Seventh convergence order solvers free of derivatives for solving equations in Banach space

*Santhosh George, **Ioannis K. Argyros

*National Institute of Technology Karnataka **Cameron University

*sgeorge@nitk.edu.in, **iargyros@cameron.edu

Abstract

We study a seventh convergence order solver introduced earlier on the $j-$dimensional Euclidean space for solving systems of equations. We use hypotheses only on the divided differences of order one in contrast to the earlier study using hypotheses on derivatives reaching up to order eight although these derivatives do not appear on the solver. This way we expand the applicability of the solver, and in the more general setting of Banach space valued operators. Numerical examples complement the theoretical results.

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Introduction

Finding a solution x_* of the equation

$$\mathcal{F}(x) = 0, \tag{0.1}$$

where $\mathcal{F} : \Omega \longrightarrow \mathcal{E}_2$ is Fréchet differentiable operator is an important problem due to its wide application in many fields [1–24]. Here and below $\Omega \subset \mathcal{E}_1$ be nonempty, open, and $\mathcal{E}_1, \mathcal{E}_2$ be Banach spaces.

This paper is devoted to the study of the seventh order developed in [1] (for $\mathcal{E}_1 = \mathcal{E}_2 = \mathbb{R}^j$) given as

$$\begin{aligned} x_0 \in \Omega,\, y_n &= x_n - [u_n, x_n; \mathcal{F}]^{-1}\mathcal{F}(x_n) \\ z_n &= x_n - A_n^{-1}\mathcal{F}(y_n) \\ x_{n+1} &= z_n - B_n^{-1}\mathcal{F}(z_n), \end{aligned} \tag{0.2}$$

where

$$A_n = [y_n, x_n; \mathcal{F}] + [y_n, u_n; \mathcal{F}] - [u_n, x_n; \mathcal{F}],$$

$$B_n = [z_n, x_n; \mathcal{F}] + [z_n, y_n; \mathcal{F}] - [y_n, x_n; \mathcal{F}]$$

and $u_n = x_n + \mathcal{F}(x_n)$. Methods (0.2) was studied in [1] using conditions on eight order derivative, and Taylor series(although these derivatives do not appear in solver (0.2)). The hypotheses on the eight order derivatives limit the usage of solver (0.2).

As an academic example: Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}$, $\Omega = [-\frac{1}{2}, \frac{3}{2}]$. Define $\mathcal{F}$ on Ω by

$$\mathcal{F}(x) = x^3 \log x^2 + x^5 - x^4$$

Then, we have $x_* = 1$, and

$$\mathcal{F}'(x) = 3x^2 \log x^2 + 5x^4 - 4x^3 + 2x^2,$$

$$\mathcal{F}''(x) = 6x \log x^2 + 20x^3 - 12x^2 + 10x,$$

$$\mathcal{F}'''(x) = 6 \log x^2 + 60x^2 = 24x + 22.$$

Obviously $\mathcal{F}'''(x)$ is not bounded on Ω. So, the convergence of solver (0.2) not guaranteed by the analysis in [11–13].

Other problems with the usage of solver (0.2) are: no information on how to choose x_0; bounds on $\|x_n - x_*\|$ and information on the location of x_*. All these are addressed in this paper by only using conditions on the first derivative, and in the more general setting of Banach space valued operators. That is how, we expand the applicability of solver (0.2). To avoid the usage of Taylor series and high convergence order derivatives, we rely on the computational order of convergence (COC) or the approximate computational order of convergence (ACOC) [1,2,4].

The layout of the rest of the paper includes: the local convergence in Section 2, and the example in Section 3.

1 local convergence analysis

Let $\alpha \geq 0$, $\beta \geq 0$ and set $\gamma = \max\{\alpha, \beta\}$. Consider function $\varphi_0 : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ to be continuous, increasing with $\varphi_0(0,0) = 0$.

Assume equation

$$\varphi_0(\beta t, t) = 1 \tag{1.1}$$

has a minimal positive solution ρ_0. Consider functions $\varphi, \varphi_1 : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ continuous, increasing with $\varphi(0,0) = 0$. Define functions $\bar{\varphi}_1, \bar{\bar{\varphi}}_1$ on the interval $[0, \rho_0)$ by

$$\bar{\varphi}_1(t) = \frac{\varphi(\alpha t, t)}{1 - \varphi_0(\beta t, t)}$$

and

$$\bar{\bar{\varphi}}_1(t) = \bar{\varphi}_1(t) - 1.$$

By these definitions $\bar{\bar{\varphi}}_1(0) = -1$ and $\bar{\bar{\varphi}}_1(t) \longrightarrow \infty$ with $t \longrightarrow \rho_0^-$. Then, the intermediate value theorem assures the existence of at least one solution of equation $\bar{\bar{\varphi}}_1(t) = 0$ in $(0, \rho_0)$. Denote by r_1 the minimal such solution.

Assume equation

$$p(t) = 1 \tag{1.2}$$

has a minimal positive solution ρ_p, where

$$p(t) = \varphi_1(\bar{\varphi}_1(t)t + \beta t, 0) + \varphi_0(\bar{\varphi}_1(t)t, \beta t).$$

Set

$$\rho = \min\{\rho_0, \rho_p\}.$$

Define functions $\bar{\varphi}_2, \bar{\bar{\varphi}}_2$ on the interval $[0, \rho)$ by

$$\bar{\varphi}_2(t) = \frac{\alpha g(t)\bar{\varphi}(t)}{1 - p(t)}$$

and

$$\bar{\bar{\varphi}}_2(t) = \bar{\varphi}_2(t) - 1,$$

where

$$g(t) = \varphi_1(\bar{\varphi}_1(t)t + \beta t, 0) + \varphi(\bar{\varphi}_1(t)t + \beta t, t).$$

Using these definitions $\bar{\bar{\varphi}}_2(0) = -1$, and $\bar{\bar{\varphi}}_2(t) \longrightarrow \infty$ with $t \longrightarrow \rho^-$. Denote by r_2 the minimal solution of equation $\bar{\bar{\varphi}}_2(t) = 0$ in $(0, \rho)$.

Assume equation

$$q(t) = 1 \tag{1.3}$$

has a minimal positive solution ρ_q, where

$$q(t) = \varphi_1(\bar{\varphi}_2(t)t + \bar{\varphi}_1(t)t, 0) + \varphi(0, \bar{\varphi}_1(t)t).$$

Set

$$\rho_1 = \min\{\rho, \rho_q\}.$$

Define functions $\bar{\varphi}_3, \bar{\bar{\varphi}}_3$ on the interval $[0, \rho_1)$ by

$$\bar{\varphi}_3(t) = \frac{\alpha q(t)\bar{\varphi}_2(t)}{1 - q(t)}$$

and

$$\bar{\bar{\varphi}}_3(t) = \bar{\varphi}_3(t) - 1.$$

By these definitions $\bar{\bar{\varphi}}_3(0) = -1$ and $\bar{\bar{\varphi}}_3(t) \longrightarrow \infty$ with $t \longrightarrow \rho_1^-$. Denote by r_3 the minimal solution of equation $\bar{\bar{\varphi}}_3(t) = 0$ in $(0, \rho_1)$.

Define a radius of convergence r by

$$r = \min\{r_j\}, \ j = 1, 2, 3. \tag{1.4}$$

Then, we have

$$0 \leq \varphi_0(\beta t, t) < 1 \tag{1.5}$$

$$0 \leq p(t) < 1 \tag{1.6}$$

$$0 \leq q(t) < 1 \tag{1.7}$$

and

$$0 \leq \bar{\varphi}_j(t) < 1 \tag{1.8}$$

for all $t \in [0, r)$.

Here and below $S(x, \eta)$ stand for the open ball in $\mathcal{E}_1$ with center x and radius $\eta > 0$ and $\bar{S}(x, \eta)$ stand for the closure of $S(x, \eta)$. The following conditions (A) are used in the local convergence analysis that follows:

(a1) $\mathcal{F} : \Omega \longrightarrow \mathcal{E}_2$ is continuous, $[., .; \mathcal{F}] : \Omega \times \Omega \longrightarrow \mathcal{E}_2$ is a divided difference of order one, and there exists $x_* \in \Omega$ such that $\mathcal{F}(x_*) = 0$ and $\mathcal{F}'(x_*)^{-1} \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)$.

(a2) $\varphi_0 : [0, \infty) \times [0, \infty)$ is continuous, increasing with $\varphi_0(0, 0) = 0$ such that for all $x, y \in \Omega$

$$\|\mathcal{F}'(x_*)^{-1}([x, y; \mathcal{F}] - \mathcal{F}'(x_*))\| \leq \varphi_0(\|x - x_*\|, \|y - x_*\|).$$

Set $\Omega_0 = \Omega \cap S(x_*, \rho_0)$ where ρ_0 is given in (1.1.

(a3) There exist continuous and increasing functions $\varphi, , \varphi_1 : [0, \rho_0) \times [0, \rho_0) \longrightarrow [0, \infty)$ with $\varphi(0, 0) = \varphi_1(0, 0) = 0$ such that for each $x, y, z, w \in \Omega_0$

$$\|\mathcal{F}'(x_*)^{-1}([y, x; \mathcal{F}] - [z, x_*; \mathcal{F}])\| \leq \varphi(\|y - z\|, \|x - x_*\|),$$

$$\|\mathcal{F}'(x_*)^{-1}([y, x; \mathcal{F}] - [z, w; \mathcal{F}])\| \leq \varphi_1(\|y - z\|, \|x - w\|)$$

and for $\alpha \geq 0, \beta \geq 0$ and $x \in \Omega_0$

$$\|\mathcal{F}(x)\| \leq \alpha,$$

$$\|I + [x, x_*; \mathcal{F}]\| \leq \beta.$$

(a4) $\bar{S}(x_*, \gamma r) \subseteq \Omega$, where r is given in (1.4), ρ_p, ρ_q and ρ_1 given previously exist, and $\gamma = \max\{\alpha, \beta\}$.

(a5) There exists $\bar{r} \geq r$ such that

$$\varphi_0(0, \bar{r}) < 1 \text{ or } \varphi_0(\bar{r}, 0) < 1.$$

Set $\Omega_1 = \Omega \cap \bar{S}(x_*, \bar{r})$.

Next, we present the local convergence analysis of method (0.2) based on the preceding notation and conditions (A).

THEOREM 1.1 *Under the conditions (A) further suppose that $x_0 \in S(x_*, r) - \{x_*\}$. Then, the following items hold*

$$\{x_n\} \subset S(x_*, r) \tag{1.9}$$

$$\lim_{n \longrightarrow \infty} x_n = x_*, \tag{1.10}$$

$$\|y_n - x_*\| \leq \bar{\varphi}_1(\|x_n - x_*\|)\|x_n * -x_*\| \leq \|x_n - x_*\| < r, \tag{1.11}$$

$$\|z_n - x_*\| \leq \bar{\varphi}_2(\|x_n - x_*\|)\|x_n * -x_*\| \leq \|x_n - x_*\|, \tag{1.12}$$

$$\|x_{n+1} - x_*\| \leq \bar{\varphi}_3(\|x_n - x_*\|)\|x_n * -x_*\| \leq \|x_n - x_*\|, \tag{1.13}$$

and x_* is the only solution of equation $\mathcal{F}(x) = 0$ in the set Ω_1, where functions $\bar{\varphi}_j, j = 1, 2, 3,$ and Ω_1 are defined previously.

Proof. The proof is based on mathematical induction. Let $x, y \in S(x_*, r)$. Then, using (a1), (1.4), (1.5), and (a3), we have in turn

$$\begin{aligned} \|\mathcal{F}'(x_*)^{-1}([x + \mathcal{F}(x), x; \mathcal{F}] - \mathcal{F}'(x_*))\| &\leq \varphi_0(\|x + \mathcal{F}(x) - x_*\|, \|x - x_*\|) \\ &\leq \varphi_0(\beta\|x - x_*\|, \|x - x_*\|) \\ &\leq \varphi_0(\beta r, r) < 1, \end{aligned} \tag{1.14}$$

which together with the Banach lemma on invertible operators [16] show $[x+\mathcal{F}(x),x;\mathcal{F}]^{-1}\in\mathcal{L}(\mathcal{E}_2,\mathcal{E}_1)$, and

$$\|[x+\mathcal{F}(x),x;\mathcal{F}]^{-1}\mathcal{F}'(x_*)\|\leq\frac{1}{1-\varphi_0(\beta\|x-x_*\|,\|x-x_*\|)},\tag{1.15}$$

where we also used

$$\|x+\mathcal{F}(x)-x_*\|\leq\|(I+[x,x_*;\mathcal{F}])(x-x_*)\|\leq\beta\|x-x_*\|\leq\beta r,\tag{1.16}$$

so $x+\mathcal{F}(x)\in S(x_*,\beta r)\subseteq S(x_*,\gamma r)\subseteq\Omega$. The point y_0 is well defined by the first substep of method (0.2) for $n=0$. Using (1.4), (1.8) (for $j=1$), (a3), (1.15), and the first substep of method (0.2) for $n=0$, we get in turn that

$$\begin{aligned}\|y_0-x_*\| &= \|x_0-x_*-[u_0,x_0;\mathcal{F}]^{-1}\mathcal{F}(x_0)\|\\ &\leq \|[u_0,x_0;\mathcal{F}]^{-1}\mathcal{F}'(x_*)\|\\ &\quad\times\|\mathcal{F}'(x_*)^{-1}([u_0,x_0;\mathcal{F}]-[x_0,x_*;\mathcal{F}])\|\|x_0-x_*\|\\ &\leq \frac{\varphi(\alpha\|x_0-x_*\|,\|x_0-x_*\|)\|x_0-x_*\|}{1-\varphi_0(\beta\|x_0-x_*\|,\|x_0-x_*\|)}\\ &= \bar{\varphi}_1(\|x_0-x_*\|)\|x_0-x_*\|\leq\|x_0-x_*\|<r\end{aligned}\tag{1.17}$$

showing $y_0\in S(x_*,r)$ and (1.11) for $n=0$. We must show $A_0^{-1}\mathcal{L}(\mathcal{E}_2,\mathcal{E}_1)$. By (1.4), (1.6), (a3) and (1.17), we get in turn that

$$\begin{aligned}\|\mathcal{F}'(x_*)^{-1}(A_0-\mathcal{F}'(x_*))\| &\leq \|\mathcal{F}'(x_*)^{-1}([y_0,x_0;\mathcal{F}]-[x_0+\mathcal{F}(x_0),x_0;\mathcal{F}])\|\\ &\quad+\|\mathcal{F}'(x_*)^{-1}([y_0,x_0+\mathcal{F}(x_0);\mathcal{F}]-\mathcal{F}'(x_*))\|\\ &\leq \varphi_1(\|y_0-x_0-\mathcal{F}(x_0)\|,0)\\ &\quad+\varphi_0(\|y_0-x_*\|,\|x_0+\mathcal{F}(x_0)-x_*\|)\\ &\leq p(\|x_0-x_*\|)<1,\end{aligned}$$

so $A_0^{-1}\in\mathcal{L}(\mathcal{E}_2,\mathcal{E}_1)$,

$$\|A_0^{-1}\mathcal{F}'(x_*)\|\leq\frac{1}{1-p(\|x_0-x_*\|)},\tag{1.18}$$

and z_0 is well defined by the second substep of method (0.2) for $n=0$. We also need that estimate

$$\begin{aligned}&\|\mathcal{F}'(x_*)^{-1}([y_0,x_0;\mathcal{F}]+[y_0,x_0+\mathcal{F}(x_0);\mathcal{F}]\\ &\quad-[x_0+\mathcal{F}(x_0),x_0;\mathcal{F}]-[y_0,x_*;\mathcal{F}])\|\\ \leq\;& \|\mathcal{F}'(x_*)^{-1}([y_0,x_0;\mathcal{F}]-[x_0+\mathcal{F}(x_0),x_0;\mathcal{F}])\|\\ &\quad+\|\mathcal{F}'(x_*)^{-1}([y_0,x_0+\mathcal{F}(x_0);\mathcal{F}]-[y_0,x_*;\mathcal{F}])\|\\ \leq\;& \varphi_1(\|y_0-x_0-\mathcal{F}(x_0)\|,0)\\ &\quad+\varphi(\|x_0+\mathcal{F}(x_0)-y_0\|,\|x_0-x*\|)\\ \leq\;& \varphi_1(\bar{\varphi}(\|x_0-x_*\|)\|x_0-x_*\|+\beta\|x_0-x_*\|,0)\\ &\quad+\varphi(\bar{\varphi}(\|x_0-x_*\|)\|x_0-x_*\|+\beta\|x_0-x_*\|,\|x_0-x_*\|)\\ =\;& g(\|x_0-x_*\|).\end{aligned}\tag{1.19}$$

Then, by the second substep of method (0.2), we can write the identity

$$\begin{aligned}z_0-x_* &= y_0-x_*-A_0^{-1}\mathcal{F}(y_0)\\ &= [A_0^{-1}\mathcal{F}'(x_*)]\\ &\quad\times[\mathcal{F}'(x_*)^{-1}(A_0-[y_0,x_*;\mathcal{F}])(y_0-x_*)],\end{aligned}\tag{1.20}$$

so by (1.4), (1.8) (for $j = 2$), (1.17)-(1.20), we obtain in turn that

$$\begin{aligned}\|z_0 - x_*\| &\leq \|A_0^{-1}\mathcal{F}'(x_*)\| \\ &\quad \times\|\mathcal{F}'(x_*)^{-1}(A_0 - [y_0, x_*; \mathcal{F}])\|\|y_0 - x_*\| \\ &\leq \frac{\alpha g(\|x_0 - x_*\|)\bar{\varphi}_1(\|x_0 - x_*\|)\|x_0 - x_*\|}{1 - p(\|x_0 - x_*\|)} \\ &= \bar{\varphi}_2(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\|, \end{aligned} \tag{1.21}$$

so $z_0 \in S(x_*, r)$, and (1.12) holds for $n = 0$. Next, we must show $B_0^{-1} \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)$. We have in turn that

$$\begin{aligned} & \|\mathcal{F}'(x_*)^{-1}(B_0 - \mathcal{F}'(x_*))\| \\ = & \|\mathcal{F}'(x_*)^{-1}([z_0, x_0; \mathcal{F}] + [z_0, y_0; \mathcal{F}] \\ & -[y_0, x_0; \mathcal{F}] - [z_0, x_*; \mathcal{F}])\| \\ \leq & \|\mathcal{F}'(x_*)^{-1}([z_0, x_0; \mathcal{F}] - [y_0, x_0; \mathcal{F}])\| \\ & +\|\mathcal{F}'(x_*)^{-1}([z_0, y_0; \mathcal{F}] - [z_0, x_*; \mathcal{F}])\| \\ \leq & \varphi_1(\|(z_0 - x_*) + (x_* - y_0)\|, 0) + \varphi(0, \|y_0 - x_*\|) \\ \leq & \varphi_1(\bar{\varphi}_2(\|x_0 - x_*\|)\|x_0 - x_*\| + \bar{\varphi}_1(\|x_0 - x_*\|)\|x_0 - x_*\|, 0) \\ & +\varphi(0, \bar{\varphi}_1(\|x_0 - x_*\|)\|x_0 - x_*\|) = q(\|x_0 - x_*\|) < 1, \end{aligned} \tag{1.22}$$

so $B_0^{-1}\mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)$,

$$\|B_0^{-1}\mathcal{F}'(x_*)\| \leq \frac{1}{1 - q(\|x_0 - x_*\|)}, \tag{1.23}$$

and x_1 is well defined by the last substep of method (0.2). We also need the estimate

$$\begin{aligned} & \|\mathcal{F}'(x_*)^{-1}(B_0 - [z_0, x_*; \mathcal{F}])\| \\ = & \|\mathcal{F}'(x_*)^{-1}([z_0, x_0; \mathcal{F}] + [z_0, y_0; \mathcal{F}] \\ & -[y_0, x_0; \mathcal{F}] - [z_0, x_*; \mathcal{F}])\| \\ \leq & \|\mathcal{F}'(x_*)^{-1}([z_0, x_0; \mathcal{F}] - [y_0, x_0; \mathcal{F}])\| \\ & +\|\mathcal{F}'(x_*)^{-1}([z_0, y_0; \mathcal{F}] - [z_0, x_*; \mathcal{F}])\| \\ \leq & \varphi_1(\|z_0 - x_*) + (x_* - y_0)\|, 0) + \varphi(0, \|y_0 - x_*\|) \\ \leq & \varphi_1(\bar{\varphi}_2(\|x_0 - x_*\|)\|x_0 - x_*\| + \bar{\varphi}_1(\|x_0 - x_*\|\|x_0 - x_*\|, 0) \\ & +\varphi(0, \bar{\varphi}_1(\|x_0 - x_*\|))\|x_0 - x_*\| = q(\|x_0 - x_*\|). \end{aligned} \tag{1.24}$$

Then, by the last substep of method (0.2), we write

$$\begin{aligned} x_1 - x_* &= z_0 - x_* - B_0^{-1}\mathcal{F}(z_0) \\ &= [B_0^{-1}\mathcal{F}'(x_*)][\mathcal{F}'(x_*)^{-1}(B_0 - [z_0, x_*; \mathcal{F}])(z_0 - x_*)]. \end{aligned} \tag{1.25}$$

Hence, by (1.4), (1.8) (for $j = 3$), (1.17), (1.21) and (1.23)-(1.25)

$$\begin{aligned}\|x_1 - x_*\| &\leq \|B_0^{-1}\mathcal{F}'(x_*)\| \\ &\quad \times\|\mathcal{F}'(x_*)^{-1}(B_0 - [z_0, x_*; \mathcal{F}])\|\|z_0 - x_*\| \\ &\leq \frac{q(\|x_0 - x_*\|)\bar{\varphi}_2(\|x_0 - x_*\|)\|x_0 - x_*\|}{1 - q(\|x_0 - x_*\|)} \\ &= \bar{\varphi}_3(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\|, \end{aligned} \tag{1.26}$$

so $x_1 \in S(x_*, r)$ and (1.13) holds for $n = 0$. The induction is finished, if x_0, y_0, z_0, x_1 are replaced by x_i, y_i, z_i, x_{i+1} in the preceding estimates. It then follows from the estimate

$$\|x_{i+1} - x_*\| \leq c\|x_i - x_*\| < r, \tag{1.27}$$

that $\lim_{i\longrightarrow\infty} x_i = x_*$ and $x_{i+1} \in S(x_*, r)$, where $c = \bar{\varphi}_3(\|x_0 - x_*\|) \in [0, 1)$. Let $D = [x_*, y_*; \mathcal{F}]$, where $y_* \in \Omega_1$ with $\mathcal{F}(y_*) = 0$. In view of (a2) and (a5)

$$\begin{aligned} \|\mathcal{F}'(x_*)^{-1}(D - \mathcal{F}'(x_*))\| &\leq \varphi_0(0, \|y_* - x_*\|) \\ &\leq \varphi_0(0, \bar{r}) < 1, \end{aligned}$$

so $D^{-1} \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)$. Finally, using the identity

$$0 = \mathcal{F}(x_*) - \mathcal{F}(y_*) = D(x_* - y_*), \tag{1.28}$$

we get $x_* = y_*$.

□

REMARK 1.2 *(a) The local results can be used for projection solvers such as Arnoldi's solver, the generalized minimum residual solver(GMREM), the generalized conjugate solver(GCM) for combined Newton/finite projection solvers and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy [1–5].*

(b) It is worth noticing that solver (0.2) is not changing when we use the conditions of the preceding Theorem instead of the stronger conditions used in [1]. Moreover, we can compute the computational order of convergence (COC) defined as

$$\xi = \ln\left(\frac{\|x_{n+1} - x_*\|}{\|x_n - x_*\|}\right) / \ln\left(\frac{\|x_n - x_*\|}{\|x_{n-1} - x_*\|}\right)$$

or the approximate computational order of convergence (ACOC) [9, 10]

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence, but not higher order derivatives are used.

2 Numerical example

We present the following example to test the convergence criteria. We define the divided difference, by

$$[x, y; \mathcal{F}] = \int_0^1 \mathcal{F}'(y + \theta(x - y))d\theta.$$

EXAMPLE 2.1 *Let* $\mathcal{E}_1 = \mathcal{E}_2 = \mathbb{R}^3$, $\Omega = U(0, 1), x_* = (0, 0, 0)^T$ *and define* $\mathcal{F}$ *on* Ω *by*

$$\mathcal{F}(x) = \mathcal{F}(u_1, u_2, u_3) = (e^{u_1} - 1, \frac{e-1}{2}{u_2}^2 + u_2, u_3)^T. \tag{2.1}$$

For the points $u = (u_1, u_2, u_3)^T$, the Fréchet derivative is given by

$$\mathcal{F}'(u) = \begin{pmatrix} e^{u_1} & 0 & 0 \\ 0 & (e-1)u_2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using the norm of the maximum of the rows $x_* = (0,0,0)^T$ and since $\mathcal{F}'(x_*) = diag(1,1,1)$, we get $\varphi_0(s,t) = \frac{e-1}{2}(s+t)$, $\varphi(s,t) = \frac{1}{2}(e^{\frac{1}{e-1}}s + (e-1)t), \varphi_1(s,t) = \frac{1}{2}e^{\frac{1}{e-1}}(s+t), \alpha = \alpha(t) = e^{\frac{1}{e-1}}t$ or $\alpha = e^{\frac{1}{e-1}}, \beta = \beta(t) = 2 + \varphi(t,t)$.

$$r_1 = 0.0259288,\ r_2 = 0.0179402,\ r_3 = 0.188239.$$

EXAMPLE 2.2 *Let $\mathcal{E}_1 = \mathcal{E}_2 = C[0,1], \Omega = \bar{U}(0,1)$. Define function F on Ω by*

$$F(w)(x) = w(x) - 5\int_0^1 x\theta w(\theta)^3 d\theta.$$

Then, the Fréchet-derivative is given by

$$F'(w(\xi))(x) = \xi(x) - 15\int_0^1 x\theta w(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in \Omega.$$

Then, we have $\varphi_0(s,t) = \frac{15}{4}(s+t)$, $\varphi(s,t) = \frac{1}{2}(15s + 7.5t), \varphi_1(s,t) = \frac{15}{2}(s+t), \alpha = \alpha(t) = 15t$ or $\alpha = 15, \beta = \beta(t) = 2 + \varphi(t,t)$. Then, the radius of convergence are given by

$$r_1 = 0.0454057,\ r_2 = 0.0284351, r_3 = 0298118.$$

EXAMPLE 2.3 *Returning back to the motivational example given at the introduction of this study, we get $\varphi_0(s,t) = \varphi(s,t) = \varphi_1(s,t) = \frac{1}{2}(96.662907)(s+t), \alpha = \alpha(t) = 1.0631t,\ \beta = \beta(t) = 2 + \varphi(s,t)$. Then, the radius of convergence are given by*

$$r_1 = 0.00464539,\ r_2 = 0.0036858,\ r_3 = 0.000368962.$$

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