Robust RLS Wiener Fixed-Interval Smoother in Linear Discrete-Time Stochastic Systems with Uncertain Parameters

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Abstract

This paper proposes the robust RLS Wiener filter and fixed-interval smoothing algorithms based on the innovation approach. As a result, the robust RLS Wiener filtering algorithm is same as the existing robust RLS Wiener filtering algorithm. The estimation accuracy of the fixed-interval smoother is compared with the robust RLS Wiener filter and the following fixed-interval smoothers. In the proposed robust RLS Wiener fixed-interval smoother, the case, where the observed value is replaced with the robust filtering estimate of the signal, is also simulated. (1) The RLS Wiener fixed-interval smoother in which the filtering estimate of the state is replaced with the robust RLS Wiener filtering estimate. (2) The RTS (Rauch-Tung-Strieber) fixed-interval smoother in which the filtering estimate of the state is replaced with the robust RLS Wiener filtering estimate. (3) The H_∞ RLS Wiener fixed-interval smoother and the H_∞ RLS Wiener filter. (4) The RLS Wiener fixed-interval smoother in which the filtering estimate of the state is replaced with the robust RLS Wiener filtering estimate and the observed value is replaced with the robust RLS Wiener filtering estimate of the signal. From the simulation results, the most feasible estimation technique for the fixed-interval smoothing estimate is the RLS Wiener fixed-interval smoother. Here, the robust filtering estimate is used and the observed value is replaced with the robust filtering estimate.

Keywords: Robust RLS Wiener filter; fixed-interval smoother; discrete-time stochastic systems; uncertain parameters; autoregressive model

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1. Introduction

The robust estimation problem has been studied extensively over the last three decades for the systems with unknown parameters [1]- [5]. For example, in Zhu et al. [6] and Yang et al. [7], the robust Kalman filter is proposed for the systems with norm-bounded parameter uncertainties in both the state and output matrices in linear discrete-time stochastic systems. In Duane et al. [8], the robust filter is devised for the systems with polytypic uncertainty. In Wang et al. [9], an adaptive robust Kalman filtering algorithm is proposed in linear time-varying systems with stochastic parametric uncertainties. The minimization of the mean-square error is reduced to linear matrix inequalities (LMIs).

Along with the RLS Wiener estimation technique with the invariant imbedding method, Nakamori [10] proposed the robust recursive least-squares (RLS) Wiener filter and fixed-point smoother in linear discrete-time systems with uncertain parameters in the system and observation matrices. It is a characteristic of the robust estimators that the degraded signal process is fitted to the finite order autoregressive (AR) model. The estimation accuracy of the robust RLS Wiener filter [10] is superior in estimation accuracy to the robust Kalman filter by Zhu et al. [6]. Nakamori also proposed the robust RLS Wiener FIR filter [11]. This paper, from the point to improve the estimation accuracy, examines to develop the robust RLS Wiener filter and fixed-interval smoother based on the innovation theory. As a result, the robust RLS Wiener filter, derived in this paper, is same as the robust RLS Wiener filter [10]. The robust RLS Wiener filter [10] and fixed-interval smoother in this paper use the information. (1) The covariance function of the state for the degraded signal. (2) The cross-covariance function of the state for the signal with the state for the degraded signal. (3) The observation matrices for the signal and the degraded signal. (4) The system matrices for the signal and the degraded signal. (5) The variance of the white observation noise. The observation matrix and the system matrix for the degraded signal are obtained by fitting the degraded signal to the finite-order AR model.

A numerical simulation example compares the estimation accuracy of the proposed robust RLS Wiener fixed-interval smoother with the robust RLS Wiener filter [10], the RLS Wiener fixed-interval smoother [12], the RTS (Rauch-Tung-Strieber) fixed-interval smoother [13], [14], the RLS Wiener H_∞ filter [15] and the RLS Wiener H_∞ fixed-interval smoother [16]. In the RLS Wiener fixed-interval smoother [12] and the RTS fixed-interval smoother, the filtering estimate by the robust RLS Wiener filter [10] is used. In addition, in the RLS Wiener fixed-interval smoother [12] and the proposed robust RLS Wiener fixed-interval smoother, the case, where the observed value is replaced with the robust filtering estimate of the signal, is also simulated.

2. Degraded system including uncertain parameters and flawless state-space model

Let an m-dimensional observation equation and an n-dimensional state equation be described by

$$\begin{aligned} &\breve{y}(k) = \breve{z}(k) + v(k), \breve{z}(k) = \bar{H}(k)\bar{x}(k), \bar{H}(k) = H + \Delta H(k),\\ &\bar{x}(k+1) = \bar{\Phi}(k)\bar{x}(k) + \Gamma w(k), \bar{\Phi}(k) = \Phi + \Delta\Phi(k),\\ &E[v(k)v^T(s)] = R\delta_K(k-s), E[w(k)w^T(s)] = Q\delta_K(k-s) \end{aligned} \tag{1}$$

in linear discrete-time stochastic systems with uncertain parameters [10]. $\Delta H(k)$ and $\Delta\Phi(k)$ represent the perturbations including uncertain parameters. Here, $v(k)$ is the white observation noise with the variance R. $w(k)$ is the white input noise with the variance Q. Their auto-covariance functions are given in (1) with the Kronecker delta function $\delta_K(k-s)$. The state equation, which generates $\bar{x}(k+1)$, contains the uncertain quantity $\Delta\Phi(k)$ in the system matrix $\bar{\Phi}(k)$. In addition, in the observation equation the observation matrix $\bar{H}(k)$ contains the uncertain quantity $\Delta H(k)$. Hence, the degraded signal $\breve{z}(k)$ in (1) has a deviation from the nominal signal $z(k)$ along with the flawless state-space model (2), which does not have any uncertain quantities. In (1), as the sum of the degraded signal $\breve{z}(k)$ and the observation noise $v(k)$, the observed value $\breve{y}(k)$ is given. The flawless state-space model without any uncertain quantities $\Delta H(k)$ and $\Delta\Phi(k)$ in (1) is written as

$$y(k) = z(k) + v(k), z(k) = Hx(k),$$
$$x(k+1) = \Phi x(k) + \Gamma w(k). \tag{2}$$

In (2), $z(k)$ represents the signal to be estimated. H denotes an m by n observation matrix, $x(k)$ is the state and $v(k)$ is the white observation noise with the auto-covariance function given in (1). The auto-covariance function of the input noise $w(k)$ is also given in (1). It is assumed that the signal and the observation noise are zero-mean and mutually independent stochastic processes. The purpose of this paper is to design the robust RLS Wiener fixed-interval smoother in estimating the signal $z(k)$ with the observed value $\breve{y}(k)$ without using any information concerning the uncertain quantities $\Delta\Phi(k)$ and $\Delta H(k)$.

As in [10] let the degraded signal $\breve{z}(k)$ be represented by the N −th order AR model of

$$\breve{z}(k) = -a_1\breve{z}(k-1) - a_2\breve{z}(k-2)\cdots - a_N\breve{z}(k-N) + \breve{e}(k),$$
$$E[\breve{e}(k)\breve{e}^T(s)] = \breve{Q}\delta_K(k-s). \tag{3}$$

By introducing the state $\breve{x}(k)$ in (4), $\breve{z}(k)$ is expressed as

$$\breve{z}(k) = \breve{H}\breve{x}(k),$$
$$\breve{x}(k) = \begin{bmatrix} \breve{x}_1(k) \\ \breve{x}_2(k) \\ \vdots \\ \breve{x}_{N-1}(k) \\ \breve{x}_N(k) \end{bmatrix} = \begin{bmatrix} \breve{z}(k) \\ \breve{z}(k+1) \\ \vdots \\ \breve{z}(k+N-2) \\ \breve{z}(k+N-1) \end{bmatrix}, \tag{4}$$
$$\breve{H} = [I_{m\times m} \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0].$$

From (3) and (4) the state equation for the state $\breve{x}(k)$ is given by

$$\begin{bmatrix} \breve{x}_1(k+1) \\ \breve{x}_2(k+1) \\ \vdots \\ \breve{x}_{N-1}(k+1) \\ \breve{x}_N(k+1) \end{bmatrix} = \begin{bmatrix} 0 & I_{m\times m} & 0 & \cdots & 0 \\ 0 & 0 & I_{m\times m} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_{m\times m} \\ -\breve{a}_N & -\breve{a}_{N-1} & -\breve{a}_{N-2} & \cdots & -\breve{a}_1 \end{bmatrix} \begin{bmatrix} \breve{x}_1(k) \\ \breve{x}_2(k) \\ \vdots \\ \breve{x}_{N-1}(k) \\ \breve{x}_N(k) \end{bmatrix}$$
$$+ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I_{m\times m} \end{bmatrix} \zeta(k), \zeta(k) = \breve{e}(k+N), E[\zeta(k)\zeta^T(s)] = \breve{Q}\delta_K(k-s). \tag{5}$$

Let $\breve{K}(k,s) = \breve{K}(k-s)$ represent the auto-covariance function of the state $\breve{x}(k)$ in wide-sense stationary stochastic systems [17]. Hence, $\breve{K}(k,s)$ has the form of

$$\breve{K}(k,s) = \begin{cases} A(k)B^T(s), 0 \le s \le k, \\ B(k)A^T(s), 0 \le k \le s, \end{cases}$$
$$A(k) = \breve{\Phi}^k, B^T(s) = \breve{\Phi}^{-s}\breve{K}(s,s). \tag{6}$$

Here, $\breve{\Phi}$ represents the system matrix for the state $\breve{x}(k)$. The system matrix $\breve{\Phi}$ in the state equation (5) is given by

$$\breve{\Phi}=\begin{bmatrix} 0 & I_{m\times m} & 0 & \cdots & 0 \\ 0 & 0 & I_{m\times m} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_{m\times m} \\ -\breve{a}_N & -\breve{a}_{N-1} & -\breve{a}_{N-2} & \cdots & -\breve{a}_1 \end{bmatrix}. \quad (7)$$

Also, by putting $K_{\breve{z}}(k,s)=K_{\breve{z}}(k-s)=E[\breve{z}(k)\breve{z}^T(s)]$, the auto-variance function $\breve{K}(k,k)$ of the state $\breve{x}(k)$ is described by

$$\begin{aligned} \breve{K}(k,k) &= E\left[\begin{bmatrix} \breve{z}(k) \\ \breve{z}(k+1) \\ \vdots \\ \breve{z}(k+N-2) \\ \breve{z}(k+N-1) \end{bmatrix} \times [\breve{z}^T(k) \quad \breve{z}^T(k+1) \quad \cdots \quad \breve{z}^T(k+N-2) \quad \breve{z}^T(k+N-1)]\right] \\ &= \begin{bmatrix} K_{\breve{z}}(0) & K_{\breve{z}}(-1) & \cdots & K_{\breve{z}}(-N+2) & K_{\breve{z}}(-N+1) \\ K_{\breve{z}}(1) & K_{\breve{z}}(0) & \cdots & K_{\breve{z}}(-N+3) & K_{\breve{z}}(-N+2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_{\breve{z}}(N-2) & K_{\breve{z}}(N-3) & \cdots & K_{\breve{z}}(0) & K_{\breve{z}}(-1) \\ K_{\breve{z}}(N-1) & K_{\breve{z}}(N-2) & \cdots & K_{\breve{z}}(1) & K_{\breve{z}}(0) \end{bmatrix}. \end{aligned} \quad (8)$$

By using the expression of $K_{\breve{z}}(k-s)$, the Yule-Walker equation for the AR parameters is formulated as

$$\begin{aligned} \widehat{K}(k,k)\begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_{N-1}^T \\ a_N^T \end{bmatrix} &= -\begin{bmatrix} K_{\breve{z}}^T(1) \\ K_{\breve{z}}^T(2) \\ \vdots \\ K_{\breve{z}}^T(N-1) \\ K_{\breve{z}}^T(N) \end{bmatrix}, \\ \widehat{K}(k,k) &= \begin{bmatrix} K_{\breve{z}}(0) & K_{\breve{z}}(1) & \cdots & K_{\breve{z}}(N-2) & K_{\breve{z}}(N-1) \\ K_{\breve{z}}^T(1) & K_{\breve{z}}(0) & \cdots & K_{\breve{z}}(N-3) & K_{\breve{z}}(N-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_{\breve{z}}^T(N-2) & K_{\breve{z}}^T(N-3) & \cdots & K_{\breve{z}}(0) & K_{\breve{z}}(1) \\ K_{\breve{z}}^T(N-1) & K_{\breve{z}}^T(N-2) & \cdots & K_{\breve{z}}^T(1) & K_{\breve{z}}(0) \end{bmatrix}. \end{aligned} \quad (9)$$

3. Robust RLS Wiener fixed-interval smoothing problem based on innovation approach

Let the fixed-interval smoothing estimate $\hat{x}(k,L)$ of the state $x(k)$ be given by

$$\hat{x}(k,L)=\sum_{i=1}^{L} g\,(k,i)\breve{v}(i) \quad (10)$$

in terms of the innovation process $\breve{v}(i)=\breve{y}(k)-\breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1), 1\le i\le L$. Here, $\hat{\breve{x}}(k-1,k-1)$ represents the filtering estimate of the state $\breve{x}(k-1)$. $g(k,i)$ donotes a time-varying impulse response function and L is the fixed interval. Let us consider the fixed-interval smoothing problem, which minimizes the mean-square value (MSV)

$$J = E[||x(k) - \hat{x}(k,L)||^2] \tag{11}$$

of the fixed-interval smoothing error $x(k) - \hat{x}(k,L)$. From an orthogonal projection lemma [17],

$$x(k) - \sum_{i=1}^{L} g\,(k,i)\breve{v}(i) \perp \breve{v}(s), 1 \le s \le L, \tag{12}$$

the impulse response function satisfies the Wiener-Hop equation

$$E[x(k)\breve{v}^T(s)] = \sum_{i=1}^{L} g\,(k,i)E[\breve{v}(i)\breve{v}^T(s)]. \tag{13}$$

Here, '⊥' denotes the notation of the orthogonality. Let, in terms of $g_0(s,i)$, the filtering estimate $\hat{\breve{x}}(s,s)$ of the state $\breve{x}(s)$ be expressed by

$$\hat{\breve{x}}(s,s) = \sum_{i=1}^{s} g_0\,(s,i)\breve{v}(i). \tag{14}$$

Substituting (1) and (14) into (13), from (4), and using $E[x(k)\breve{y}^T(s)] = K_{x\breve{z}}(k,s) = K_{x\breve{x}}(k,s)\breve{H}^T$ and introducing the variance of the innovation process $\Pi(s) = E[\breve{v}(s)\breve{v}^T(s)]$, we obtain

$$g(k,s)\Pi(s) = K_{x\breve{x}}(k,s)\breve{H}^T - \sum_{i=1}^{s-1} g\,(k,i)\Pi(i)g_0^T(s,i)\breve{H}^T. \tag{15}$$

Here, $K_{x\breve{x}}(k,s)$ represent the cross-covariance function of the state $x(k)$ with the degraded signal $\breve{x}(s)$, $E[x(k)\breve{x}^T(s)]$. Let $K_{x\breve{x}}(k,s)$ be represented by

$$K_{x\breve{x}}(k,s) = \begin{cases} \alpha(k)\beta^T(s), 0 \le s \le k, \\ \gamma(k)\delta^T(s), 0 \le k \le s, \end{cases}$$
$$\alpha(k) = \Phi^k, \beta^T(s) = \Phi^{-s}K_{x\breve{x}}(s,s), \gamma(k) = K_{x\breve{x}}(k,k)(\breve{\Phi}^T)^{-k}, \delta^T(s) = (\breve{\Phi}^T)^s. \tag{16}$$

Here, Φ denotes the system matrix for the state $x(k)$. Similarly, to (15), the optimal impulse response function $g_0(k,s)$ satisfies

$$g_0(k,s)\Pi(s) = \breve{K}(k,s)\breve{H}^T - \sum_{i=1}^{s-1} g\,(k,i)\Pi(i)g_0^T(s,i)\breve{H}^T. \tag{17}$$

4 Autoregressive models for signal process

Let the signal process be expressed in terms of the J −th order AR model

$$z(k) = -a_1 z(k-1) - a_2 z(k-2) - \cdots - a_J z(k-J) + w(k). \tag{18}$$

It is seen that the observation matrix H and the state equation for the state $x(k)$ in (2) are given by

$$H = [I_{m\times m} \quad 0 \quad 0 \quad \cdots \quad 0], \tag{19}$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{J-1}(k+1) \\ x_J(k+1) \end{bmatrix} = \begin{bmatrix} 0 & I_{m\times m} & 0 & \cdots & 0 \\ 0 & 0 & I_{m\times m} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_{m\times m} \\ -a_J & -a_{J-1} & -a_{J-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{J-1}(k) \\ x_J(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I_{m\times m} \end{bmatrix} w(k), E[w(k)w^T(s)] = Q\delta_K(k-s), \Gamma = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I_{m\times m} \end{bmatrix}. \tag{20}$$

5. Robust RLS Wiener fixed-interval smoothing and filtering algorithms

Under the preliminary formulation on the robust fixed-interval smoothing problem for the signal $z(k)$ and the uncertain and certain systems in sections 2, 3, and 4, Theorem 1 presents the robust RLS Wiener filtering and fixed-interval smoothing algorithms. Here, the robust RLS Wiener filtering algorithm is same as that in [10].

Theorem 1 Let the state equation and the observation equation, including the uncertain quantities $\Delta\Phi$ and ΔH respectively, be given by (1). Let Φ and H represent the system and observation matrices respectively for the signal $z(k)$. Let $\breve{\Phi}$ and $\breve{H}$ represent the system and observation matrices respectively for the degraded signal $\breve{z}(k)$, which is fitted to the AR model (3) of the order N. Let the variance $\breve{K}(k,k)$ of the state $\breve{x}(k)$ for the degraded signal $\breve{z}(k)$ and the cross-variance function $K_{x\breve{x}}(k,k)$ of the state $x(k)$ for the signal $z(k)$ with the state $\breve{x}(k)$ for the degraded signal $\breve{z}(k)$ Be given. Let the variance of the white observation noise $v(k)$ be R. Then, the robust RLS Wiener fixed-interval smoothing and filtering algorithms for the signal $z(k)$ consist of (21) -(31) in linear discrete-time stochastic systems.

Filtering estimate of the signal $z(k)$: $\hat{z}(k,k)$

$$\hat{z}(k,k) = H\hat{x}(k,k) \tag{21}$$

Filtering estimate of the state $x(k)$: $\hat{x}(k,k)$

$$\hat{x}(k,k) = \Phi\hat{x}(k-1,k-1) + g(k,k)(\breve{y}(k) - \breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1)), \quad \hat{x}(0,0) = 0 \tag{22}$$

Filter gain for the filtering estimate $\hat{x}(k,k)$ in (22): $g(k,k)$

$$\begin{aligned} g(k,k) &= [K_{x\breve{z}}(k,k) - \Phi S(k-1)\breve{\Phi}^T\breve{H}^T] \\ &\times \{R + \breve{H}[\breve{K}(k,k) - \breve{\Phi}S_0(L-1)\breve{\Phi}^T]\breve{H}^T\}^{-1}, \\ K_{x\breve{z}}(k,k) &= K_{x\breve{x}}(k,k)\breve{H}^T \end{aligned} \tag{23}$$

Filtering estimate of $\breve{x}(k)$: $\hat{\breve{x}}(k,k)$

$$\hat{\breve{x}}(k,k) = \breve{\Phi}\hat{\breve{x}}(k-1,k-1) + g_0(k,k)(\breve{y}(k) - \breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1)),$$
$$\hat{\breve{x}}(0,0) = 0 \quad (24)$$

Filter gain for the filtering estimate $\hat{\breve{x}}(k,k)$ in (24): $g_0(k,k)$

$$g_0(k,k) = [\breve{K}(k,k)\breve{H}^T - \breve{\Phi}S_0(k-1)\breve{\Phi}^T\breve{H}^T]$$
$$\times \{R + \breve{H}[\breve{K}(k,k) - \breve{\Phi}S_0(L-1)\breve{\Phi}^T]\breve{H}^T\}^{-1} \quad (25)$$

Auto-variance function of the filtering estimate $\hat{\breve{x}}(k,k)$: $S_0(k) = E[\hat{\breve{x}}(k,k)\hat{\breve{x}}^T(k,k)]$

$$S_0(k) = \breve{\Phi}S_0(k-1)\breve{\Phi}^T + g_0(k,k)\breve{H}[\breve{K}(k,k) - \breve{\Phi}S_0(k-1)\breve{\Phi}^T],$$
$$S_0(0) = 0 \quad (26)$$

Cross-variance function of $\hat{x}(k,k)$ with $\hat{\breve{x}}(k,k)$: $S(k) = E[\hat{x}(k,k)\hat{\breve{x}}^T(k,k)]$

$$S(k) = \Phi S(k-1)\breve{\Phi}^T + g(k,k)\breve{H}[\breve{K}(k,k) - \breve{\Phi}S_0(k-1)\breve{\Phi}^T],$$
$$S(0) = 0 \quad (27)$$

Fixed-interval smoothing estimate of the signal $z(k)$: $\hat{z}(k,L)$

$$\hat{z}(k,L) = H\hat{x}(k,L) \quad (28)$$

Fixed-interval smoothing estimate of the state $x(k)$: $\hat{x}(k,L)$

$$\hat{x}(k,L) = \hat{x}(k,k) + (K_{x\breve{x}}(k,k) - S(k))\breve{q}_1(k+1,L) \quad (29)$$

Backward equation for $\breve{q}_1(k,L)$ from $\breve{q}_1(k+1,L)$:

$$\breve{q}_1(k,L) = \breve{\Phi}^T\breve{q}_1(k+1,L) + \breve{\Phi}^T\breve{H}^T\Pi^{-1}(k)(\breve{y}(k) - \breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1))$$
$$-\breve{\Phi}^T\breve{H}^T g_0^T(k,k)\,\breve{q}_1(k+1,L),\ \breve{q}_1(L+1,L) = 0 \quad (30)$$

Variance of the innovation process $\breve{v}(k)$: $\Pi(k)$

$$\Pi(k) = R + \breve{H}[\breve{K}(k,k) - \breve{\Phi}S_0(L-1)\breve{\Phi}^T]\breve{H}^T \quad (31)$$

Proof of Theorem 1 is deferred to the appendix. The robust RLS Wiener filtering algorithm, proposed in Theorem 1 based on the innovation theory, is same as the robust RLS Wiener filtering algorithm [10].

The necessary conditions for the stability of the robust filtering and fixed-interval smoothing algorithms are as follows.

1. All the real parts in the eigenvalues of the matrix Φ are negative.

2. All the real parts in the eigenvalues of the matrix $\breve{\Phi} - g_0(k,k)\breve{H}\breve{\Phi}$ are negative.

3. $R + \breve{H}[\breve{K}(k,k) - \breve{\Phi}S_0(L-1)\breve{\Phi}^T]\breve{H}^T > 0$

The fixed-interval smoothing error variance $\tilde{P}_z(k,L)$ of the signal $z(k)$ is shown in section 6.

6. Fixed-interval smoothing error variance of the signal

This section discusses on the existence of the fixed-interval smoothing estimate $\hat{z}(k,L)$. The variance $\tilde{P}_z(k,L)$ of the fixed-interval smoothing error $z(k) - \hat{z}(k,L)$ is shown as

$$\tilde{P}_z(k,L) = E[(z(k) - \hat{z}(k,L))(z(k) - \hat{z}(k,L))^T]. \tag{32}$$

(32) might be written as

$$\begin{aligned} &\tilde{P}_z(k,L) = HK_x(k,k)H^T - P_{\hat{z}}(k,L), \\ &P_{\hat{z}}(k,L) = E[\hat{z}(k,L)\hat{z}^T(k,L)] \\ &= HE[\hat{x}(k,L)\hat{x}^T(k,L)]H^T \\ &= HP_{\hat{x}}(k,L)H^T, P_{\hat{x}}(k,L) = E[\hat{x}(k,L)\hat{x}^T(k,L)]. \end{aligned} \tag{33}$$

Here, $K_x(k,k)$ represents the variance of the state $x(k)$. From (29), (A-38), (A-39), (A-51) and (A-52), the auto-variance function of the fixed-interval estimate $\hat{x}(k,L)$, $P_{\hat{x}}(k,L)$, is described by

$$\begin{aligned} &P_{\hat{x}}(k,L) = P_{\hat{x}}(k,k) + (K_{x\breve{x}}(k,k-S(k))E[\breve{q}_1(k+1,L)\breve{q}_1^T(k+1,L)] \\ &\times (K_{x\breve{x}}(k,k-S(k))^T \\ &= P_{\hat{x}}(k,k) + (K_{x\breve{x}}(k,k-S(k))\bar{q}(k+1,L)(K_{x\breve{x}}(k,k-S(k))^T, \\ &\bar{q}(k+1,L) = E[\breve{q}_1(k+1,L)\breve{q}_1^T(k+1,L)]. \end{aligned} \tag{34}$$

Here, $\bar{q}(k,L)$ is calculated in the backward of time k by

$$\begin{aligned} &\bar{q}(k,L) = \bar{\Phi}(k)\bar{q}(k+1,L)\bar{\Phi}(k)^T + \bar{G}(\mathrm{k})\Pi(\mathrm{k})\bar{G}(k)^T, \\ &\bar{q}(L+1,L) = 0, \\ &\bar{\Phi}(k) = (\breve{\Phi}^T - \breve{\Phi}^T\breve{H}^T g_0^T(k,k)), \bar{G}(k) = \breve{\Phi}^T\breve{H}^T\Pi^{-1}(k). \end{aligned} \tag{35}$$

(34) shows that the relationship $0 \le K_x(k,k) - P_{\hat{x}}(k,L) \le K_x(k,k) - P_{\hat{x}}(k,k)$ holds. This means that the MSV of the fixed-interval smoothing error $x(k) - \hat{x}(k,L)$ is less than the MSV of the filtering error $x(k) - \hat{x}(k,k)$. Since $\tilde{P}_z(k,L) \ge 0$ and the variance $P_{\hat{z}}(k,L)$ of the fixed-interval smoothing estimate $\hat{z}(k,L)$ is also positive semidefinite

$$P_{\hat{z}}(k,L) \ge 0, \tag{36}$$

it follows that

$$0 \le P_{\tilde{z}}(k,L) \le HK_x(k,k)H^T. \tag{37}$$

(37) shows that the variance of the fixed-interval smoothing error is upper bounded by the variance of the signal and lower bounded by the zero matrix. This validates the existence of the robust fixed-interval smoothing estimate $\hat{z}(k,L)$ of the signal $z(k)$. In section 7 the estimation accuracy of the proposed robust RLS Wiener fixed-interval smoother is compared with the robust RLS Wiener filter [10], the RLS Wiener fixed-interval smoother [12], the RTS fixed-interval smoother [13], [14], the RLS Wiener H_∞ filter [15] and the H_∞ RLS Wiener fixed-interval smoother [16] from the numerical aspect.

7. A numerical simulation example

In linear discrete-time stochastic systems, let a scalar observation equation and a state equation for $x(k)$ be given by

$$\begin{aligned} &y(k) = z(k) + v(k), z(k) = Hx(k), H = [1 \quad 0],\\ &x(k+1) = \Phi x(k) + \Gamma w(k), x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix},\\ &\Phi = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}, \Gamma = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, a_1 = -0.1, a_2 = -0.8\\ &E[v(k)v(s)] = R\delta_K(k-s), E[w(k)w(s)] = Q\delta_K(k-s), Q = 0.5^2. \end{aligned} \tag{38}$$

The observation noise $v(k)$ is a zero-mean white noise process. From (1) let the state-space model containing the uncertain quantities be described by

$$\begin{aligned} &\breve{y}(k) = \breve{z}(k) + v(k), \breve{z}(k) = \overline{H}(k)\bar{x}(k), \bar{x}(k) = \begin{bmatrix} \bar{x}_1(k) \\ \bar{x}_2(k) \end{bmatrix},\\ &\bar{x}(k+1) = \overline{\Phi}(k)\bar{x}(k) + \Gamma w(k), \overline{\Phi}(k) = \Phi + \Delta\Phi(k),\ \Gamma = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\\ &\Delta\Phi(k) = \begin{bmatrix} 0 & 0 \\ \Delta_2(k) & \Delta_1(k) \end{bmatrix}, \Delta_1(k) = -0.1, \Delta_2(k) = 0.1,\\ &\overline{H}(k) = H + \Delta\mathrm{H}(k), \Delta\mathrm{H}(k) = [0 \quad \Delta_3(k)], \Delta_3(k) = 0.1. \end{aligned} \tag{39}$$

It should be noted that the uncertain quantities $\Delta\Phi(k)$ and $\Delta\mathrm{H}(k)$ are unknown. It is a task to estimate the signal $z(k)$ recursively in terms of the observed value $\breve{y}(k)$, which is given as the sum of the degraded signal $\breve{z}(k)$ and the observation noise $v(k)$. Let $\breve{z}(k)$ be fitted to the 10-th order AR model, expressed by

$$\begin{aligned} &\breve{z}(k) = -\breve{a}_1\breve{z}(k-1) - \breve{a}_2\breve{z}(k-2) - \cdots - \breve{a}_N\breve{z}(k-N) + \breve{e}(k),\\ &E[\breve{e}(k)\breve{e}(s)] = \breve{Q}\delta_K(k-s), N = 10. \end{aligned} \tag{40}$$

In this example, the state equation for $\breve{x}(k)$, given by (5), corresponds to the case of $m = 1$. The relationship $\breve{K}(k,s) = \breve{K}(k-s)$ represents the auto-covariance function of the state $\breve{x}(k)$ in wide-sense stationary stochastic systems. $\breve{K}(k,s)$ is expressed in the form of the semi-degenerate function (6). $\breve{\Phi}$ in (7) represents the system matrix for the state $\breve{x}(k)$. Also, from $K_{\breve{z}}(k-s) = K_{\breve{z}}(s-k) = E[\breve{z}(k)\breve{z}(s)]$ for the scalar degraded signal $\breve{z}(k)$, the auto-variance function $\breve{K}(k,k)$ of the state $\breve{x}(k)$ is expressed as

$$\breve{K}(k,k) = E\left[\begin{bmatrix} \breve{z}(k) \\ \breve{z}(k+1) \\ \vdots \\ \breve{z}(k+N-2) \\ \breve{z}(k+N-1) \end{bmatrix} \times [\breve{z}(k) \quad \breve{z}(k+1) \quad \cdots \quad \breve{z}(k+N-2) \quad \breve{z}(k+N-1)]\right]$$
$$= \begin{bmatrix} K_{\breve{z}}(0) & K_{\breve{z}}(1) & \cdots & K_{\breve{z}}(N-2) & K_{\breve{z}}(N-1) \\ K_{\breve{z}}(1) & K_{\breve{z}}(0) & \cdots & K_{\breve{z}}(N-3) & K_{\breve{z}}(N-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_{\breve{z}}(N-2) & K_{\breve{z}}(N-3) & \cdots & K_{\breve{z}}(0) & K_{\breve{z}}(1) \\ K_{\breve{z}}(N-1) & K_{\breve{z}}(N-2) & \cdots & K_{\breve{z}}(1) & K_{\breve{z}}(0) \end{bmatrix}. \tag{41}$$

Let $K_{z\breve{z}}(k,s) = E[z(k)\breve{z}(s)]$ represent the cross-covariance function between the signal $z(k)$ and the degraded signal $\breve{z}(s)$. From (4) and (38), the cross-covariance function $K_{x\breve{x}}(k,s)$ is expressed as

$$K_{x\breve{x}}(k,s) = \Phi^{k-s}K_{x\breve{x}}(s,s), 0 \le s \le k,$$
$$K_{x\breve{x}}(k,k) = E\left[\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} [\breve{z}(k) \quad \breve{z}(k+1) \quad \cdots \quad \breve{z}(k+N-2) \quad \breve{z}(k+N-1)]\right]$$
$$= \begin{bmatrix} E[x_1(k)\breve{z}(k)] & E[x_1(k)\breve{z}(k+1)] \\ E[x_2(k)\breve{z}(k)] & E[x_2(k)\breve{z}(k+1)] \end{bmatrix}$$
$$\begin{matrix} \cdots & E[x_1(k)\breve{z}(k+N-2)] & E[x_1(k)\breve{z}(k+N-1)] \\ \cdots & E[x_2(k)\breve{z}(k+N-2)] & E[x_2(k)\breve{z}(k+N-1)] \end{matrix}$$
$$= \begin{bmatrix} E[z(k)\breve{z}(k)] & E[z(k)\breve{z}(k+1)] \\ E[z(k+1)\breve{z}(k)] & E[z(k+1)\breve{z}(k+1)] \end{bmatrix} \tag{42}$$
$$\begin{matrix} \cdots & E[z(k)\breve{z}(k+N-2)] & E[z(k)\breve{z}(k+N-1)] \\ \cdots & E[z(k+1)\breve{z}(k+N-2)] & [z(k+1)\breve{z}(k+N-1)] \end{matrix}$$
$$= \begin{bmatrix} K_{z\breve{z}}(k,k) & K_{z\breve{z}}(k,k+1) \\ K_{z\breve{z}}(k+1,k) & K_{z\breve{z}}(k+1,k+1) \end{bmatrix}$$
$$\begin{matrix} \cdots & K_{z\breve{z}}(k,k+N-2) & K_{z\breve{z}}(k,k+N-1) \\ \cdots & K_{z\breve{z}}(k+1,k+N-2) & K_{z\breve{z}}(k+1,k+N-1) \end{matrix}.$$

The AR parameters $\breve{a}_1, \breve{a}_2, \cdots, \breve{a}_{N-1}, \breve{a}_N$ in (40) are calculated by the Yule-Walker equation.

$$\begin{bmatrix} K_{\breve{z}}(0) & K_{\breve{z}}(1) & \cdots & K_{\breve{z}}(N-2) & K_{\breve{z}}(N-1) \\ K_{\breve{z}}(1) & K_{\breve{z}}(0) & \cdots & K_{\breve{z}}(N-3) & K_{\breve{z}}(N-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_{\breve{z}}(N-2) & K_{\breve{z}}(N-3) & \cdots & K_{\breve{z}}(0) & K_{\breve{z}}(1) \\ K_{\breve{z}}(N-1) & K_{\breve{z}}(N-2) & \cdots & K_{\breve{z}}(1) & K_{\breve{z}}(0) \end{bmatrix} \begin{bmatrix} \breve{a}_1 \\ \breve{a}_2 \\ \vdots \\ \breve{a}_{N-1} \\ \breve{a}_N \end{bmatrix} = \begin{bmatrix} -K_{\breve{z}}(1) \\ -K_{\breve{z}}(2) \\ \vdots \\ -K_{\breve{z}}(N-1) \\ -K_{\breve{z}}(N) \end{bmatrix}$$

Substituting H, $\breve{H}$, Φ, $\breve{\Phi}$, $K_{x\breve{x}}(k,k)$, $\breve{K}(k,k)$ and R into the robust RLS Wiener fixed-interval smoothing and filtering algorithms of Theorem 1, the fixed-interval smoothing and filtering estimates are calculated recursively.

In evaluating $\breve{\Phi}$ in (7) for $m=1$, $\breve{K}(k,k)$ in (41) and $K_{x\breve{x}}(k,k)$ in (42), 2,000 number of the signal and degraded signal data are used.

Fig.1 illustrates the signal $z(k)$ and its degraded signal $\breve{z}(k)$ vs. k. In comparison with the signal, the degraded signal is influenced by the uncertain parameters in the system matrix $\overline{\Phi}(k)$ and the observation matrix $\overline{H}(k)$ in (39). Fig.2 illustrates the signal, the RLS Wiener filtering estimate $\hat{z}(k,k)$, the RLS Wiener fixed-interval smoothing estimate $\hat{z}(k,L)$, for the fixed interval $\mathrm{L}=200$, by Nakamori et al. [12], vs. k when the white Gaussian observation noise obeys $\mathrm{N}(0,0.3^2)$. In the calculation of the fixed-interval smoothing estimate, the filtering estimate by the robust RLS Wiener filter [10] is used. Fig.3 illustrates the mean-square values (MSVs) of the filtering errors $z(k)-\hat{z}(k,k)$ by the robust RLS Wiener filter [10] and the fixed-interval smoothing errors $z(k)-\hat{z}(k,L)$ by the RTS fixed-interval smoother [13], [14] vs. fixed interval L, $50\le L\le 250$, $1\le k\le L$, for the white Gaussian observation noises $\mathrm{N}(0,0.1^2)$, $\mathrm{N}(0,0.3^2)$, $\mathrm{N}(0,0.5^2)$ and $\mathrm{N}(0,0.7^2)$. Here, the RTS fixed-interval smoother uses the filtering estimates calculated by the robust RLS Wiener filter [10]. For the white Gaussian observation noise $\mathrm{N}(0,0.1^2)$, The estimation accuracy of the fixed-interval smoother improves that of the robust filter by a little. For the white Gaussian observation noise $\mathrm{N}(0,0.3^2)$ with $L=50$, the estimation accuracy of the fixed-interval smoother is superior to that of the robust filter [10]. Fig.4 illustrates the MSVs of the filtering and fixed-interval smoothing errors by the RLS Wiener H_∞ filter [15] and the RLS Wiener H_∞ fixed-interval smoother [16] vs. L. Compared with the H_∞ filtering estimate, the estimation accuracy of the H_∞ fixed-interval smoothing estimate is improved for $\mathrm{L}=150,200$ for the observation noise $\mathrm{N}(0,0.5^2)$ and for $\mathrm{L}=100,150$ for the observation noise $\mathrm{N}(0,0.7^2)$. However, from Fig.3 and Fig.4, the estimation accuracy of the RLS Wiener H_∞ filter is inferior to that of the robust RLS Wiener filter. Fig.5 illustrates the MSVs of the filtering and fixed-interval smoothing errors by the robust RLS Wiener filter [10] and the RLS Wiener fixed-interval smoother [12] vs. L. The RLS Wiener fixed-interval smoother is superior in estimation accuracy to the robust RLS Wiener filter particularly for the white Gaussian observation noise $\mathrm{N}(0,0.7^2)$. For the white Gaussian observation noises $\mathrm{N}(0,0.1^2)$, $\mathrm{N}(0,0.3^2)$ and $\mathrm{N}(0,0.5^2)$, The estimation accuracy of the robust RLS Wiener filter is superior to the RLS Wiener fixed-interval smoother. Here, in the calculation of the fixed-interval smoothing estimate, the filtering estimate by the robust RLS Wiener filter [10] is used. Fig.6 illustrates the MSVs of the filtering and fixed-interval smoothing errors by the robust RLS Wiener filter [10] and the robust RLS Wiener fixed-interval smoother of Theorem 1 vs. L. Compared with the robust filtering

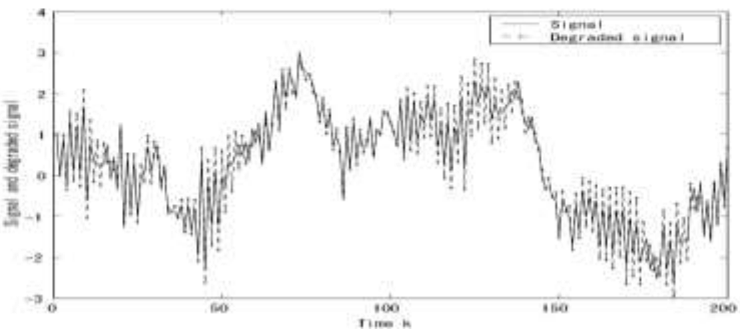

Fig.1 Signal and degraded signal.

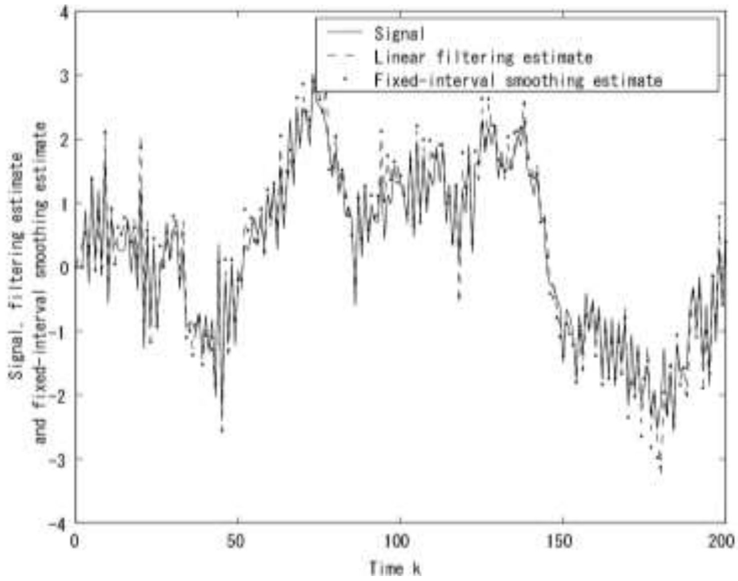

Fig.2 Signal, robust RLS Wiener filtering estimate by Nakamori [10] and fixed-interval smoothing estimate by Nakamori et al. [12] for white Gaussian observation noise $\mathrm{N}(0, 0.3^2)$.

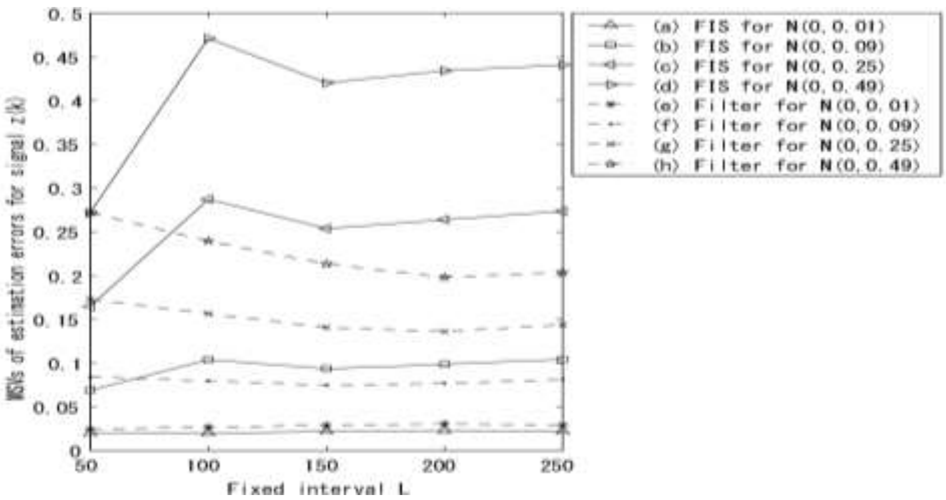

Fig.3 MSVs of filtering and fixed-interval smoothing errors by robust RLS Wiener filter [10] and RTS fixed-interval smoother [13], [14] vs. fixed interval L.

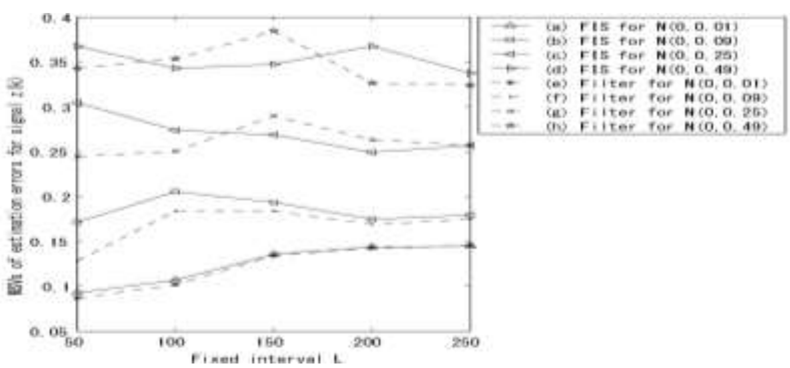

Fig.4 MSVs of filtering and fixed-interval smoothing errors by H_∞ RLS Wiener filter [15] and H_∞ RLS

Wiener fixed-interval smoother [16] vs. fixed interval L.

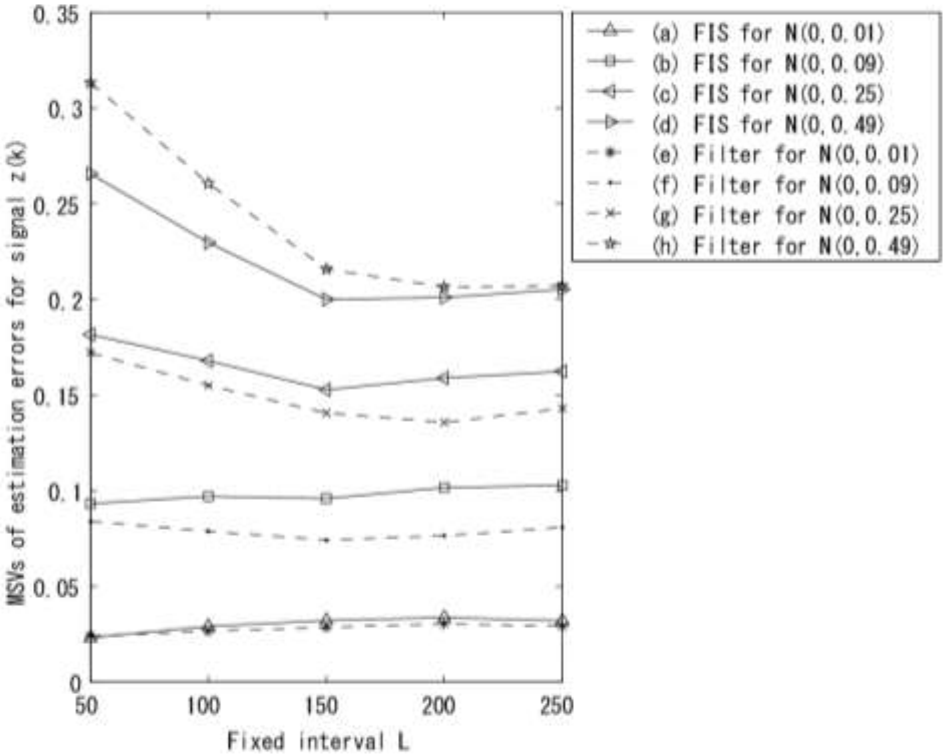

Fig.5 MSVs of filtering and fixed-interval smoothing errors by robust RLS Wiener filter [10] and RLS Wiener fixed-interval smoother [12] vs. fixed interval L.

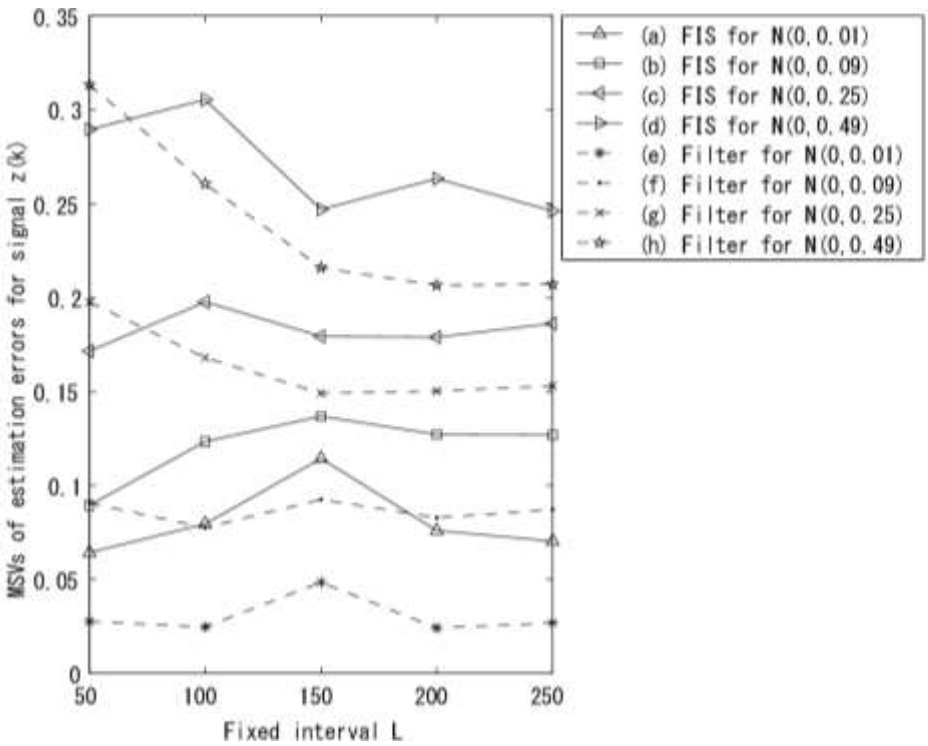

Fig.6 MSVs of filtering and fixed-interval smoothing errors by robust RLS Wiener filter [10] and RLS

Wiener fixed-interval smoother in Theorem 1 vs. fixed interval L.

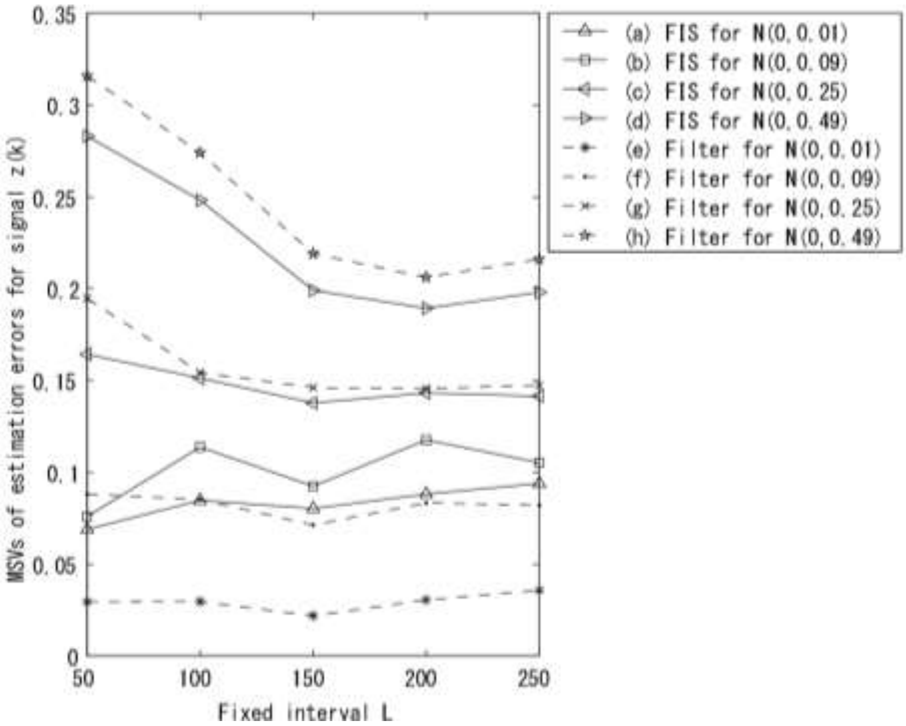

Fig.7 MSVs of filtering and fixed-interval smoothing errors by robust RLS Wiener filter [10] and robust RLS Wiener fixed-interval smoother in Theorem 1 by replacing the observed value with robust filtering estimate vs. fixed interval L.

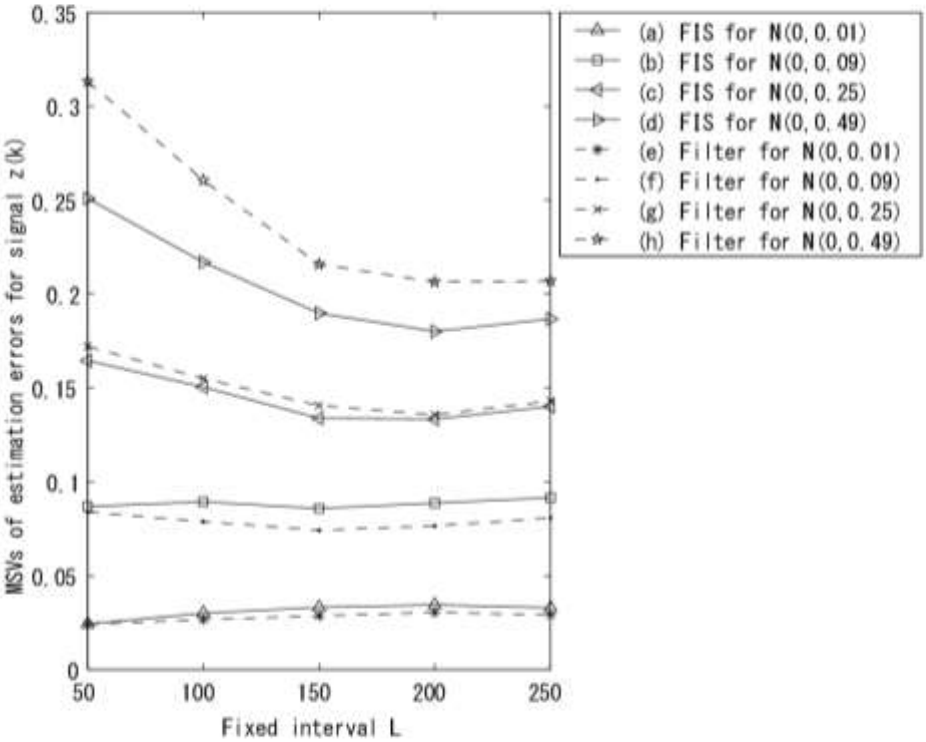

Fig.8 MSVs of the filtering and fixed-interval smoothing errors by the robust RLS Wiener filter [10] and

the RLS Wiener fixed-interval smoother [12] by replacing the filtering estimate of the state with the robust RLS Wiener filtering estimate and the observed value with the robust RLS Wiener filtering estimate vs. fixed interval L.

estimate, the estimation accuracy of the robust fixed-interval smoother is improved for $L = 50$ for the observation noises $N(0, 0.5^2)$ and $N(0, 0.7^2)$. Fig.7 illustrates the MSVs of the filtering and fixed-interval smoothing errors by the robust RLS Wiener filter [10] and the robust RLS Wiener fixed-interval smoother of Theorem 1 by replacing the observed value with the robust filtering estimate vs L. The estimation accuracy of the fixed-interval smoother is superior to the filter for the observation noises $N(0, 0.5^2)$ and $N(0, 0.7^2)$. Compared with the robust fixed-interval smoother of Fig.6, the estimation accuracy of the robust fixed-interval smoother of Fig.7 is superior for the observation noises $N(0, 0.3^2)$, $N(0, 0.5^2)$ and $N(0, 0.7^2)$. Fig.8 illustrates the MSVs of the filtering and fixed-interval smoothing errors by the robust RLS Wiener filter [10] and the RLS Wiener fixed-interval smoother [12] by replacing the filtering estimate of the state with the robust RLS Wiener filtering estimate and the observed value with the robust RLS Wiener filtering estimate vs. L. The estimation accuracy of the fixed-interval smoother is superior to the filter for the observation noises $N(0, 0.5^2)$ and $N(0, 0.7^2)$. Compared with the robust fixed-interval smoother of Fig.7, the estimation accuracy of the fixed-interval smoother of Fig.8 is superior for the observation noises $N(0, 0.1^2)$, $N(0, 0.3^2)$ and $N(0, 0.7^2)$. Here, the MSVs are evaluated by $\sum_{k=1}^{L}(z(k)-\hat{z}(k,k))^2/L$ for the filtering errors and $\sum_{k=1}^{L}(z(k)-\hat{z}(k,L))^2/L$ for the fixed-interval smoothing errors. For the less MSV of the estimation errors, the estimation accuracy becomes better.

From the above simulation results, the most feasible estimation technique for the fixed-interval smoothing estimate is the RLS Wiener fixed-interval smoother [12]. Here, the robust filtering estimate is used and the observed value is replaced with the robust filtering estimate. This result might be conjectured from the following viewpoints. (1) The RLS Wiener filter and fixed-interval smoother [12] are designed based on the precise state-space model. By replacing the degraded observed value with the filtering estimate, calculated by the robust RLS Wiener filter in [10], the estimation accuracy of the RLS Wiener filter and fixed-interval smoother [12] is improved as seen from Fig.5 and Fig.8. (2) The robust RLS Wiener estimators in Theorem 1 are designed based on the state-space model with uncertain parameters. As shown in Fig.7 and Fig.8, for the relatively small variances 0.1^2 and 0.3^2 of the observation noise, the estimation accuracy of the robust RLS Wiener fixed-interval smoother in Theorem 1 is inferior to the RLS Wiener fixed-interval smoother [12]. This degradedness of the robust RLS Wiener fixed-interval smoother might be caused by the imprecise state-space model.

8. Conclusions

This paper has proposed, in Theorem 1, the robust RLS Wiener filter and the robust RLS Wiener fixed-interval smoother based on the innovation theory. As a result, the robust RLS Wiener filtering algorithm is same as the existing robust RLS Wiener filtering algorithm. In the numerical simulation example, the estimation characteristics of the fixed-interval smoother in Theorem 1 are compared with the robust RLS Wiener filter, the

RTS fixed-interval smoother, the RLS Wiener fixed-interval smoother, the H_∞ RLS Wiener filter and the H_∞fixed-interval smoother. From the simulation results, the most feasible estimation technique for the fixed-interval smoothing estimate is the RLS Wiener fixed-interval smoother. Here, the robust filtering estimate is used, and the observed value is replaced with the robust filtering estimate.

Appendix Proof of Theorem 1

From (10) the fixed-interval smoothing estimate $\hat{x}(k,L)$ of the state $x(k)$ is expressed as

$$\hat{x}(k,L) = \sum_{i=1}^{k} g\,(k,i)\breve{v}(i) + \sum_{i=k+1}^{L} g\,(k,i)\breve{v}(i). \qquad \text{(A-1)}$$

The first term on the right-hand side represents the filtering estimate $\hat{x}(k,k)$ of the state $x(k)$. Let us introduce an auxiliary function $J_1(s)$, which satisfies

$$J_1(s)\Pi(\mathrm{s}) = \beta^T(s)\breve{H}^T - \sum_{i=1}^{s-1} J_1(i)\,\Pi(\mathrm{i})g_0^T(s,i)\breve{H}^T. \qquad \text{(A-2)}$$

From (15), (16) and (A-2), $g(k,s)$, $0 \le s \le k$, is given by

$$g(k,s) = \alpha(k)J_1(s). \qquad \text{(A-3)}$$

Let us introduce an auxiliary function $J_3(s)$, which satisfies

$$J_3(s)\Pi(\mathrm{s}) = B^T(s)\breve{H}^T - \sum_{i=1}^{s-1} J_3(i)\,\Pi(\mathrm{i})g_0^T(s,i)\breve{H}^T. \qquad \text{(A-4)}$$

From (6), (17) and (A-4), $g_0(k,s)$ is given by

$$g_0(k,s) = A(k)J_3(s), 0 \le \mathrm{s} \le \mathrm{k}\,. \qquad \text{(A-5)}$$

In (A-2), by putting $s = k$, it follows that

$$J_1(k)\Pi(\mathrm{k}) = \beta^T(k)\breve{H}^T - \sum_{i=1}^{k-1} J_1(i)\,\Pi(\mathrm{i})g_0^T(k,i)\breve{H}^T. \qquad \text{(A-6)}$$

Substituting (A-5) into (A-6) and introducing

$$r_{13}(k) = \sum_{i=1}^{k} J_1(i)\,\Pi(\mathrm{i})J_3^T(i), \qquad \text{(A-7)}$$

we have

$$J_1(k)\Pi(\mathrm{k}) = \beta^T(k)\breve{H}^T - r_{13}(k-1)A^T(k)\breve{H}^T. \qquad \text{(A-8)}$$

Subtracting $r_{13}(k-1)$ from $r_{13}(k)$, we have

$$r_{13}(k) = r_{13}(k-1) + J_1(k)\Pi(\mathrm{k})J_3^T(k), r_{13}(0) = 0. \tag{A-9}$$

From (A-3), the filtering estimate $\hat{z}(k,k)$ of the signal $z(k)$ is given by

$$\hat{z}(k,k) = H\alpha(k)e_1(k), \tag{A-10}$$

where $e_1(k)$ satisfies

$$e_1(k) = \sum_{i=1}^{k} J_1(i)\breve{v}(i). \tag{A-11}$$

Subtracting $e_1(k-1)$ from $e_1(k)$, we have

$$e_1(k) = e_1(k-1) + J_1(k)(\breve{y}(k) - \breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1)), e_1(0) = 0. \tag{A-12}$$

Now, the variance of the innovation process is given by

$$\begin{aligned} \Pi(k) &= E[v(k)v^T(k)] \\ &= \mathrm{E}[\breve{y}(k)\breve{y}^T(k)] - \breve{H}\breve{\Phi}E[\hat{\breve{x}}(k-1,k-1)\hat{\breve{x}}^T(k-1,k-1)]\breve{\Phi}^T\breve{H}^T. \end{aligned} \tag{A-13}$$

From (14) and (A-5), the filtering estimate $\hat{\breve{x}}(k,k)$ of the state $\breve{x}(k)$ is expressed as

$$\begin{aligned} \hat{\breve{x}}(k,k) &= \sum_{i=1}^{k} g_0(k,i)\breve{v}(i) \\ &= A(k)e_3(k), \end{aligned} \tag{A-14}$$

where $e_3(k)$ is given by

$$e_3(k) = \textstyle\sum_{i=1}^{k} J_3(i)\breve{v}(i). \tag{A-15}$$

Subtracting $e_3(k-1)$ from $e_3(k)$, we have

$$e_3(k) = e_3(k-1) + J_3(k)(\breve{y}(k) - \breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1)),\ e_3(0) = 0. \tag{A-16}$$

Substituting (A-14) into (A-13), we have

$$\Pi(\mathrm{k}) = R + \breve{H}\breve{K}(k,k)\breve{H}^T - \breve{H}\breve{\Phi}A(k-1)\mathrm{E}[e_3(k-1){e_3}^T(k-1)]A^T(k-1)\breve{\Phi}^T\breve{H}^T. \tag{A-17}$$

Substituting (A-15) into (A-17) and introducing

$$r_{33}(k) = \sum_{i=1}^{k} J_3(i)\Pi(i)J_3^T(i), \tag{A-18}$$

we obtain the expression for $\Pi(\mathrm{k})$ as

$$\Pi(\mathrm{k}) = R + \breve{H}\breve{K}(k,k)\breve{H}^T - \breve{H}\breve{\Phi}A(k-1)r_{33}(k-1)A^T(k-1)\breve{\Phi}^T\breve{H}^T. \tag{A-19}$$

Substituting (A-16) into (A-14), from (A-5), the filtering estimate $\hat{\breve{x}}(k,k)$ is calculated by

$$\begin{aligned}\hat{\breve{x}}(k,k) &= \mathrm{A(k)}e_3(k-1) + A(k)J_3(k)(\breve{y}(k) - \breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1))\\ &= \Phi\hat{x}(k-1,k-1) + g_0(k,k)(\breve{y}(k) - \breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1)).\end{aligned} \tag{A-20}$$

From (A-4), (A-5) and (A-18), $J_3(k)$ is developed as

$$\begin{aligned}J_3(k)\Pi(\mathrm{k}) &= B^T(k)\breve{H}^T - \sum_{i=1}^{k-1} J_3(i)\,\Pi(\mathrm{i})g_0^T(k,i)\breve{H}^T\\ &= B^T(k)\breve{H}^T - \sum_{i=1}^{k-1} J_3(i)\,\Pi(\mathrm{i}){J_3}^T(i)A^T(k)\breve{H}^T\\ &= B^T(k)\breve{H}^T - r_{33}(k-1)A^T(k)\breve{H}^T.\end{aligned} \tag{A-21}$$

Substitution of (A-21) into (A-5) yields

$$\begin{aligned}g_0(k,k) &= A(k)(B^T(k)\breve{H}^T - r_{33}(k-1)A^T(k)\breve{H}^T)\Pi^{-1}(k)\\ &= (\breve{K}(k,k)\breve{H}^T - \breve{\Phi}S_0(k-1)\breve{\Phi}^T\breve{H}^T)\Pi^{-1}(k).\end{aligned} \tag{A-22}$$

Here,

$$S_0(k) = A(k)r_{33}(k)A^T(k). \tag{A-23}$$

Subtraction of $r_{33}(k-1)$ from $r_{33}(k)$ yields

$$r_{33}(k) = r_{33}(k-1) + J_3(k)\Pi(\mathrm{k}){J_3}^T(k), r_{33}(0) = 0. \tag{A-24}$$

Substitution of (A-24) into (A-23) and using (A-5) yield (26). From (A-19) and (A-23), (31) is obtained.

From (A-1) and (A-3), the filtering estimate $\hat{x}(k,k)$ of $x(k)$ is expressed by

$$\hat{x}(k,k) = \sum_{i=1}^{k} g\,(k,i)\breve{v}(i) \tag{A-25}$$

$= \alpha(k)e_1(k).$

Here,

$$e_1(k) = \sum_{i=1}^{k} J_1(i)\breve{v}(i). \quad \text{(A-26)}$$

Subtraction of $e_1(k-1)$ from $e_1(k)$ yields

$$e_1(k) = e_1(k) + J_1(k)\breve{v}(k), e_1(0) = 0. \quad \text{(A-27)}$$

Substitution of (A-27) into (A-25), from (A-3), yields

$$\hat{x}(k,k) = \Phi\hat{x}(k-1,k-1) + \alpha(k)J_1(k)\breve{v}(k)$$

$$=\Phi\hat{x}(k-1,k-1) + g(k,k)(\breve{y}(k) - \breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1)), \quad \text{(A-28)}$$

$$\hat{x}(0,0) = 0.$$

Substituting (A-8) into (A-3) and introducing

$$S(k) = \alpha(k)r_{13}(k)A^T(k), \quad \text{(A-29)}$$

we get

$$g(k,k) = \alpha(k)(\beta^T(k)\breve{H}^T - r_{13}(k-1)A^T(k)\breve{H}^T)\Pi^{-1}(k)$$

$$=(K_{x\breve{z}}(k,k) - \Phi S(k-1)\breve{\Phi}^T\breve{H}^T)\,\Pi^{-1}(k). \quad \text{(A-30)}$$

Substituting (A-9) into (A-29), we have

$$S(k) = \alpha(k)(r_{13}(k-1) + J_1(k)\Pi(\mathrm{k})J_3^T(k))A^T(k). \quad \text{(A-31)}$$

From (A-3), (A-5), (A-22), (A-29) and (A-31), (27) is obtained. This completes the derivation of the robust RLS Wiener filter.

Now, in terms of the filtering estimate $\hat{x}(k,k)$, the fixed-interval smoothing estimate in (A-1) is written as

$$\hat{x}(k,L) = \hat{x}(k,k) + \sum_{i=k+1}^{L} g(k,i)\breve{v}(i). \quad \text{(A-32)}$$

$g(k,s)$, for $0 \le \mathrm{k} \le \mathrm{s}$, satisfies

$$g(k,s)\Pi(s) = K_{x\breve{x}}(k,s)\breve{H}^T - \sum_{i=1}^{k} g(k,i)\Pi(i)g_0^T(s,i)\breve{H}^T$$

$$-\sum_{i=k+1}^{s-1} g(k,i)\Pi(i)g_0^T(s,i)\breve{H}^T \quad \text{(A-33)}$$

$$=\gamma(\mathrm{k})\delta^T(s)\breve{H}^T - \sum_{i=1}^{k} g\,(k,i)\Pi(i)J_3^T(i)A^T(s)\breve{H}^T - \sum_{i=k+1}^{s-1} g\,(k,i)\Pi(i)g_0^T(s,i)\breve{H}^T$$

$$=\gamma(\mathrm{k})\delta^T(s)\breve{H}^T - \alpha(k)r_{13}(k)A^T(s)\breve{H}^T - \sum_{i=k+1}^{s-1} g\,(k,i)\Pi(i)g_0^T(s,i)\breve{H}^T.$$

By introducing

$$\Delta_1(k,s)\Pi(s) = \delta^T(s)\breve{H}^T - \sum_{i=k+1}^{s-1}\Delta_1(k,i)\,\Pi(i)g_0^T(s,i)\breve{H}^T, \tag{A-34}$$

and

$$\Delta_2(k,s)\Pi(s) = A^T(s)\breve{H}^T - \sum_{i=k+1}^{s-1}\Delta_2(k,i)\,\Pi(i)g_0^T(s,i)\breve{H}^T, \tag{A-35}$$

$g(k,s)$ is expressed as

$$g(k,s) = \gamma(k)\Delta_1(k,s) - \alpha(\mathrm{k})r_{13}(\mathrm{k})\Delta_2(\mathrm{k},\mathrm{s}), 0 \le \mathrm{k} \le \mathrm{s}. \tag{A-36}$$

Substituting (A-36) into (A-32), we have

$$\hat{x}(k,L) = \hat{x}(k,k) + \sum_{i=k+1}^{L}(\gamma(k)\Delta_1(k,i) - \alpha(\mathrm{k})r_{13}(\mathrm{k})\Delta_2(\mathrm{k},\mathrm{i}))\,\breve{v}(i). \tag{A-37}$$

By introducing functions

$$q_1(k+1,L) = \sum_{i=k+1}^{L}\Delta_1(k,i)\,\breve{v}(i), \tag{A-38}$$

$$q_2(k+1,L) = \sum_{i=k+1}^{L}\Delta_2(k,i)\,\breve{v}(i), \tag{A-39}$$

the fixed-interval smoothing estimate $\hat{x}(k,L)$ is given by

$$\hat{x}(k,L) = \hat{x}(k,k) + \gamma(k)q_1(k+1,L) - \alpha(\mathrm{k})r_{13}(\mathrm{k})q_2(k+1,L). \tag{A-40}$$

Subtraction of $\Delta_1(k,s)$ from $\Delta_1(k+1,s)$ yields

$$\begin{aligned}(\Delta_1(k+1,s) - \Delta_1(k,s))\Pi(s)\\ &= \Delta_1(k,k+1)\Pi(k+1)g_0^T(s,k+1)\breve{H}^T \\ &- \sum_{i=k+2}^{s-1}(\Delta_1(k+1,i) - \Delta_1(k,i))\,\Pi(i)g_0^T(s,i)\breve{H}^T.\end{aligned} \tag{A-41}$$

From (A-5), (A-41) is written as

$$(\Delta_1(k+1,s)-\Delta_1(k,s))\Pi(s) \\ =\Delta_1(k,k+1)\Pi(k+1)J_3^T(k+1)A^T(s)\breve{H}^T \\ -\sum_{i=k+2}^{s-1}(\Delta_1(k+1,i)-\Delta_1(k,i))\,\Pi(i)g_0^T(s,i)\breve{H}^T. \quad \text{(A-42)}$$

From (A-34), it is seen that

$$\Delta_1(k,k+1)\Pi(k+1)=\delta^T(k+1)\breve{H}^T-\sum_{i=k+1}^{k}\Delta_1(k,i)\,\Pi(i)g_0^T(k+1,i)\breve{H}^T \\ =\delta^T(k+1)\breve{H}^T. \quad \text{(A-43)}$$

From (A-35), (A-42) and (A-43), we obtain

$$\Delta_1(k+1,s)-\Delta_1(k,s)=\delta^T(k+1)\breve{H}^TJ_3^T(k+1)\,\Delta_2(k+1,s). \quad \text{(A-44)}$$

In a similar fashion, following relationships are derived.

$$\Delta_2(k,k+1)\Pi(k+1)=A^T(k+1)\breve{H}^T \quad \text{(A-45)}$$

$$\Delta_2(k+1,s)-\Delta_2(k,s)=A^T(k+1)\breve{H}^TJ_3^T(k+1)\,\Delta_2(k+1,s). \quad \text{(A-46)}$$

Subtracting $q_1(k,L)$ from $q_1(k+1,L)$ and referring to (A-39) and (A-44), we have

$$q_1(k+1,L)-q_1(k,L)=-\Delta_1(k-1,k)\breve{v}(i) \\ +\sum_{i=k+1}^{L}(\Delta_1(k,i)-\Delta_1(k-1,i))\breve{v}(i) \\ =-\Delta_1(k-1,k)\breve{v}(i)+\delta^T(k)\breve{H}^TJ_3^T(k)\,q_2(k+1,L). \quad \text{(A-47)}$$

From (A-38), it is clear that

$$q_1(L+1,L)=\textstyle\sum_{i=L+1}^{L}\Delta_1(k,i)\,\breve{v}(i)=0. \quad \text{(A-48)}$$

Similarly, we obtain following relationships

$$q_2(k+1,L)-q_2(k,L)=-A^T(k)\breve{H}^T\Pi^{-1}(k)\breve{v}(i) \\ +A^T(k)\breve{H}^TJ_3^T(k)\,q_2(k+1,L),\ \ q_2(L+1,L)=0. \quad \text{(A-49)}$$

From (A-38) and (A-39), (A-37) is written as

$$\hat{x}(k,L) = \hat{x}(k,k) + \gamma(k)q_1(k+1,L) - \alpha(k)r_{13}(k)q_2(k+1,L). \tag{A-50}$$

By putting

$$\breve{q}_1(k+1,L) = (A^T(k))^{-1}q_1(k+1,L), \tag{A-51}$$

$$\breve{q}_2(k+1,L) = (A^T(k))^{-1}q_2(k+1,L), \tag{A-52}$$

we obtain (30), from (A-47) and (A-49), after some manipulations, by noting the relationship $\breve{q}_1(k+1,L) = \breve{q}_2(k+1,L)$.

(Q.E.D.)

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