Precise Asymptotics in Wichura's Law of Iterated Logarithm

Gooty Divanji*, Kokkada Vidyalaxmi and K. N. Raviprakash

Department of Studies in Statistics, Manasagangotri, University of Mysore, Mysuru-570006-Karnataka-India.

gootydivan@yahoo.co.in, vidyalaxmi.k@gmail.com and raviprakashmaiya@gmail.com

Abstract

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with a common distribution function $F = P(X \leq x)$ in the domain of attraction of a asymmetric stable law, with index α, $1 < \alpha < 2$ and set $S_n = \sum_{k=1}^{n} X_k$. We prove that,

$$\lim_{\varepsilon \to 0} \sqrt{\varepsilon} \sum_{n \geq 3} \frac{1}{n} P\left(S_n \leq (\theta_\alpha - \varepsilon) A_n\right) = \frac{1}{2\sqrt{2\alpha}},$$

where $A_n = n^{\frac{1}{\alpha}} (\log \log n)^{\frac{\alpha-1}{\alpha}}$, $\theta_\alpha = (B(\alpha))^{\frac{1-\alpha}{\alpha}}$ and $B(\alpha) = (1-\alpha)\, \alpha^{\frac{\alpha}{1-\alpha}} \left(cos \frac{\pi\alpha}{2}\right)^{\frac{1}{\alpha-1}}$.

Key words and phrases: Precise Asymptotics, Rate of Convergence, Law of Iterated Logarithm, Asymmetric Stable Law, Domain of Attraction.

2010 Mathematics Subject Classification: Primary 60G50; Secondary 60E15, 60F15

1 Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with a common distribution function $F = P(X \leq x)$. Set $S_n = \sum_{k=1}^{n} X_k$, for $n \geq 1$. In a well known innovative work of Hsu and Robbins [11] introduced the concept of 'complete convergence' and proved that the sequence of arithmetic means of i.i.d. r.v.s converges completely to the expected value of random variables, provided the mean and the variance exist. More precisely, they had shown that,

$$\sum_{n=1}^{\infty} P\left(|S_n| \geq \varepsilon n\right) < \infty, \quad \text{for } \varepsilon > 0,$$

if $EX = 0$ and $EX^2 < \infty$. The converse was proved by Erdos [4] and [5]. Later Baum and Katz [1] extended the Hsu-Robbins-Erdos theorem in more general form. Which states that, for all $p < 2$ and $r \geq p$,

$$\sum_{n=1}^{\infty} n^{\left(\frac{r}{p}-2\right)} P\left(|S_n| \geq \varepsilon n^{\frac{1}{p}}\right) < \infty, \quad \text{for } \varepsilon > 0 \tag{1.1}$$

if and only if $E|X|^r < \infty$.

Notice that when $r = 2$ and $p = 1$ the result reduces to the theorem of Hsu-Robbins-Erdos. For $r = p = 1$, we rediscover the famous theorem of Spitzer [16]. For $r > 0$ and $p = 1$ the result was earlier proved by Katz [12].

In view of the fact that the sums tend to infinity as $\varepsilon \to 0$, it is of interest to find the exact rate in terms of

CC BY

ε that yield non-trivial limits. Such results are referred to as precise asymptotics in the laws of large numbers. The first step in this direction was done by Heyde [10], who proved that,

$$\lim_{\varepsilon \to 0} \varepsilon^2 \sum_{n=1}^{\infty} P\left(|S_n| \geq \varepsilon n\right) = EX^2,$$

whenever $EX = 0$ and $EX^2 < \infty$. Later Chen [2] generalized for the case $p < 2$ and $r \geq 2$. Spataru [15] for $r = p = 1$ and Gut and Spataru [7], for $1 \leq p < r < \alpha$, they did not assume finite variance, but the distribution of the summands belongs to the normal domain of attraction to a stable distribution with index $\alpha \in (1, 2]$. Later, this result was also proved for the case $0 < p < \alpha < 1$ by Gut and Steinebach [9].

Further, for finite variance, Gut and Spataru [8] established similar results to the classical law of iterated logarithm. Later Li et. al. [13] improved and generalized the results of Gut and Spataru [8]. We can observe that when variance is infinite, the above results of Gut and Spataru [8] and Li et. al. [13] are no longer valid. In the literature we have law of iterate logarithm for power normalization and linear normalization results. Wichura's [17] law of iterate logarithm deals with linear normalization. In this work, we use Wichura's law of iterate logarithm to study precise asymptotics in law of iterate logarithm for more general class of distributions with infinite variance. In brief let$\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed non-negative random variables. whose common distribution function F is in the domain of attraction of a completely asymmetric stable law G_α, with index α, $1 < \alpha < 2$. Note that, under this assumption on F, we have $\frac{S_n}{B_n} \xrightarrow{D} Y_\alpha$, the asymmetric stable random variables Y_α, where $B_n \to \infty$ as $n \to \infty$.We envisage that our results are having much importance in share market, determination of prices of shares, fluctuations exceeding a certain level in SENSEX have major role to play in determining the share value and are to be predicted by considering extreme items.

We know from Wichura [17] that,

$$\liminf_{n \to \infty} \frac{S_n}{A_n} = \theta_\alpha \quad \text{a.s.,} \tag{1.2}$$

where $A_n = n^{\frac{1}{\alpha}} \left(\log\log n\right)^{\frac{\alpha-1}{\alpha}}$, $\theta_\alpha = \left(B(\alpha)\right)^{\frac{1-\alpha}{\alpha}}$ and $B(\alpha) = (1-\alpha)\,\alpha^{\frac{\alpha}{1-\alpha}} \left(cos\frac{\pi\alpha}{2}\right)^{\frac{1}{\alpha-1}}$.

Then for any sufficiently small $\varepsilon > 0$, (1.2) implies $P\left(S_n \leq (\theta_\alpha - \varepsilon)\, A_n \text{ i.o.}\right) = 0$ and $P\left(S_n \leq (\theta_\alpha + \varepsilon)\, A_n \text{ i.o.}\right) = 1$, where a.s. and i.o. mean almost surely and infinitely often respectively. It is our interest to study the precise asymptotic behavior of the sums $\sum_{n\geq 3} \frac{1}{n} P\left(S_n \leq (\theta_\alpha - \varepsilon)\, A_n\right)$, when $\varepsilon \to 0$, with the assumption that the distribution function F is in the domain of attraction of a completely asymmetric stable law, with index α, $1 < \alpha < 2$. Throughout the paper C with or without a suffix or a super suffix stand for a positive constant, possibly varying from place to place. Similarly, $[x]$ shall denote the largest integer which is less than or equal to x. The symbol $\sim$ denotes the asymptotics. i.e. $f(t) \sim g(t)$, for $t \to t_0$, if $\lim_{t \to t_0} \frac{f(t)}{g(t)} = 1$. $\stackrel{D}{=}$ indicates distributionally same. In the sequel we use the following Wichura's lemma in proof of Proposition 3.1.

Lemma 1.1. *(Wichura, M.J. [17])*

For each $C > 0$, $P\left(\frac{S_n}{A_n} \leq d(\alpha)C\right) = \exp\left\{-(1+o(1))\left(\frac{C}{\theta_a}\right)^{\lambda} \log\log n\right\}$, *where* $S_n = \sum_{k=1}^{n} X_k$, $A_n = \theta_\alpha n^{\frac{1}{\alpha}}\left(\log\log n\right)^{\frac{\alpha-1}{\alpha}}$, $\frac{\alpha}{|\alpha-1|^{\frac{1}{\lambda}}}$, $\lambda = \frac{\alpha}{\alpha-1}$, $d(\alpha) = 1$, *if* $\alpha < 1$ *and* = *-1, if* $\alpha > 1$ *and* $\alpha \neq 1$.

Now we are ready to state our result.

Theorem 1.1. *Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed non-negative random variables with a common distribution function F is in the domain of attraction of a completely asymmetric stable law, with index α, $1 < \alpha < 2$. Then*

$$\lim_{\varepsilon \to 0} \sqrt{\varepsilon} \sum_{n \geq 3} \frac{1}{n} P\left(S_n \leq (\theta_\alpha - \varepsilon) A_n\right) = \frac{1}{2\sqrt{2\alpha}},$$

where $A_n = n^{\frac{1}{\alpha}} (\log \log n)^{\frac{\alpha-1}{\alpha}}$, $\theta_\alpha = (B(\alpha))^{\frac{1-\alpha}{\alpha}}$ *and* $B(\alpha) = (1-\alpha)\,\alpha^{\frac{\alpha}{1-\alpha}} \left(cos \frac{\pi\alpha}{2}\right)^{\frac{1}{\alpha-1}}$.

The proof of the theorem will be carried out in two stages. In section 2, F is assumed to be positive stable distribution function, while section 3 will be devoted to the distribution function F is in the domain of attraction of a completely asymmetric stable law with index α, $1 < \alpha < 2$.

2 F is positive stable law

Proposition 2.1. *By Wichura's theorem (1.2), we have, if F is positive stable law with index α, $1 < \alpha < 2$, then we have,*

$$\lim_{\varepsilon \to 0} \sqrt{\varepsilon} \sum_{n \geq 3} \frac{1}{n} P\left(S_n \leq (\theta_\alpha - \varepsilon) A_n\right) = \frac{1}{2\sqrt{2\alpha}},$$

where $A_n = n^{\frac{1}{\alpha}} (\log \log n)^{\frac{\alpha-1}{\alpha}}$.

Proof. We have

$$\begin{aligned}\sum_{n \geq 3} \frac{1}{n} P\left(S_n \leq (\theta_\alpha - \varepsilon) A_n\right) &= \sum_{n \geq 3} \frac{1}{n} P\left(S_n \leq (\theta_\alpha - \varepsilon)\, n^{\frac{1}{\alpha}} (\log \log n)^{\frac{\alpha-1}{\alpha}}\right) \\ &= \sum_{n \geq 3} \frac{1}{n} P\left(X_1 \leq (\theta_\alpha - \varepsilon) (\log \log n)^{\frac{\alpha-1}{\alpha}}\right).\end{aligned}$$

Since $\frac{S_n}{n^{\frac{1}{\alpha}}} \stackrel{D}{=} X_1$ and using the asymptotic formula IV of Mijnheer [14, Theorem 2.1.7], one gets,

$$\begin{aligned}\sum_{n \geq 3} \frac{1}{n} P\left(S_n \leq (\theta_\alpha - \varepsilon) A_n\right) &= \sum_{n \geq 3} \frac{1}{n} P\left(X_1 \leq (\theta_\alpha - \varepsilon) (\log \log n)^{\frac{\alpha-1}{\alpha}}\right) \\ &= \sum_{n \geq 3} \frac{1}{n} F_{X_1}\left((\theta_\alpha - \varepsilon) (\log \log n)^{\frac{\alpha-1}{\alpha}}\right) \\ \sim \left(\frac{2}{\alpha}\right)^{\frac{1}{2}} &\sum_{n \geq 3} \frac{1}{n} P\left(U \geq (2B(\alpha))^{\frac{1}{2}} (\theta_\alpha - \varepsilon)^{-\frac{\alpha}{2(1-\alpha)}} (\log \log n)^{\frac{1}{2}}\right)\end{aligned} \tag{2.1}$$

where U is a standard normal random variable. Using θ_α and $B(\alpha)$, we get,

$$\sum_{n \geq 3} \frac{1}{n} P\left(S_n \leq (\theta_\alpha - \varepsilon) A_n\right) \sim \left(\frac{2}{\alpha}\right)^{\frac{1}{2}} \sum_{n \geq 3} \frac{1}{n} P\left(U \geq \sqrt{\delta (\log \log n)}\right),$$

where $\delta = 2\left(\frac{\theta_\alpha}{\theta_\alpha - \varepsilon}\right)^{\frac{\alpha}{1-\alpha}}$. As U is standard normal random variable, we set $\psi(x) = 2(1 - \Phi(x)), x \geq 0$, where $\Phi(x)$ is a standard normal distribution function and hence one gets that

$$\sum_{n \geq 3} \frac{1}{n} P\left(S_n \leq (\theta_\alpha - \varepsilon) A_n\right) \sim \left(\frac{2}{\alpha}\right)^{\frac{1}{2}} \sum_{n \geq 3} \frac{1}{n} \frac{\psi\left(\sqrt{\delta} \log \log n\right)}{2}$$

$$= \left(\frac{1}{\sqrt{2\alpha}}\right) \sum_{n\geq 3} \frac{1}{n} \psi\left(\sqrt{\delta \log\log n}\right). \tag{2.2}$$

For $m \geq 1$, on account of the Euler-Maclaurin sum formula [Cramer, [3], p-124], we have,

$$\begin{aligned} \sum_{n=3}^{m} \frac{1}{n} P\left(S_n \leq (\theta_\alpha - \varepsilon) A_n\right) &= \frac{1}{\sqrt{2\alpha}} \sum_{n=3}^{m} \frac{1}{n} \psi\left(\sqrt{\delta \log\log n}\right) \\ &= \frac{1}{\sqrt{2\alpha}} \left\{ \int_3^m \frac{1}{x} \psi\left(\sqrt{\delta \log\log x}\right) dx + \frac{1}{2m} \psi\left(\sqrt{\delta \log\log x}\right) \right. \\ &\quad \left. + \frac{1}{2}\psi(\delta) - \int_3^m P_1(x) d\left(\frac{1}{x}\psi\left(\sqrt{\delta\log\log x}\right)\right) \right\}, \end{aligned} \tag{2.3}$$

where $P_1(x) = [x] - x + \frac{1}{2}$. Taking limit $m \to \infty$ in (2.3) we get,

$$\begin{aligned} \sum_{n\geq 3} \frac{1}{n} P\left(S_n \leq (\theta_\alpha - \varepsilon) A_n\right) = \frac{1}{\sqrt{2\alpha}} &\left\{ \int_3^\infty \frac{1}{x} \psi\left(\sqrt{\delta \log\log x}\right) dx + \frac{1}{2}\psi(\delta) \right. \\ &\left. - \int_3^\infty P_1(x) d\left(\frac{1}{x}\psi\left(\sqrt{\delta\log\log x}\right)\right) \right\}. \end{aligned} \tag{2.4}$$

Observe that

$$\begin{aligned} d\left(x^{-1}\psi\left(\sqrt{\delta\log\log x}\right)\right) &= \frac{\delta\psi'\left(\sqrt{\delta\log\log x}\right)}{2x^2\log x\sqrt{\delta\log\log x}} - \frac{\psi\left(\sqrt{\delta\log\log x}\right)}{x^2} \\ &= \frac{\sqrt{\delta}\psi'\left(\sqrt{\delta\log\log x}\right)}{2x^2\log x\sqrt{\log\log x}} - \frac{\psi\left(\sqrt{\delta\log\log x}\right)}{x^2}. \end{aligned}$$

We have

$$\begin{aligned} &\int_3^\infty P_1(x) d\left(x^{-1}\psi\left(\sqrt{\delta\log\log x}\right)\right) \\ &= \frac{\sqrt{\delta}}{2}\int_3^\infty P_1(x)\frac{\psi'\left(\sqrt{\delta\log\log x}\right)}{x^2\log x\sqrt{\log\log x}}dx - \int_3^\infty P_1(x)\frac{\psi\left(\sqrt{\delta\log\log x}\right)}{x^2}dx, \end{aligned} \tag{2.5}$$

where $\delta = 2\left(\frac{\theta_\alpha}{\theta_\alpha - \varepsilon}\right)^{\frac{\alpha}{1-\alpha}}$. Since $|P_1(x)| \leq \frac{1}{2}$, we get

$$\left|\int_3^\infty P_1(x)\frac{\psi\left(\sqrt{\delta\log\log x}\right)}{x^2}dx\right| \leq \frac{1}{2}\int_3^\infty \frac{dx}{x^2} = 0.16666. \tag{2.6}$$

Put $y = \sqrt{\log\log x}$ in first part of right hand side (R.H.S.) of (2.5), we get

$$\left|\frac{\sqrt{\delta}}{2}\int_3^\infty P_1(x)\frac{\psi'\left(\sqrt{\delta\log\log x}\right)}{x^2\log x\sqrt{\log\log x}}dx\right| \leq C\int_{\sqrt{\log\log 3}}^\infty \frac{dy}{e^{e^{y2}}}, \tag{2.7}$$

where C is positive constant. We know that the density of a stable distribution is bounded and using the fact $e^{e^{y2}} > e^{1+y^2}$ we get

$$\int_{\sqrt{\log\log 3}}^\infty \frac{dy}{e^{e^{y2}}} \leq \int_{\sqrt{\log\log 3}}^\infty e^{-1-y^2}dy = \frac{\sqrt{\pi}}{2e}\left(1 - \mathrm{erf}\left(\sqrt{\log\log 3}\right)\right),$$

where erf(x)= Gauss error function of $x = \frac{2}{\pi}\int\limits_0^x e^{-t^2}dt$.
Since $\sqrt{\log\log 3} = 0.30667$ and $\text{erf}(\log\log 3) = 0.33549$, we have

$$\int\limits_{\sqrt{\log\log 3}}^{\infty} \frac{dy}{e^{e^{y^2}}} \leq 0.21664. \tag{2.8}$$

Substitute (2.8) in (2.7), one can find some constant $C_1(> C)$ such that

$$\left| \frac{\sqrt{\delta}}{2} \int\limits_3^{\infty} P_1(x) \frac{\psi^{'}\left(\sqrt{\delta\log\log x}\right)}{x^2\log x\sqrt{\log\log x}} dx \right| \leq C_1. \tag{2.9}$$

Substitute (2.6) and (2.9) in (2.5) and taking $\varepsilon \to 0$, we can claim that

$$\lim_{\varepsilon\to 0}\int\limits_3^{\infty} P_1(x) d\left(x^{-1}\psi\left(\sqrt{\delta\log\log x}\right)\right) = 0.$$

Proof of Proposition 2.1 will be completed if we show that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\sum_{n\geq 3}\frac{1}{n}P\left(S_n \leq (\theta_\alpha - \varepsilon)A_n\right) \\ &\quad = \frac{1}{\sqrt{2\alpha}}\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\int\limits_3^{\infty}\frac{1}{x}\psi\left(\sqrt{\delta\log\log x}\right)dx = \frac{1}{2\sqrt{2\alpha}}.\end{aligned} \tag{2.10}$$

Again put $y = \sqrt{\delta\log\log x}$ in (2.10) and proceed as in (2.7) one gets that

$$\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\sum_{n\geq 3}\frac{1}{n}P\left(S_n \leq (\theta_\alpha - \varepsilon)A_n\right) = \frac{1}{\sqrt{2\alpha}}\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\int\limits_{\left(\sqrt{\delta\log\log 3}\right)}^{\infty}\psi(y)d\left(e^{\frac{y^2}{\delta}}\right).$$

Using integration by parts, we get

$$\begin{aligned}&\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\sum_{n\geq 3}\frac{1}{n}P\left(S_n \leq (\theta_\alpha - \varepsilon)A_n\right) \\ &\quad = \frac{1}{\sqrt{2\alpha}}\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\left[\psi(y)e^{\frac{y^2}{\delta}}\Big|_{\sqrt{\delta\log\log 3}}^{\infty} - \int\limits_{\left(\sqrt{\delta\log\log 3}\right)}^{\infty}\psi^{'}(y)\left(e^{\frac{y^2}{\delta}}\right)dy\right],\end{aligned}$$

which yields,

$$\begin{aligned}&\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\sum_{n\geq 3}\frac{1}{n}P\left(S_n \leq (\theta_\alpha - \varepsilon)A_n\right) \\ &= \left\{\frac{1}{\sqrt{2\alpha}}\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\left[-\psi\left(\sqrt{\delta\log\log 3}\right)e^{\frac{\left(\sqrt{\delta\log\log 3}\right)^2}{\delta}}\right]\right. \\ &\qquad \left. + \frac{1}{\sqrt{2\alpha}}\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\int\limits_{\left(\sqrt{\delta\log\log 3}\right)}^{\infty}\psi^{'}(y)\left(e^{\frac{y^2}{\delta}}\right)dy\right\} \\ &= A + B \quad \text{(say)},\end{aligned} \tag{2.11}$$

where $A = -\frac{1}{\sqrt{2\alpha}} \lim_{\varepsilon \to 0} \sqrt{\varepsilon} \left[\psi \left(\sqrt{\delta \log \log 3} \right) e^{\frac{(\delta \log \log 3)}{\delta}} \right]$ and
$B = \frac{1}{\sqrt{2\alpha}} \lim_{\varepsilon \to 0} \sqrt{\varepsilon} \int_{\left(\sqrt{\delta \log \log 3}\right)}^{\infty} \psi^{'}(y) \left(e^{\frac{y^2}{\delta}} \right) dy.$
Observe that $A = -\frac{1}{\sqrt{2\alpha}} \lim_{\varepsilon \to 0} \sqrt{\varepsilon} \psi \left(\sqrt{\delta \log \log 3} \right) (\log 3) = 0.$

We know that $\psi^{'}(y)$ is probability density function of standard normal variable and hence,

$$B = \frac{1}{\sqrt{2\alpha}} \lim_{\varepsilon \to 0} \sqrt{\varepsilon} \frac{1}{\sqrt{2\pi}} \int_{\left(\sqrt{\delta \log \log 3}\right)}^{\infty} e^{\frac{-y^2}{2} + \frac{y^2}{\delta}} dy$$

Put $\frac{-y^2}{2} + \frac{y^2}{\delta} = \frac{-y^2}{2\sigma^2}$ and $\frac{y}{\sigma} = u$, where $\sigma^2 = \frac{\delta}{\delta - 2}$ and $\delta = 2 \left(\frac{\theta_\alpha}{\theta_\alpha - \varepsilon} \right)^{\frac{\alpha}{1-\alpha}}$ in the above expression, we get

$$B = \frac{1}{\sqrt{2\alpha}} \lim_{\varepsilon \to 0} \sqrt{\varepsilon} \frac{1}{\sqrt{2\pi}} \int_{\left(\sqrt{\delta \log \log 3}\right)}^{\infty} e^{\frac{-u^2}{2}} \sigma du$$

Since $(\frac{\theta_\alpha}{\theta_\alpha - \varepsilon})^{\frac{\alpha}{1-\alpha}} = (\frac{\theta_\alpha - \varepsilon}{\theta_\alpha})^{\frac{\alpha}{\alpha-1}} = (1 - \frac{\varepsilon}{\theta_\alpha})^{\frac{\alpha}{\alpha-1}} < 1$, where $1 < \alpha < 2$.
Now choose $\varepsilon_1 \to 0$ as $\varepsilon \to 0$ such that $(\frac{\theta_\alpha}{\theta_\alpha - \varepsilon})^{\frac{\alpha}{1-\alpha}} < 1 = (1 - \varepsilon_1)$ (say)
Then we have $\delta = 2(\frac{\theta_\alpha}{\theta_\alpha - \varepsilon})^{\frac{\alpha}{1-\alpha}} = 2(1 - \varepsilon_1)$ and $\sigma^2 = \frac{\delta}{\delta - 2} = \frac{2(1-\varepsilon_1)}{2(1-\varepsilon_1)-2} = \frac{1-\varepsilon_1}{-\varepsilon_1} = 1 - \frac{1}{\varepsilon_1} > \frac{1}{\varepsilon_1}$. i.e., $\sigma^2 > \frac{1}{\varepsilon_1}$ implies $\sigma > \frac{1}{\sqrt{\varepsilon_1}}$. Similarly $\delta = 2(1 - \varepsilon_1) > 2\varepsilon_1$. Therefore

$$B = \frac{1}{\sqrt{2\alpha}} \lim_{\varepsilon_1 \to 0} \sqrt{\varepsilon_1} \frac{1}{\sqrt{\varepsilon_1}} \frac{1}{\sqrt{2\pi}} \int_{\left(\sqrt{2\varepsilon_1 \log \log 3}\right)}^{\infty} e^{\frac{-u^2}{2}} du$$

$$\sim \frac{1}{\sqrt{2\alpha}} \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{\frac{-u^2}{2}} du = \frac{1}{\sqrt{2\alpha}} \frac{1}{2} = \frac{1}{2\sqrt{2\alpha}}.$$

Substitute A and B in (2.11), we get,

$$\lim_{\varepsilon \to 0} \sqrt{\varepsilon} \sum_{n \geq 3} \frac{1}{n} P\left(S_n \leq (\theta_\alpha - \varepsilon) A_n\right) = \frac{1}{2\sqrt{2\alpha}}.$$

Therefore proof of the Proposition 2.1 is completed. □

3 When F is attracted to a positive asymmetric stable law

Recall that under the assumption made on F, we have $\frac{S_n}{B_n} \xrightarrow{w} Y_\alpha$; the asymmetric stable random variable with distribution function G_α with index α, $1 < \alpha < 2$, where $B_n \to \infty$ as $n \to \infty$. We know that only stable distributions have non-empty domains of attraction. Moreover, moments of all order $\gamma < \alpha$ exits, i.e. $E|X|^\gamma < \infty$, for all $\gamma < \alpha$. Further, note that B_n can be taken as $B_n = n^{\frac{1}{\alpha}} L(n)$, where $L(n)$ is slowly varying at infinity. (See Mijnheer [1975], page 16).

Define, for any $M > 1$, $b(\varepsilon, M) = \exp \left\{ \sqrt{\frac{M^2}{\varepsilon}} \right\}$ for convenience.

Proposition 3.1. *We have*

$$\lim_{\varepsilon \to 0} \sqrt{\varepsilon} \sum_{n > b(\varepsilon, M)} \frac{1}{n} P\left(S_n \leq (\theta_\alpha - \varepsilon) A_n\right) = 0.$$

Proof. Since $\{X_n, n \geq 1\}$ is a sequence of independent and identically distributed random variables with a common distribution function F is in the domain of attraction of a positive asymmetric stable law with index α, $1 < \alpha < 2$ and $A_n = n^{\frac{1}{\alpha}} (\log\log n)^{\frac{\alpha-1}{\alpha}}$, then using lemma (1.1), for sufficiently small $\varepsilon > 0$,

$$
\begin{aligned}
P\left(S_n \leq (\theta_\alpha - \varepsilon) A_n\right) &= P\left(S_n \leq (\theta_\alpha - \varepsilon) n^{\frac{1}{\alpha}} (\log\log n)^{\frac{\alpha-1}{\alpha}}\right) \\
&= \exp\left\{-(1+o(1))\left(\frac{\theta_\alpha - \varepsilon}{\theta_\alpha}\right)^{\lambda} \log\log n\right\} \\
&= (\log n)^{-(1+o(1))\left(\frac{\theta_\alpha-\varepsilon}{\theta_\alpha}\right)^{\lambda}},
\end{aligned}
$$

where $\lambda = \frac{\alpha}{\alpha-1}$. Thus for n large enough,

$$
\begin{aligned}
&\sqrt{\varepsilon} \sum_{n > b(\varepsilon, M)} \frac{1}{n} P\left(S_n \leq (\theta_\alpha - \varepsilon) A_n\right) \\
&= \sqrt{\varepsilon} \sum_{n > b(\varepsilon, M)} \frac{1}{n(\log n)^{(1+o(1))\left(\frac{\theta_\alpha-\varepsilon}{\theta_\alpha}\right)^{\lambda}}} \qquad (3.1)
\end{aligned}
$$

Since $n > b(\varepsilon, M) = \exp\left\{\sqrt{\frac{M^2}{\varepsilon}}\right\}$ implies

$$
\begin{aligned}
&\sqrt{\varepsilon} \sum_{n > b(\varepsilon, M)} \frac{1}{n(\log n)^{(1+o(1))\left(\frac{\theta_\alpha-\varepsilon}{\theta_\alpha}\right)^{\lambda}}} \\
&< \sqrt{\varepsilon} \left(\sqrt{\frac{\varepsilon}{M^2}}\right)^{(1+o(1))\left(\frac{\theta_\alpha-\varepsilon}{\theta_\alpha}\right)^{\lambda}} \frac{1}{\exp\left\{\sqrt{\frac{M^2}{\varepsilon}}\right\}}. \qquad (3.2)
\end{aligned}
$$

Substitute (3.2) in (3.1) one gets that

$$
\begin{aligned}
&\sqrt{\varepsilon} \sum_{n > b(\varepsilon, M)} \frac{1}{n} P\left(S_n \leq (\theta_\alpha - \varepsilon) A_n\right) \\
&< \sqrt{\varepsilon} \left(\sqrt{\frac{\varepsilon}{M^2}}\right)^{(1+o(1))\left(\frac{\theta_\alpha-\varepsilon}{\theta_\alpha}\right)^{\lambda}} \frac{1}{\exp\left\{\sqrt{\frac{M^2}{\varepsilon}}\right\}} \\
&\to 0 \text{ as } \varepsilon \to 0.
\end{aligned}
$$

Hence the proof of Proposition 3.1 is completed. □

Proposition 3.2. *We have*

$$
\lim_{\varepsilon \to 0} \sqrt{\varepsilon} \sum_{n > b(\varepsilon, M)} \frac{1}{n} G_\alpha\left((\theta_\alpha - \varepsilon)(\log\log n)^{\frac{\alpha-1}{\alpha}}\right) = 0.
$$

Proof. Since G_α is a positive asymmetric stable distribution function with exponent α, $1 < \alpha < 2$ and using Feller [6, Theorem1 on page 448], for some positive constant C, we have

$$\begin{aligned}
&\sqrt{\varepsilon} \sum_{n>b(\varepsilon,M)} \frac{1}{n} G_\alpha \left((\theta_\alpha - \varepsilon) (\log\log n)^{\frac{\alpha-1}{\alpha}} \right) \\
&\quad = \sqrt{\varepsilon} \sum_{n>b(\varepsilon,M)} \frac{1}{n} P \left(X_1 \leq (\theta_\alpha - \varepsilon) (\log\log n)^{\frac{\alpha-1}{\alpha}} \right) \\
&\quad \sim C\sqrt{\varepsilon} \sum_{n>b(\varepsilon,M)} \frac{1}{n} \exp\left\{ - \left((\theta_\alpha - \varepsilon) (\log\log n)^{\frac{\alpha-1}{\alpha}} \right)^{-\alpha} \right\} \\
&\quad < C\sqrt{\varepsilon} \sum_{n>b(\varepsilon,M)} \frac{1}{n(\log n)^{(\theta_\alpha-\varepsilon)^{-\alpha}}}.
\end{aligned}$$

Since $b(\varepsilon, M) = \exp\left\{ \sqrt{\frac{M^2}{\varepsilon}} \right\}$ and $n > b(\varepsilon, M)$ we get

$$\frac{1}{n(\log n)^{(\theta_\alpha-\varepsilon)^{-\alpha}}} < \sqrt{\frac{\varepsilon}{M^2}} \frac{1}{\exp\left\{ \sqrt{\frac{M^2}{\varepsilon}} \right\}}.$$

Therefore

$$\begin{aligned}
&\sqrt{\varepsilon} \sum_{n>b(\varepsilon,M)} \frac{1}{n} G_\alpha \left((\theta_\alpha - \varepsilon) (\log\log n)^{\frac{\alpha-1}{\alpha}} \right) \\
&\quad < C\sqrt{\varepsilon} \sqrt{\frac{\varepsilon}{M^2}} \frac{1}{\exp\left\{ \sqrt{\frac{M^2}{\varepsilon}} \right\}} \\
&\quad < C \frac{\varepsilon}{M} \exp\left\{ \sqrt{\frac{M^2}{\varepsilon}} \right\} \to 0 \text{ as } \varepsilon \to 0.
\end{aligned}$$

Hence proof of Proposition 3.2 is completed. □

Proposition 3.3. *We have*

$$\lim_{\varepsilon\to 0} \sqrt{\varepsilon} \sum_{n\leq b(\varepsilon,M)} \frac{1}{n} \left| P \left(S_n \leq (\theta_\alpha - \varepsilon) n^{\frac{1}{\alpha}} (\log\log n)^{\frac{\alpha-1}{\alpha}} \right) - G_\alpha \left((\theta_\alpha - \varepsilon) (\log\log n)^{\frac{\alpha-1}{\alpha}} \right) \right| = 0,$$

where G_α is a positive asymmetric stable law with index α, $1 < \alpha < 2$.

Proof. Put

$$\triangle_n = \sup_{\varepsilon<\theta_\alpha} \left| P \left(S_n \leq (\theta_\alpha - \varepsilon) n^{\frac{1}{\alpha}} (\log\log n)^{\frac{\alpha-1}{\alpha}} \right) - G_\alpha \left((\theta_\alpha - \varepsilon) (\log\log n)^{\frac{\alpha-1}{\alpha}} \right) \right|.$$

From Proposition 3.1 and Proposition 3.2 we have the following estimates,

$P \left(S_n \leq (\theta_\alpha - \varepsilon) n^{\frac{1}{\alpha}} (\log\log n)^{\frac{\alpha-1}{\alpha}} \right) \sim (\log n)^{-(1+o(1))\left(\frac{\theta_\alpha-\varepsilon}{\theta_\alpha}\right)^\lambda}$ and

$G_\alpha \left((\theta_\alpha - \varepsilon) (\log\log n)^{\frac{\alpha-1}{\alpha}} \right) = P \left(X_1 \leq (\theta_\alpha - \varepsilon) (\log\log n)^{\frac{\alpha-1}{\alpha}} \right) \sim (\log n)^{-(\theta_\alpha-\varepsilon)^{-\alpha}}$

Using these estimates, one can observe that, $\lim_{n\to\infty} \triangle_n = 0$ and hence, we have

$$\sqrt{\varepsilon} \sum_{n \le b(\varepsilon,M)} \frac{1}{n} \left| P\left(S_n \le (\theta_\alpha - \varepsilon)\, n^{\frac{1}{\alpha}} (\log\log n)^{\frac{\alpha-1}{\alpha}} \right) - G_\alpha \left((\theta_\alpha - \varepsilon) (\log\log n)^{\frac{\alpha-1}{\alpha}} \right) \right|$$

$$\le \sqrt{\varepsilon} \sum_{n \le b(\varepsilon,M)} \frac{\triangle_n}{n}$$

$$\le \sqrt{\varepsilon} \log b(\varepsilon, M) \frac{1}{\log b(\varepsilon, M)} \sum_{n \le b(\varepsilon,M)} \frac{\triangle_n}{n}.$$

Since $b(\varepsilon, M) = \exp\left\{ \sqrt{\frac{M^2}{\varepsilon}} \right\}$,

$$\sqrt{\varepsilon} \sum_{n \le b(\varepsilon,M)} \frac{1}{n} \left| P\left(S_n \le (\theta_\alpha - \varepsilon)\, n^{\frac{1}{\alpha}} (\log\log n)^{\frac{\alpha-1}{\alpha}} \right) - G_\alpha \left((\theta_\alpha - \varepsilon) (\log\log n)^{\frac{\alpha-1}{\alpha}} \right) \right|$$

$$\le M \frac{1}{\log b(\varepsilon, M)} \sum_{n \le b(\varepsilon,M)} \frac{\triangle_n}{n} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Hence proof of Proposition 3.3 is completed. □

4 Proof of the Theorem 1.1.

Writing,

$$\sum_{n\ge 3} \tfrac{1}{n} P\left(S_n \le (\theta_\alpha - \varepsilon) A_n\right) = \sum_{n\ge 3} \tfrac{1}{n} G_\alpha \left((\theta_\alpha - \varepsilon) (\log\log n)^{\frac{\alpha-1}{\alpha}} \right)$$
$$+ \sum_{n \le b(\varepsilon,M)} \tfrac{1}{n} \left(P\left(S_n \le (\theta_\alpha - \varepsilon) A_n\right) - G_\alpha \left((\theta_\alpha - \varepsilon) (\log\log n)^{\frac{\alpha-1}{\alpha}} \right) \right)$$
$$- \sum_{n > b(\varepsilon,M)} \tfrac{1}{n} G_\alpha \left((\theta_\alpha - \varepsilon) (\log\log n)^{\frac{\alpha-1}{\alpha}} \right) + \sum_{n > b(\varepsilon,M)} \tfrac{1}{n} P\left(S_n \le (\theta_\alpha - \varepsilon) A_n\right),$$

where $A_n = n^{\frac{1}{\alpha}} (\log\log n)^{\frac{\alpha-1}{\alpha}}$ and proof of the theorem follows from Propositions 2.1, 3.1, 3.2 and 3.3.

Acknowledgements: The authors expressing their deep gratitude to the referee for his / her valuable comments on an earlier version of the article which improve the quality of our article.

References

[1] Baum, L. E. and Katz, M., Convergence rates in the law of large numbers. *Trans. Amer. Math. Soc,* 1965, 120: 108-123

[2] Chen, R., A remark on the tail probability of a distribution. *J. Multivariate Anal,* 1978, 8: 328-333

[3] Cramer, H., Mathematical Methods of Statistics, *Princeton University Press. Princeton, N.J.*, 1946

[4] Erdös, P., On a theorem of Hsu and Robbins. *Ann. Math. Statist,* 1949, 20: 286-291

[5] Erdös, P., Remark on my paper On a theorem of Hsu and Robbins. *Ann. Math. Statist,* 1950, 21: 138

[6] Feller, W., An Introduction to Probability Theory and its Applications, 4^{th} *ed. Wiley, New York*, Vol. 2, 1986

[7] Gut, A. and Spataru, A., Precise asymptotics in the Baum–Katz and Davis law of large numbers *J. Math. Anal. Appl.* 2000a, 248: 233-246

[8] Gut, A. and Spataru, A., Precise Asymptotics in the law of the iterated logarithm, *Ann. Probab,* 2000b, 28: 1870-1883

[9] Gut, A. and Steinebach, J., Convergence rates and precise asymptotics for renewal counting processes and some first passage times, *Technical report, Uppsala University*, 2004

[10] Heyde, C. C., A supplement to the strong law of large numbers,*J. Appl. Probab.* 1975, 12: 173-175

[11] Hsu, P. L. and Robbins, H., Complete convergence and the law of large numbers. *Proc. Nat. Acad. Sci. U.S.A,* 1947, 33: 25-31

[12] Katz, M., The probability in the tail of a distribution, *Ann. Math. Statist,* 1963, 34: 312-318

[13] Li, D.L., Nguyen, B.E and Rosalsky, A., A supplement to precise asymptotics in the law of the iterated logarithm, *J. Math. Anal. Appl,* 2005, 302: 84-96

[14] Mijnheer, Sample Path Properties of Stable Process, Mathematisch, Amsterdam, 1975

[15] Spataru, A., Precise asymptotics in Spitzer's law of large numbers, *J. Theoret. Probab.* 1999, 12: 811-819

[16] Spitzer, F., A combinatorial lemma and its applications to probability theory, *Trans. Amer. Math. Soc,* 1956, 82: 323-339

[17] Wichura, M.J., Functional laws of the iterated logarithm for the partial sums of i.i.d. random variables in the domain of attraction of a completely asymmetric stable law, *Ann. Probab,* 1974, 2: 1108-1138