{1}{3}} & 3t^2_{n+\frac{1}{3}} & 4t^3_{n+\frac{1}{3}} & 5t^4_{n+\frac{1}{3}} & 6t^5_{n+\frac{1}{3}} & 7t^6_{n+\frac{1}{3}} \\ 0 & 1 & 2t_{n+\frac{2}{3}} & 3t^2_{n+\frac{2}{3}} & 4t^3_{n+\frac{2}{3}} & 5t^4_{n+\frac{2}{3}} & 6t^5_{n+\frac{2}{3}} & 7t^6_{n+\frac{2}{3}} \\ 0 & 1 & 2t_{n+1} & 3t^2_{n+1} & 4t^3_{n+1} & 5t^4_{n+1} & 6t^5_{n+1} & 7t^6_{n+1} \\ 0 & 1 & 2t_{n+\frac{4}{3}} & 3t^2_{n+\frac{4}{3}} & 4t^3_{n+\frac{4}{3}} & 5t^4_{n+\frac{4}{3}} & 6t^5_{n+\frac{4}{3}} & 7t^6_{n+\frac{4}{3}} \\ 0 & 1 & 2t_{n+\frac{5}{3}} & 3t^2_{n+\frac{5}{3}} & 4t^3_{n+\frac{5}{3}} & 5t^4_{n+\frac{5}{3}} & 6t^5_{n+\frac{5}{3}} & 7t^6_{n+\frac{5}{3}} \\ 0 & 1 & 2t_{n+2} & 3t^2_{n+2} & 4t^3_{n+2} & 5t^4_{n+2} & 6t^5_{n+2} & 7t^6_{n+2} \end{bmatrix}$$

Solving (2.7), for $a_j's, j = 0(1)7$ and substituting back into equation (2.4) gives a continuous linear multistep method of the form,

$$y(t) = \alpha_{\frac{5}{3}}(t) y_{n+\frac{5}{3}} + h \sum_{j=0}^{2} \beta_j(t) f_{n+j}, \; j = 0\left(\frac{1}{3}\right)2 \tag{2.8}$$

where

$$\begin{bmatrix} \alpha_0(t) = 1 \\ \beta_0(t) = \frac{1}{1391040}\begin{pmatrix} 209952t^7 - 1711206t^6 + 5688144t^5 - 9913995t^4 \\ +9675792t^3 - 5202792t^2 + 1391040t - 137575 \end{pmatrix} \\ \beta_{\frac{1}{3}}(t) = \frac{1}{7245}\begin{pmatrix} 6561t^7 - 51030t^6 + 158193t^5 - 246645t^4 \\ +197316t^3 - 68040t^2 + 3625 \end{pmatrix} \\ \beta_{\frac{2}{3}}(t) = \frac{1}{1391040}\begin{pmatrix} 3149280t^7 - 23320710t^6 + 67482072t^5 - 95111415t^4 \\ +64978200t^3 - 17010000t^2 - 171875 \end{pmatrix} \\ \beta_1(t) = \frac{1}{86940}\begin{pmatrix} 262440t^7 - 1845585t^6 + 4997538t^5 - 6475140t^4 \\ +4009320t^3 - 982800t^2 + 44375 \end{pmatrix} \\ \beta_{\frac{4}{3}}(t) = \frac{1}{463680}\begin{pmatrix} 1049760t^7 - 6991110t^6 + 17799264t^5 - 21662235t^4 \\ +12791520t^3 - 3061800t^2 - 124375 \end{pmatrix} \\ \beta_{\frac{5}{3}}(t) = \frac{1}{86940}\begin{pmatrix} 78732t^7 - 494991t^6 + 1194102t^5 - 1394820t^4 \\ +802872t^3 - 190512t^2 + 15325 \end{pmatrix} \\ \beta_2(t) = \frac{1}{463680}\begin{pmatrix} 69984t^7 - 413910t^6 + 957096t^5 - 1087695t^4 \\ +617064t^3 - 146160t^2 + 5125 \end{pmatrix} \end{bmatrix} \tag{2.9}$$

and t is given by

$$t = \frac{x - x_n}{h} \tag{2.10}$$

We then evaluate equation (2.8) at $t = \frac{1}{3}\left(\frac{1}{3}\right)2$, thus obtaining a new method of the the form (2.5) given by,

$$\left[\begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ y_{n+1} \\ y_{n+\frac{4}{3}} \\ y_{n+\frac{5}{3}} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-\frac{1}{3}} \\ y_{n-\frac{2}{3}} \\ y_{n-1} \\ y_{n-\frac{4}{3}} \\ y_{n-\frac{5}{3}} \\ y_n \end{bmatrix} \\ +h \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{428149}{417312} \\ 0 & 0 & 0 & 0 & 0 & \frac{8419}{86940} \\ 0 & 0 & 0 & 0 & 0 & \frac{3043}{30912} \\ 0 & 0 & 0 & 0 & 0 & \frac{6346}{65205} \\ 0 & 0 & 0 & 0 & 0 & \frac{27515}{278208} \\ 0 & 0 & 0 & 0 & 0 & \frac{907}{9660} \end{bmatrix} + h \begin{bmatrix} \frac{2713}{7245} & \frac{-410461}{1391040} & \frac{67477}{260820} & \frac{-69737}{463680} & \frac{4381}{86940} & \frac{-30701}{4173120} \\ \frac{752}{1449} & \frac{-4447}{86940} & \frac{4112}{21735} & \frac{-3629}{28980} & \frac{976}{21735} & \frac{-199}{28980} \\ \frac{81}{161} & \frac{6201}{51520} & \frac{3803}{9660} & \frac{-8289}{51520} & \frac{171}{3220} & \frac{-1207}{154560} \\ \frac{3712}{7245} & \frac{1784}{21735} & \frac{39232}{65205} & \frac{58}{7245} & \frac{832}{21735} & \frac{-416}{65205} \\ \frac{725}{1449} & \frac{34375}{278208} & \frac{8875}{17388} & \frac{24875}{92736} & \frac{3065}{17388} & \frac{-1025}{92736} \\ \frac{432}{805} & \frac{27}{3220} & \frac{1744}{2415} & \frac{27}{3220} & \frac{432}{805} & \frac{907}{9660} \end{bmatrix} \end{array}\right] \tag{2.11}$$

Equation (2.11) is the new method capable of approximating QREs of the form (1.1). Note however, that the method is implicit in nature; this means that, it requires some starting values before it can be efficiently implemented. Thus, starting values for $y_{n+j},\ j = \frac{1}{3}\left(\frac{1}{3}\right)2$ are predicted with the aid of Taylor series up to the order of each individual method.

3 ANALYSIS OF BASIC PROPERTIES OF THE METHOD

In this section, the analysis of basic properties of the newly derived method shall be carried out. These properties include; order of accuracy, consistency, root condition, convergence, symmetry and region of absolute stability.

3.1 Order of Accuracy and Error Constant of the Method

Let the linear operator $\ell\{y(t):h\}$ be defined on the method (2.5) when $i = 0$ by the expression,

$$\ell\{y(t):h\} = A^{(0)}\mathbf{Y}_m - \sum_{i=0}^{2} \frac{(jh)^{(i)}}{i!} E\mathbf{y}_n^{(i)} + h\left[d_i\mathbf{f}(\mathbf{y}_n) + b_i\mathbf{F}(\mathbf{Y}_m)\right] \tag{3.12}$$

It is evident from equation (3.12), that expanding $\mathbf{Y}_m$ and $\mathbf{F}(\mathbf{Y}_m)$ in Taylor's series and comparing the coefficients of h gives

$$\ell\{y(t):h\} = \overline{c}_0 y(t) + \overline{c}_0 hy'(t) + ... + \overline{c}_p h^p y^p(t) + \overline{c}_{p+1} h^{p+1} y^{p+1}(t) + ... \tag{3.13}$$

Definition 3.4 [5].

The linear operator ℓ *and the associated block method* (2.5) *are said to be of accurate order* p *if* $\overline{c}_0 = \overline{c}_1 = \overline{c}_2 = ... = \overline{c}_p = 0$, $\overline{c}_{p+1} \neq 0$.

The parameter $\overline{c}_{p+1}$ is called the error constant and implies that the truncation error is given by,

$$T_{n+k} = C_{p+1} h^{p+1} y^{p+1}(t) + O(h^{p+2}) \qquad (3.14)$$

$$\begin{bmatrix} \sum\limits_{j=0}^{\infty} \frac{\left(\frac{1}{3}h\right)^j}{j!} - y_n - \frac{428149}{417312} h y_n' - \sum\limits_{j=0}^{\infty} \frac{h^{j+1}}{j!n} y_n^{j+1} \begin{bmatrix} \frac{2713}{7245}\left(\frac{1}{3}\right)^j - \frac{410461}{1391040}\left(\frac{2}{3}\right)^j + \frac{67477}{260820}(1)^j \\ -\frac{69737}{463680}\left(\frac{4}{3}\right)^j + \frac{4381}{86940}\left(\frac{5}{3}\right)^j - \frac{30701}{4173120}(2)^j \end{bmatrix} \\ \sum\limits_{j=0}^{\infty} \frac{\left(\frac{2}{3}h\right)^j}{j!} - y_n - \frac{8419}{86940} h y_n' - \sum\limits_{j=0}^{\infty} \frac{h^{j+1}}{j!n} y_n^{j+1} \begin{bmatrix} \frac{752}{1449}\left(\frac{1}{3}\right)^j - \frac{4447}{86940}\left(\frac{2}{3}\right)^j + \frac{4112}{21735}(1)^j \\ -\frac{3629}{28980}\left(\frac{4}{3}\right)^j + \frac{976}{21735}\left(\frac{5}{3}\right)^j - \frac{199}{28980}(2)^j \end{bmatrix} \\ \sum\limits_{j=0}^{\infty} \frac{(h)^j}{j!} - y_n - \frac{3043}{30912} h y_n' - \sum\limits_{j=0}^{\infty} \frac{h^{j+1}}{j!n} y_n^{j+1} \begin{bmatrix} \frac{81}{161}\left(\frac{1}{3}\right)^j + \frac{6201}{51520}\left(\frac{2}{3}\right)^j + \frac{3803}{9660}(1)^j \\ -\frac{8289}{51520}\left(\frac{4}{3}\right)^j + \frac{171}{3220}\left(\frac{5}{3}\right)^j - \frac{1207}{154560}(2)^j \end{bmatrix} \\ \sum\limits_{j=0}^{\infty} \frac{\left(\frac{4}{3}h\right)^j}{j!} - y_n - \frac{6346}{65205} h y_n' - \sum\limits_{j=0}^{\infty} \frac{h^{j+1}}{j!n} y_n^{j+1} \begin{bmatrix} \frac{3712}{7245}\left(\frac{1}{3}\right)^j + \frac{1784}{21735}\left(\frac{2}{3}\right)^j + \frac{39232}{65205}(1)^j \\ +\frac{58}{7245}\left(\frac{4}{3}\right)^j + \frac{832}{21735}\left(\frac{5}{3}\right)^j - \frac{416}{65205}(2)^j \end{bmatrix} \\ \sum\limits_{j=0}^{\infty} \frac{\left(\frac{5}{3}h\right)^j}{j!} - y_n - \frac{27515}{278208} h y_n' - \sum\limits_{j=0}^{\infty} \frac{h^{j+1}}{j!n} y_n^{j+1} \begin{bmatrix} \frac{725}{1449}\left(\frac{1}{3}\right)^j + \frac{34375}{278208}\left(\frac{2}{3}\right)^j + \frac{8875}{17388}(1)^j \\ +\frac{24875}{92736}\left(\frac{4}{3}\right)^j + \frac{3065}{17388}\left(\frac{5}{3}\right)^j - \frac{1025}{92736}(2)^j \end{bmatrix} \\ \sum\limits_{j=0}^{\infty} \frac{(2h)^j}{j!} - y_n - \frac{907}{9660} h y_n' - \sum\limits_{j=0}^{\infty} \frac{h^{j+1}}{j!n} y_n^{j+1} \begin{bmatrix} \frac{432}{805}\left(\frac{1}{3}\right)^j + \frac{27}{3220}\left(\frac{2}{3}\right)^j + \frac{1744}{2415}(1)^j \\ +\frac{27}{3220}\left(\frac{4}{3}\right)^j + \frac{432}{805}\left(\frac{5}{3}\right)^j + \frac{907}{9660}(2)^j \end{bmatrix} \end{bmatrix} = 0 \qquad (3.15)$$

Therefore, if we compare the coefficients of h, the order and the error constant of the method are given by

$\overline{c}_0 = \overline{c}_1 = \overline{c}_2 = \overline{c}_3 = \overline{c}_4 = \overline{c}_5 = \overline{c}_6 = 0$

and

$\overline{c}_7 = \left[3.5672 \times 10^{-6} \;\; 4.9438 \times 10^{-6} \;\; 4.7926 \times 10^{-6} \;\; 4.8807 \times 10^{-6} \;\; 4.7663 \times 10^{-6} \;\; 5.1121 \times 10^{-6}\right]^T$

respectively. Thus, the newly derived method (2.11) is of accurate uniform sixth order.

3.2 Root Condition and Zero Stability of the Method

Definition 3.5 [5]

The *block method* (2.5) *is said to satisfy root condition, if the roots* $z_s, s = 1, 2, ..., k$ *of the first characteristic polynomial* $\rho(z)$ *defined by* $\rho(z) = \det(zA^{(0)} - E)$ *satisfies* $|z_s| \leq 1$ *and every root satisfying* $|z_s| = 1$ *have multiplicity not exceeding the order of the differential equation. The method* (2.5) *is said to be zero-stable if it satisfies the root condition. Moreover, as* $h \to 0, \rho(z) = z^{r-\mu}(z-1)^{\mu}$, *where* μ *is the order of the differential equation,* r *is the order of the matrices* $A^{(0)}$ *and* E *.*

We shall now verify whether or not the new method (2.11) satisfies root condition.

$$\rho(z) = \left| z \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right| = 0 \qquad (3.16)$$

Fron equation (3.16), $\rho(z) = z^5(z-1) = 0 \Longrightarrow z_1 = z_2 = z_3 = z_4 = z_5 = 0,\ z_6 = 1$. Hence, the method (2.11) is said to satisfy root condition.

Theorem 3.6 [5]

The necessary and sufficient condition for the method given by (2.5) *to be zero-stable is that it satisfies the root condition.*

This therefore implies that the new method (2.11) is zero-stable.

3.3 Consistency of the Method

Suffice to say that the consistency of a method controls the magnitude of the local truncation error which is committed at each stage of the computation, [30].Thus, the method (2.11) is consistent since it has uniform order $p = 6 \geq 1$.

3.4 Convergence of the Method

The method (2.11) is convergent by consequence of Dahlquist theorem below.

Theorem 3.7 [31]

The necessary and sufficient conditions that a continuous LMM be convergent are that it be consistent and zero-stable.

3.5 Stability Region of the Method

Definition 3.8 [5]

The linear multistep method (2.5) *is said to have region of absolute stability* R_A*, where* R_A *is a region of the complex* $\bar{h}-$*plane, if it is absolutely stable for all* $\bar{h} \in R_A$*. The intersection of* R_A *with the real axis is called the interval of absolute stability.*

The stability region of the new method (2.11) is given by the expression,

$$\left[\begin{array}{l} \bar{h}(w) = h^6\left(\frac{8}{39123}w^6 - \frac{1348847}{314940150}w^5\right) - h^5\left(\frac{79}{27945}w^6 + \frac{17751889}{787350375}w^5\right) \\ \quad +h^4\left(\frac{29}{1242}w^6 - \frac{49712251}{299943000}w^5\right) - h^3\left(\frac{475}{3726}w^6 + \frac{24256997}{52490025}w^5\right) \\ \quad +h^2\left(\frac{569}{1242}w^6 - \frac{67419419}{69986700}w^5\right) - h\left(\frac{206}{207}w^6 + \frac{208}{207}w^5\right) + w^6 - w^5 \end{array} \right] \qquad (3.17)$$

The stability region is shown in Figure 3.1.

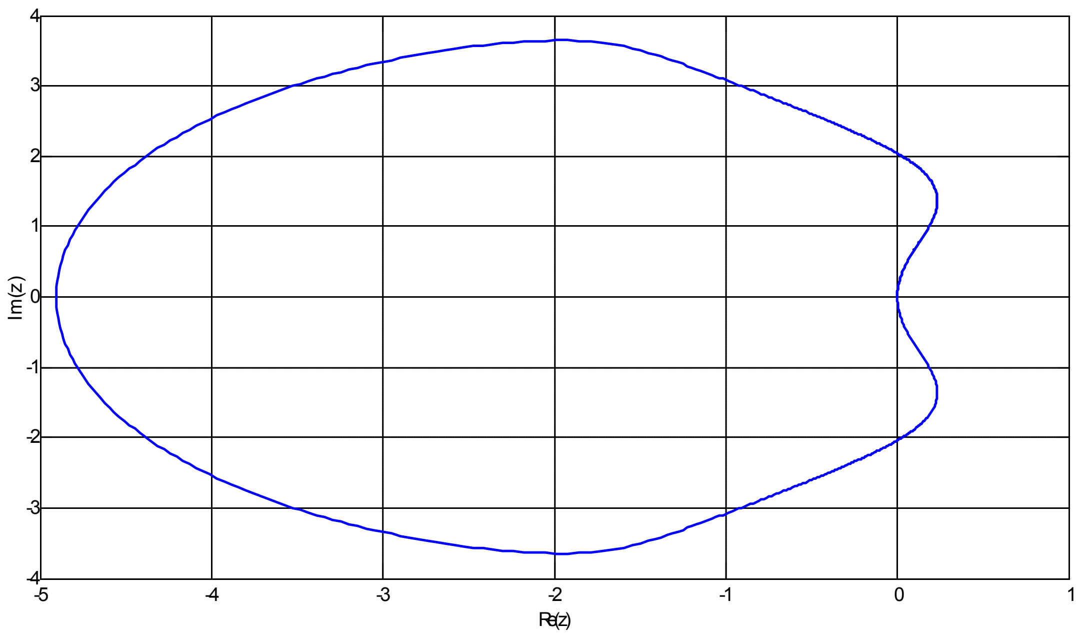

Figure 1:

The region of absolute stability in Figure 3.1 above is $A(0)$-stable. The stable region is the interior of the curve while the unstable region consists of the complex plane outside the enclosed figure.

4 RESULTS AND DISCUSSION

4.1 Numerical Experiments

The newly derived two-step method (2.11) shall be applied in the approximation of QREs of the form (1.1).The results obtained shall be juxtaposed along other results obtained by some authors. This is aimed at showing that the newly derived method performs better than some existing methods. We shall employ the notations below in the result tables.

ERR=|Exact Solution - Computed Solution|

Exec.t/ sec- Execution time per seconds

EBN-Error in [28]

EJS-Error in [32]

Problem 4.9

Consider the QRE of the form,

$$y'(t) = 1 + 2y(t) - y^2(t), \; y(0) = 0 \tag{4.18}$$

whose exact solution is given by,

$$y(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2t} + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right) \tag{4.19}$$

Source: [32]

Applying the newly derived method on the Problem 4.9, we obtain the result presented in Table 4.14 below.

Problem 4.10

Consider the QRE of the form,

$$y'(t) = 1 - y^2(t), \; y(0) = 0 \tag{4.20}$$

whose exact solution is given by,

$$y(t) = \frac{e^{2t} - 1}{e^{2t} + 1} \tag{4.21}$$

Source: [32]

Applying the newly derived method on the Problem 4.10, we obtain the result presented in Table 4.15 below.

Problem 4.11

Consider the QRE of the form,

$$y'(t) = -\frac{1}{1+t},\ y(0) = 1 \tag{4.22}$$

whose exact solution is given by,

$$y(t) = \frac{1}{1+t} \tag{4.23}$$

Source: [32]

Applying the newly derived method on the Problem 4.11, we obtain the result presented in Table 4.16 below.

Problem 4.12

Consider the QRE of the form,

$$y'(t) = 10 + 3y(t) - y^2(t),\ y(0) = 0 \tag{4.24}$$

whose exact solution is given by,

$$y(t) = -2 + \frac{14}{5 + 2e^{7t}} \tag{4.25}$$

Source: [28]

Applying the newly derived method on the Problem 4.12, we obtain the result presented in Table 4.17 below.

Problem 4.13 :

Consider the QRE of the form,

$$y'(t) = y^2(t) - 1,\ y(0) = 0 \tag{4.26}$$

whose exact solution is given by,

$$y(t) = -\tanh(t) \tag{4.27}$$

Source: [32]

Applying the newly derived method on the Problem 4.13, we obtain the result presented in Table 4.18 below.

Table 4.14 : Result for Problem 4.9

t	Exact Solution	Computed Solution	ERR	EJS	Exec.t/ sec
0.1000	0.1102951969169624	0.1102951969169602	$2.248202e-015$	$3.201692e-011$	0.0578
0.2000	0.2419767996211095	0.2419767996210897	$1.978973e-014$	$3.758708e-010$	0.8654
0.3000	0.3951048486603790	0.3951048486603087	$7.033263e-014$	$1.438245e-009$	0.9026
0.4000	0.5678121662929394	0.5678121662927802	$1.592060e-013$	$3.354903e-009$	1.2517
0.5000	0.7560143934313766	0.7560143934311164	$2.601253e-013$	$5.573525e-009$	1.3003
0.6000	0.9535662164719240	0.9535662164716031	$3.208545e-013$	$6.855140e-009$	1.3394
0.7000	1.1529489669796247	1.1529489669793307	$2.939871e-013$	$6.041839e-009$	1.4128
0.8000	1.3463636553683767	1.3463636553681937	$1.829648e-013$	$3.168413e-009$	1.4819
0.9000	1.5269113132806256	1.5269113132805694	$5.617729e-014$	$1.336715e-010$	1.5488
1.0000	1.6894983915943844	1.6894983915943751	$9.325873e-015$	$1.492398e-009$	1.6019

Table 4.15 : Result for Problem 4.10

t	Exact Solution	Computed Solution	ERR	EJS	Exec.t/ sec
0.1000	0.0996679946249559	0.0996679946249558	$9.714451e-017$	$1.149081e-014$	0.0536
0.2000	0.1973753202249041	0.1973753202249041	$8.326673e-017$	$6.716849e-014$	0.3744
0.3000	0.2913126124515911	0.2913126124515911	$0.000000e+000$	$1.833533e-013$	0.8300
0.4000	0.3799489622552251	0.3799489622552249	$2.220446e-016$	$3.386180e-013$	0.8614
0.5000	0.4621171572600101	0.4621171572600099	$2.220446e-016$	$4.861112e-013$	1.2087
0.6000	0.5370495669980355	0.5370495669980354	$1.110223e-016$	$5.798695e-013$	1.5045
0.7000	0.6043677771171638	0.6043677771171635	$3.330669e-016$	$5.948575e-013$	1.5499
0.8000	0.6640367702678492	0.6640367702678487	$5.551115e-016$	$5.327960e-013$	1.7292
0.9000	0.7162978701990248	0.7162978701990242	$5.551115e-016$	$4.161116e-013$	1.9368
1.0000	0.7615941559557653	0.7615941559557650	$3.330669e-016$	$2.745582e-013$	2.0353

Table 4.16 : Result for Problem 4.11

t	Exact Solution	Computed Solution	ERR	EJS	Exec.t/ sec
0.1000	0.9090909090909091	0.9090909090909090	$1.110223e-016$	$3.8296e-07$	0.0586
0.2000	0.8333333333333333	0.8333333333333329	$3.330669e-016$	$3.8296e-07$	0.3116
0.3000	0.7692307692307691	0.7692307692307687	$3.330669e-016$	$5.7951e-07$	0.3475
0.4000	0.7142857142857141	0.7142857142857134	$6.661338e-016$	$6.8133e-07$	0.3830
0.5000	0.6666666666666665	0.6666666666666663	$2.220446e-016$	$7.3394e-07$	0.7965
0.6000	0.6249999999999998	0.6250000000000000	$2.220446e-016$	$7.6091e-07$	1.2168
0.7000	0.5882352941176469	0.5882352941176474	$5.551115e-016$	$7.7483e-07$	1.4326
0.8000	0.5555555555555554	0.5555555555555558	$4.440892e-016$	$7.8257e-07$	1.5336
0.9000	0.5263157894736840	0.5263157894736850	$9.992007e-016$	$7.8799e-07$	1.5967
1.0000	0.4999999999999998	0.5000000000000002	$4.440892e-016$	$7.9326e-07$	1.6501

Table 4.17 : Result for Problem 4.12

t	Exact Solution	Computed Solution	ERR	EBN	Exec.t/ sec
0.1000	1.1229599550199865	1.1229599535565202	$1.463466e-009$	1.5×10^{-6}	0.2713
0.2000	2.3303636672393440	2.3303636702315602	$2.992216e-009$	3.2×10^{-6}	0.4742
0.3000	3.3592985913921902	3.3592986263236813	$3.493149e-008$	8.0×10^{-7}	0.6429
0.4000	4.0762561998939519	4.0762562765451094	$7.665116e-008$	3.2×10^{-6}	0.8785
0.5000	4.5086402379423145	4.5086403319614563	$9.401914e-008$	3.7×10^{-6}	1.0359
0.6000	4.7470598637518684	4.7470599483651021	$8.461323e-008$	9.7×10^{-7}	1.1531
0.7000	4.8720664654895476	4.8720665291990306	$6.370948e-008$	1.0×10^{-6}	1.1871
0.8000	4.9358801511182646	4.9358801941869093	$4.306864e-008$	8.5×107	1.2208
0.9000	4.9680115179081819	4.9680115451016569	$2.719348e-008$	2.1×10^{-7}	1.3892
1.0000	4.9840783622386375	4.9840783786466476	$1.640801e-008$	1.4×10^{-6}	1.5685

Table 4.18 : Result for Problem 4.13

t	Exact Solution	Computed Solution	ERR	EJS	Exec.$t/\sec$
0.1000	-0.0996679946249559	-0.0996679946249558	$6.938894e-017$	$1.147693e-014$	0.2854
0.2000	-0.1973753202249041	-0.1973753202249041	$8.326673e-017$	$6.714074e-014$	0.3183
0.3000	-0.2913126124515911	-0.2913126124515911	$0.000000e+000$	$1.834088e-013$	0.5569
0.4000	-0.3799489622552251	-0.3799489622552249	$2.220446e-016$	$3.385625e-013$	0.7590
0.5000	-0.4621171572600100	-0.4621171572600099	$1.110223e-016$	$4.861112e-013$	0.9315
0.6000	-0.5370495669980356	-0.5370495669980354	$2.220446e-016$	$5.798695e-013$	1.0671
0.7000	-0.6043677771171637	-0.6043677771171635	$2.220446e-016$	$5.947465e-013$	1.2196
0.8000	-0.6640367702678492	-0.6640367702678487	$5.551115e-016$	$5.329071e-013$	1.3512
0.9000	-0.7162978701990247	-0.7162978701990242	$4.440892e-016$	$4.160006e-013$	1.5328
1.0000	-0.7615941559557652	-0.7615941559557650	$2.220446e-016$	$2.744471e-013$	1.5674

4.2 Discussion of Results

In view of the results obtained in the tables above, it is obvious that the newly derived method (2.11) performs better than the results of the existing methods in view of the results obtained. These results further buttress the fact that the method is convergent because the computed solutions converge toward the exact solutions at each point of integration. It is also clear from the tables that the execution time per seconds needed to generate results are micro (very small), showing that the method generates results very fast. Thus, the method is said to highly efficient.

5 CONCLUSION

In this paper, a highly efficient uniform sixth-order method (2.11) has been formulated for the solution of QREs of the form (1.1) via collocation and interpolation procedure. The basic properties of the newly derived method were also analysed, thus; showing that the method is zero-stable, consistent and convergent. The method developed was found to be $A(0)$-stable and that is why it performed extremely well on the nonlinear QREs. The results obtained on the application of the method clearly shows that it is computationally reliable.

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