Some Formulae Involving Complete Elliptic Integral

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Abstract

In this paper we have developed some formulae involving Elliptic Integral.The formulae are suppose to be new.

Key Words : Elliptic Integrals,Dilogarithm, Boolean Algebra.

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1. Introduction

Yurry A. Brychkov[Brychkov p.155-156(4.1.6)] has established the following formulae

$$\int_0^a \frac{1}{\sqrt{(a^2-x^2)}}\, sin^{-1}(bx)\ dx = \frac{1}{2}[Li_2(ab)+Li_2(-ab)],\quad [|\arg(1-a^2b^2)|<\pi]. \tag{1.1}$$

$$\int_0^a \sqrt{(a^2-x^2)}\, sin^{-1}(bx)\ dx = \frac{a}{4b}\left\{\frac{1-a^2b^2}{2ab}\ln\frac{1+ab}{1-ab}+ab[Li_2(ab)-Li_2(-ab)]-1\right\},\quad [|\arg(1-a^2b^2)|<\pi]. \tag{1.2}$$

$$\int_0^a x\,\sqrt{(a^2-x^2)}\, sin^{-1}(bx)\ dx = \frac{a^2}{9b}[2(1-2a^2b^2)D(ab)-(1-3a^2b^2)K(ab)],\quad [|\arg(1-a^2b^2)|<\pi]. \tag{1.3}$$

$$\int_0^a \frac{x}{\sqrt{(a^2-x^2)}}\, sin^{-1}(bx)\ dx = a^2b[K(ab)-D(ab)],\quad [|\arg(1-a^2b^2)|<\pi]. \tag{1.4}$$

The incomplete elliptic integral of the first kind F is defined as

$$F(\psi,k)=F(\psi\mid k^2)=F(\sin\psi;k)=\int_0^\psi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}. \tag{1.5}$$

This is the trigonometric form of the integral; substituting $t=\sin\theta, x=\sin\psi$,one obtains Jacobi's form:

$$F(x;k)=\int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{1.6}$$

Equivalently, in terms of the amplitude and modular angle one has:

$$F(\psi\backslash\alpha)=F(\psi,\sin\alpha)=\int_0^\psi \frac{d\theta}{\sqrt{1-(\sin\theta\sin\alpha)^2}}. \tag{1.7}$$

In this notation, the use of a vertical bar as delimiter indicates that the argument following it is the "parameter" (as defined above), while the backslash indicates that it is the modular angle. The use of a semicolon implies that the argument preceding it is the sine of the amplitude:

$$F(\psi,\sin\alpha)=F(\psi\mid\ \sin^2\alpha)=F(\psi\backslash\alpha)=F(\sin\psi;\sin\alpha). \tag{1.8}$$

Incomplete elliptic integral of the second kind E is defined as

$$E(\psi,k)=E(\psi\mid k^2)=E(\sin\psi;k)=\sqrt{1-k^2\sin^2\theta}\ d\theta. \tag{1.9}$$

Substituting $t=\sin\theta$ and $x=\sin\psi$, one obtains Jacobi's form:

$$E(x;k)=\int_0^x \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\ dt. \tag{1.10}$$

Equivalently, in terms of the amplitude and modular angle:

$$E(\psi\backslash\alpha) = E(\psi, \sin\alpha) = \int_0^{\psi} \sqrt{1-(\sin\theta\sin\alpha)^2}\; d\theta. \tag{1.11}$$

Incomplete elliptic integral of the third kind Π is defined as

$$\Pi(n;\psi\backslash\alpha) = \int_0^{\psi} \frac{1}{1-n\sin^2\theta}\,\frac{d\theta}{1-(\sin\theta\sin\alpha)^2}, \tag{1.12}$$

or

$$\Pi(n;\psi\mid m) = \int_0^{\sin\psi} \frac{1}{1-nt^2}\,\frac{dt}{(1-mt^2)(1-t^2)}. \tag{1.13}$$

The number n is called the characteristic and can take on any value, independently of the other arguments.

Complete elliptic integral of the first kind is defined as

Elliptic Integrals are said to be 'complete' when the amplitude $\psi = \frac{\pi}{2}$ and therefore x=1. The complete elliptic integral of the first kind K may thus be defined as

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \tag{1.14}$$

or more compactly in terms of the incomplete integral of the first kind as

$$K(k) = F\left(\frac{\pi}{2}, k\right) = F(1;k). \tag{1.15}$$

It can be expressed as a power series

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 k^{2n} = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[P_{2n}(0)\right]^2 k^{2n}, \tag{1.16}$$

where P_n is the Legendre polynomial, which is equivalent to

$$K(k) = \frac{\pi}{2}\left[1+\left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\,.\,3}{2\,.\,4}\right)^2 k^4 + \ldots\ldots\ldots + \left\{\frac{(2n-1)!!}{(2n)!!}\right\}^2 k^{2n} + \ldots\ldots\right], \tag{1.17}$$

where $n!!$ denotes the double factorial. In terms of the Gauss hypergeometric function, the complete elliptic integral of the first kind can be expressed as

$$K(k) = \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right) \tag{1.18}$$

The complete elliptic integral of the first kind is sometimes called the quarter period. It can most efficiently be computed in terms of the arithmetic-geometric mean:

$$K(k) = \frac{\frac{\pi}{2}}{agm(1-k,1+k)}. \tag{1.19}$$

Complete elliptic integral of the second kind is defined as

The complete elliptic integral of the second kind E is proportional to the circumference of the ellipse C:

$$C = 4aE(e). \tag{1.20}$$

where a is the semi-major axis, and e is the eccentricity.
E may be defined as

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\theta}\; d\theta = \int_0^1 \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\; dt. \tag{1.21}$$

or more compactly in terms of the incomplete integral of the second kind as

$$E(k) = E\left(\frac{\pi}{2}, k\right) = E(1;k). \tag{1.22}$$

It can be expressed as a power series

$$E(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left[\frac{(2n)!}{2^{2n}(n!)^2} \right]^2 \frac{k^{2n}}{1-2n}. \tag{1.23}$$

which is equivalent to

$$E(k) = \frac{\pi}{2} \left[1 - \left(\frac{1}{2} \right)^2 \frac{k^2}{1} - \left(\frac{1\ .\ 3}{2\ .\ 4} \right)^2 \frac{k^4}{3} - \ldots\ldots\ldots - \left\{ \frac{(2n-1)!!}{(2n)!!} \right\}^2 \frac{k^{2n}}{2n-1} - \ldots\ldots \right]. \tag{1.24}$$

In terms of the Gauss hypergeometric function, the complete elliptic integral of the second kind can be expressed as

$$E(k) = \frac{\pi}{2}\ {}_2F_1\left(\frac{1}{2}, -\frac{1}{2}; 1; k^2 \right). \tag{1.25}$$

Complete elliptic integral of the third kind is defined as

The complete elliptic integral of the third kind Π can be defined as

$$\Pi(n,k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1 - n\sin^2\theta)\sqrt{1 - k^2\sin^2\theta}}. \tag{1.26}$$

$$(x)_n = x(x-1)(x-2)\ldots\ldots.(x-n+1) = \prod_{k=1}^{n}(x-k+1) = \prod_{k=0}^{n-1}(x-k) \tag{1.27}$$

The basic operations of Boolean algebra are as follows:
AND (conjunction), denoted $x \wedge y$, satisfies $x \wedge y = 1$ if $x = y = 1$, and $x \wedge y = 0$ otherwise.
OR (disjunction), denoted $x \vee y$, satisfies $x \vee y = 0$ if $x = y = 0$, and $x \vee y = 14$ otherwise.
NOT (negation), denoted $\neg x$, satisfies $\neg x = 0$ if $x = 1$ and $\neg x = 1$ if $x = 0$.

The dilogarithm $Li_2(z)$ is a special case of the polylogarithm $Li_n(z)$ for $n = 2$.
The dilogarithm can be defined as

$$Li_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}. \tag{1.28}$$

2. Main Formulae of the Integration

For $Re(a) > 0 \wedge (Im(a) = 0$

$$\int_0^a x\ \sqrt{(a^2-x^2)}\ sin^{-1}(bx)\ dx = \frac{(4a^2b^2-2)E(a^2b^2) + (3a^4b^4 - 5a^2b^2 + 2)K(a^2b^2)}{9b^3}. \tag{2.1}$$

For $Re(a) > 0 \wedge (Im(a) = 0$

$$\int_0^a x^3\ \sqrt{(a^2-x^2)}\ sin^{-1}(bx)\ dx = \frac{1}{225b^5}[(31a^4b^4 + 19a^2b^2 - 24)E(a^2b^2) + \\ +(30a^6b^6 - 23a^4b^4 - 31a^2b^2 + 24)K(a^2b^2)]. \tag{2.2}$$

$$\int_0^a x^5\ \sqrt{(a^2-x^2)}\ sin^{-1}(bx)\ dx = \frac{1}{11025b^7}[(ab-1)(ab+1)(840a^6b^6 + 241a^4b^4 + 48a^2b^2 - 720)K(a^2b^2) + \\ +(778a^6b^6 + 352a^4b^4 + 408a^2b^2 - 720)E(a^2b^2)], \quad a > 0. \tag{2.3}$$

$$\int_0^a x^7\ \sqrt{(a^2-x^2)}\ sin^{-1}(bx)\ dx = \\ = \frac{1}{99225b^9}[(ab-1)(ab+1)(5040a^8b^8 + 1586a^6b^6 + 603a^4b^4 - 120a^2b^2 - 4480)K(a^2b^2) + \\ +(4388a^8b^8 + 1727a^6b^6 + 1503a^4b^4 + 2120a^2b^2 - 4480)E(a^2b^2)], \quad a > 0. \tag{2.4}$$

For $Re(a) > 0 \wedge (Im(a) = 0$

$$\int_0^a \frac{x}{\sqrt{(a^2-x^2)}}\ sin^{-1}(bx)\ dx = \frac{(a^2b^2-1)K(a^2b^2) + E(a^2b^2)}{b}. \tag{2.5}$$

For $Re(a) > 0 \wedge (Im(a) = 0$

$$\int_0^a \frac{x^3}{\sqrt{(a^2-x^2)}}\ sin^{-1}(bx)\ dx = \frac{(5a^2b^2+2)E(a^2b^2)+2(3a^4b^4-2a^2b^2-1)K(a^2b^2)}{9b^3}. \tag{2.6}$$

For $Re(a) > 0 \wedge (Im(a) = 0$

$$\int_0^a \frac{x^5}{\sqrt{(a^2-x^2)}}\ sin^{-1}(bx)\ dx = \frac{1}{225b^5}[(ab-1)(ab+1)(120a^4b^4+43a^2b^2+24)K(a^2b^2)+$$
$$+(94a^4b^4+31a^2b^2+24)E(a^2b^2)]. \tag{2.7}$$

For $Re(a) > 0 \wedge (Im(a) = 0$

$$\int_0^a \frac{x^7}{\sqrt{(a^2-x^2)}}\ sin^{-1}(bx)\ dx = \frac{1}{3675b^7}[2(ab-1)(ab+1)(840a^6b^6+311a^4b^4+188a^2b^2+120)K(a^2b^2)+$$
$$+(1276a^6b^6+389a^4b^4+256a^2b^2+240)E(a^2b^2)]. \tag{2.8}$$

For $Re(a) > 0 \wedge (Im(a) = 0$

$$\int_0^a \frac{x^9}{\sqrt{(a^2-x^2)}}\ sin^{-1}(bx)\ dx =$$
$$= \frac{1}{99225b^9}[(ab-1)(ab+1)(40320a^8b^8+15208a^6b^6+9549a^4b^4+6600a^2b^2+4480)K(a^2b^2)+$$
$$+(30064a^8b^8+8776a^6b^6+5409a^4b^4+4360a^2b^2+4480)E(a^2b^2)]. \tag{2.9}$$

For $Re(a) > 0 \wedge (Im(a) = 0$

$$\int_0^a \frac{x^{11}}{\sqrt{(a^2-x^2)}}\ sin^{-1}(bx)\ dx = \frac{1}{2401245b^{11}}[2(ab-1)(ab+1)(443520a^{10}b^{10}+169304a^8b^8+$$
$$+108693a^6b^6+78081a^4b^4+57344a^2b^2+40320)K(a^2b^2)+(653344a^{10}b^{10}+185512a^8b^8+$$
$$+110241a^6b^6+83698a^4b^4+74368a^2b^2+80640)E(a^2b^2)]. \tag{2.10}$$

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