Optimal placement of logarithmic potential charges on an equilateral triangle

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Abstract

In this paper the author will discuss the optimal placement of logarithmic potential charges on domain shaped like an equilateral triangle. In particular, the closed form solution for the placement of the charges is desired.

Indexing terms/Keywords: logarithmic potential, symmetry breaking, two dimensional domain

Subject Classification: 31A15; 30C15

Language: English

Volume: 01 Issue: 01

Journal: MathLAB

Website: https://purkh.com

Date of Submission: 2018-03-06

Date of Acceptance: 2018-04-20

Date of Publication: 2018-04-30

1 Introduction

The placement of potential charges on conductors is one problem that has attracted attention from different fields of science for over 100 years. Beginning with the work of J. J. Thomson, with his work in modeling the distribution of electrons in atoms [2], the placement of electrons on a three-dimensional sphere has morphed into a richer study of charge placement in three-dimensional shapes as well as into two-dimensional domains.

Within this paper, the placement of potential charges on a domain bounded by an equilateral triangle will be examined. What one will find is that symmetry among the placement of charges is not preserved. The main focus of this study is the exact placement of a small number of charges, thus inviting the question of how to determine the placement of larger numbers of charges that will produce an optimal (i.e. minimal) potential within the system, thus examining the symmetrical (or lack of symmetry) of the placement of charges.

2 Methods and Definitions

Let $\mathbb{C}$ be the complex plane. We will consider all charges placed at the point $z_k = (x_k, y_k) \in \mathbb{C}$ to have unit charge. This charge will generate a logarithmic potential at the point $z = (x, y)$ defined by

$$E(z_k, z) = -\log|z_k - z|$$

where log is the complex logarithm. Note that this is the logarithm of the distance between the charges. The logarithmic potential energy between any two unit charges $z_1, z_2 \in \mathbb{C}$ is given by

$$E(z_1, z_2) = -\log|z_2 - z_1|.$$

The total energy for the system of charges $z_1, z_2, \ldots, z_n$ in $\mathbb{C}$ is given by

$$E(z_1, z_2, \ldots, z_n) = -\sum_{1 \le j < k \le n} \log|z_k - z_j|. \tag{1}$$

Electrostatic equilibrium within the system of charges occurs when this energy is minimized. Note also that equivalent to finding the minimum of the energy E, the maximum of

$$H(z_1, z_2, \ldots, z_n) = \prod_{1 \le j < k \le n} |z_k - z_j|$$

can also be found.

For our consideration, the conductor we will be investigating will be a two-dimensional conductor in the shape of an equilateral triangle. In particular, we will restrict our observed triangle to have sides of length 2. The vertices will be located at $z = 0$, $z = 2e^{\pi i/3}$, and $z = 2e^{2\pi i/3}$. All charges will thus be located in this triangular region.

3 Results and Discussion

What we are interested to see is the behavior of the potential charges as we add additional unit charges to the domain. For example, when n unit charges are placed in a circular domain, the charges will be located at the roots of unity (without loss of generality based on rotation) in order to attain an extremal energy. On a triangular domain, as charges are added the charges will eventually take on a non-symmetric positioning.

We will present the first few cases, leading up to nonsymmetry.

3.1 Two charges on the conductor

For $n = 2$, we will find the largest distance possible between the two point charges. If we place one charge z_1 at the corner $z = 0$, then we can consider z_2 in two different ways. If z_2 is on either of the sides meeting at $z = 0$, the largest possible distance is 2, which occurs when z_2 is located at either $z_2 = 2e^{i\pi/3}$ or $2e^{2i\pi/3}$.

If z_2 is on the side opposite z_1, then the range in distances possible is from $\sqrt{3} \cong 1.73$ (where $z_2 = \sqrt{3}e^{i\pi/2}$) to 2 (where z_2 is located at either $2e^{i\pi/3}$ or $2e^{2i\pi/3}$).

Thus the second charge z_2 must be located at a corner, either at the point $2e^{i\pi/3}$ or $2e^{2i\pi/3}$.

3.2 Three charges on the conductor

For $n = 3$, we will consider where the product of distances between the three point charges is to be as large as possible. Without loss of generality, we place $z_1 = 0$ and $z_2 = 2e^{2i\pi/3}$. Also, without loss of generality, we consider z_3 located on the line $y = \sqrt{3}$. Note that $|z_3 - z_1|$ varies between $\sqrt{3}$ and 2. Also note that $|z_3 - z_2|$ varies between 0 and 2.

The largest possible product will thus occur when $z_3 = 2e^{i\pi/3}$, where we have $|z_3 - z_1| = 2$ and $|z_3 - z_2| = 2$.

3.3 Four charges on the conductor

For $n = 4$, we maintain three of the charges on the respective points $z_1 = 0$, $z_2 = 2e^{2i\pi/3} = -1 + i\sqrt{3}$, and $z_3 = 2e^{i\pi/3} = 1 + i\sqrt{3}$. Without loss of generality, we will place the point $z_4 = x + i\sqrt{3}$ on the line $y = \sqrt{3}$.

Using Formula 1 have the following potential energy (in rectangular coordinate form):

$$E = -\log 8 - \frac{1}{2}\log(x^2 + 3) - \log(x + 1) - \log(1 - x).$$

If we look at the derivative of this potential energy, which is

$$\frac{dE}{dx} = -\frac{x}{x^2 + 3} - \frac{1}{x + 1} + \frac{1}{1 - x}, \tag{2}$$

then Equation 2 has three solutions: $x = -\frac{i\sqrt{15}}{3}, 0, \frac{i\sqrt{15}}{3}$. Thus the minimum energy occurs when the charge z_4 has x-coordinate of 0 and thus is located at the point $z_4 = i\sqrt{3}$. This is shown in Figure 1.

3.4 Five points on the conductor

For $n = 5$ we maintain three charges at the points $z_1 = 0$, $z_2 = 2e^{2\pi/3}$, and $z_3 = 2e^{\pi/3}$. For the other two charges, we will place them at $z_4 = r_1e^{i\pi/3}$ and $z_5 = r_2e^{2i\pi/3}$, where $0 < r_1, r_2 < 2$.

The logarithmic potential energy for this system, based on Equation 1, is given by

$$E(z_1, \ldots, z_5) = -\sum_{1 \le j < k \le 5} \log|z_j - z_k|.$$

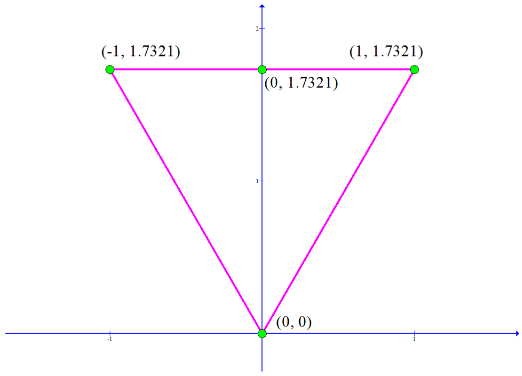

Figure 1: Triangle with 4 potential charges.

Using $z_1, \ldots, z_5$ as defined above, we have the energy equation

$$\begin{aligned} E(z_1, \ldots, z_5) = &-3\log 2 - \log r_1 - \log r_2 - \log(2 - r_1) - \log(2 - r_2) \\ &- \frac{1}{2}\log(r_1^2 - 2r_1 + 4) - \frac{1}{2}(r_2^2 - 2r_2 + 4) - \frac{1}{2}\log(r_1^2 + r_2^2 - r_1 r_2). \end{aligned} \tag{3}$$

Taking the partial derivatives of Equation 3, we have

$$\frac{\partial E}{\partial r_1} = -\frac{1}{r_1} - \frac{r_1 - 1}{r_1^2 - 2r_1 + 4} + \frac{1}{2 - r_1} - \frac{r_1 - r_2/2}{r_1^2 + r_2^2 - r_1 r_2}, \tag{4}$$

$$\frac{\partial E}{\partial r_2} = -\frac{1}{r_2} - \frac{r_2 - 1}{r_2^2 - 2r_2 + 4} + \frac{1}{2 - r_2} - \frac{r_2 - r_1/2}{r_1^2 + r_2^2 - r_1 r_2}. \tag{5}$$

To find the extrema, we will look at $\frac{\partial E}{\partial r_1} = 0$ and $\frac{\partial E}{\partial r_2} = 0$, respectively. Thus we need to look at the following equations.

$$\begin{aligned} 32r_1^2 - 44r_1^3 + 26r_1^4 - 8r_1^5 - 24r_1r_2 + 36r_1^2r_2 - 22r_1^3r_2 + 7r_1^4r_2 + 16r_2^2 \\ - 28r_1r_2^2 + 18r_1^2r_2^2 - 6r_1^3r_2 = 0 \end{aligned} \tag{6}$$

$$\begin{aligned} 32r_2^2 - 44r_2^3 + 26r_2^4 - 8r_2^5 - 24r_2r_1 + 36r_2^2r_1 - 22r_2^3r_1 + 7r_2^4r_1 + 16r_1^2 \\ - 28r_2r_1^2 + 18r_2^2r_1^2 - 6r_2^3r_1 = 0 \end{aligned} \tag{7}$$

Maple was used to find a Groebner basis for the pair of equations (6) and (7) with respect to the lexicographical order. The previous system of equations is thus transformed into an equivalent set of equations. The three equations found for this basis are given below.

$$\begin{aligned} 0 = &2416473920r_2^{13} - 51749096r_2^{18} + 4319717888r_2^{11} - 3448223104r_2^{12} + 9437184r_2^3 \\ &+ 377094144r_2^5 + 14376189r_2^{19} - 379078792r_2^{16} - 1060773888r_2^6 \\ &- 86114304r_2^4 + 2163367936r_2^7 - 3422294016r_2^8 + 4388357120r_2^9 \\ &- 4706391552r_2^{10} + 152724600r_2^{17} + 3136r_2^{23} + 806370672r_2^{15} \\ &+ 516586r_2^{21} - 1490600704r_2^{14} - 56504r_2^{22} - 3156443r_2^{20} \end{aligned} \tag{8}$$

$$
\begin{aligned}
0 = &- 1348024704768903305529 6r_2^{13} + 1534355481989658674 27r_2^{18} \\
&- 2878995086180247967782 4r_2^{11} + 2103501118543539678534 4r_2^{12} \\
&- 362289631309332 48\, r_1 r_2 - 1600498766346072883 20r_2^{3} \\
&+ 7245792626186649 6r_2^{2} - 481601764023070411161 6r_2^{5} \\
&- 3465500957248259013 7r_2^{19} + 1532693766969929894392r_2^{16} \\
&+ 1811448156546662 4r_2^{2} r_1 + 11931111425172971888640r_2^{6} \\
&+ 1260473165262342881280r_2^{4} - 21603143488833887264768r_2^{7} \\
&+ 30570726311414645515264r_2^{8} - 35324302966529956028928r_2^{9} \\
&+ 34374754875099628056576r_2^{10} - 536029502837947039196r_2^{17} \\
&- 3679902819685247159992r_2^{15} - 6542066697403087 76r_2^{21} \\
&+ 7560884742575510283248r_2^{14} + 372953351081709 44r_2^{22} \\
&+ 58264945170830369 58r_2^{20}
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
0 = &44789929301256691264723 2\, {r_2}^{13} - 524000944364043014113 1\, {r_2}^{18} \\
&+ 9498722159003804478027 52\, {r_2}^{11} - 6963808986219857958878 08\, {r_2}^{12} \\
&- 579663410094931968\, r_1 r_2 + 52821226934087877918 72\, {r_2}^{3} \\
&+ 579663410094931968\, {r_2}^{2} + 157162683826382716723200\, {r_2}^{5} \\
&+ 1192559063269146281233\, {r_2}^{19} - 5165742359496307206412 0\, {r_2}^{16} \\
&- 3888897640086851480862 72\, {r_2}^{6} - 41295116131560489517056\, {r_2}^{4} \\
&+ 704678739730206504476672\, {r_2}^{7} - 999251791842825931899904\, {r_2}^{8} \\
&+ 1157900913956622708994560\, {r_2}^{9} - 1130376567273491268870144\, {r_2}^{10} \\
&+ 18179113156399438789916\, {r_2}^{17} + 123350163368482677929848\, {r_2}^{15} \\
&+ 2289625733181025047 2\, {r_2}^{21} - 252247461336581271233648\, {r_2}^{14} \\
&- 1316837934078441152\, {r_2}^{22} - 202153837542785620590\, {r_2}^{20} \\
&+ 579663410094931968\, {r_1}^{2}
\end{aligned} \tag{10}
$$

Consider Equation 8, the first equation in the equivalent set. This equation is factorable as follows.

$$
\begin{aligned}
&{r_2}^3 (r_2 - 2)(448r_2^{14} - 4872r_2^{13} + 27478r_2^{12} - 103769r_2^{11} + 291717r_2^{10} \\
&\quad - 650038r_2^{9} + 1199112r_2^{8} - 1884864r_2^{7} + 2546528r_2^{6} - 2912128r_2^{5} \\
&\quad + 2716608r_2^{4} - 1956224r_2^{3} + 1006592r_2^{2} - 325632\, r_2 + 49152) \\
&\quad ({r_2}^2 - 2\, r_2 + 4)(7r_2^{3} - 22r_2^{2} + 36\, r_2 - 24)
\end{aligned}
$$

Equation 3.4 has one real solution in our triangular domain, given by

$$
r_2 = \frac{2}{21}\sqrt[3]{197 + 63\sqrt{89}} - \frac{136}{21}\,\frac{1}{\sqrt[3]{197 + 63\,\sqrt{89}}} + \frac{22}{21} \approx 1.22837
$$

There are no other real zeros in Equation 3.4, nor are there any other roots within Equation 9 or Equation 10 found from the Groebner basis.

Therefore the system attains equilibrium when the five charges are located at the points $z_1 = 0$, $z_2 = 2e^{2\pi/3}$, $z_3 = 2e^{\pi/3}$, and the approximate points $z_4 \approx 1.22837e^{\pi/3}$, and $z_5 \approx 1.22837e^{2\pi/3}$. The value for r_1 is identical to the computed value for r_2 based on the similar forms of Equation 6 and Equation 7 and the calculation of the Groebner basis from those equation forms.

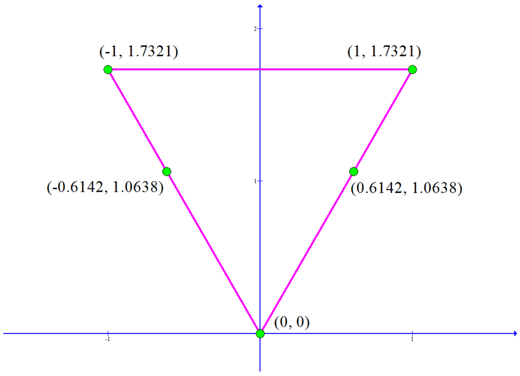

Figure 2: Triangle with 5 potential charges.

4 Conclusion

There are a number of problems involving the placement of charges on domains involving logarithmic potential. The placement of n unit charges interacting with logarithmic potential on a circle is known to be the roots of unity of $x^n - 1 = 0$ [1]. This paper attempts to address the placement of such charges on a triangular domain to obtain the minimal potential energy of the system and to find the position of these charges in a closed form not involving Schwarz-Christoffel transformations. Given the complexity of finding the position for a system with five unit charges, the question posed here is whether there is such a formula or method that will provide the location of six or more such unit charges on a equilateral triangle domain interacting with a logarithmic potential charge.

5 Conflicts of Interest

The author states that there are no relevant interests that could be perceived as conflicting.

6 Funding Statement

The author states that no grant funding was used in the preparation or creation of this article or the work described therein. This research was prepared under the employment of Texas Tech University (USA) and Prairie View A&M University (USA).

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