Some Fractional Variational Problems Involving Caputo Derivatives

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Abstract

In this paper, we study some fractional variational problems with functionals that involve some unknown functions and their Caputo derivatives. We also consider Caputo iso-perimetric problems. Generalized fractional Euler-Lagrange equations for the problems are presented. Furthermore, we study the optimality conditions for functionals depending on the unknown functions and the optimal time T. In addition, some examples are discussed.

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AMS (MOS) Subject Classification. 49K10, 26A33, 26B20.

1 Introduction

The fractional differential equations theory arises in many engineering and scientific disciplines such as mechanics, physics, chemistry, biology, economics, control theory and signal processing. For instance see, [15, 16, 17, 18, 21, 23, 25, 27].

On the other hand, the calculus of variations can be considered as an optimization branch, it is concerned with finding extrema. The calculus of variations has many applications in classical mechanics, economics, electrical engineering, urban planning and other fields. For more information and applications, see [12, 13, 24, 28, 29], and we refer the reader interested in the calculus of variations theory to [12, 20]. The fractional calculus of variations has exhausted the attention of some authors and an important research papers have been obtained in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 14, 19, 22, 26, 30].

This paper is devoted to establish necessary conditions of Euler-Lagrange type for the following fractional variational problem:

Find functions $(y_j)_{j=1,2,...,m} \in C^1([a,b])$ that maximize or minimize the functional:

$$J(y_1,...,y_m) = \int_a^b L\big(x, y_1(x), ..., y_m(x), {}_a^C D_x^{\alpha_1} y_1(x), \\ ..., {}_a^C D_x^{\alpha_m} y_m(x), {}_x^C D_b^{\beta_1} y_1(x), ..., {}_x^C D_b^{\beta_m} y_m(x)\big)dx,$$

subject to the boundary conditions:

$$y_j(a) = y_{ja},\ y_j(b) = y_{jb},$$

where $(\alpha_j)_{j=1,2,...,m}, (\beta_j)_{j=1,2,...,m}$ are in $(0,1)$.

Then, we will consider the fractional iso-perimetric problem that depends on maximizing or minimizing the above functional subject to the given boundary conditions and the fractional integral constraint:

$$\begin{aligned} I(y_1,...,y_m) &= \int_a^b F\big(x, y_1(x), ..., y_m(x), {}_a^C D_x^{\alpha_1} y_1(x), ..., {}_a^C D_x^{\alpha_m} y_m(x), \\ &\quad {}_x^C D_b^{\beta_1} y_1(x), ..., {}_x^C D_b^{\beta_m} y_m(x)\big) dx \\ &= r, \end{aligned}$$

where $r \in \mathbb{R}$.

Finally, we will study the optimality conditions for a pair functions-time $\left((y_j)_{j=1,\dots,m}, T\right) \in \left\{\left(C^1([a,b]), [a,b]\right) : y_j(a) = y_{ja}\right.$ to be an optimal solution to:

$$J'(y_1,...,y_m,T) = \int_a^T L^*\big(x, y_1(x), ..., y_m(x), {}_a^C D_x^{\alpha_1} y_1(x), ..., {}_a^C D_x^{\alpha_m} y_m(x)\big) dx.$$

2 Fractional Calculus

In this section, we recall some basic definitions which are used throughout this paper, [17, 21, 23].

Definition 1 *The left and right Riemann-Liouville fractional integral operators of order* $\alpha > 0$ *for an integrable function* y *on* $[a,b]$ *are defined respectively by:*

$$\begin{aligned} {}_a I_x^{\alpha} y(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} y(t)\, dt, \\ \text{and} \quad {}_x I_b^{\alpha} y(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} y(t)\, dt, \end{aligned} \tag{1}$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-u} u^{\alpha-1} du$.

Definition 2 *The left and right Riemann-Liouville fractional derivative operators of order* $\alpha > 0$ *for a function* $y : [a,b] \to \mathbb{R}$, *can be defined respectively, as:*

$$\begin{aligned} {}_a^{RL} D_x^{\alpha} y(x) &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^{(n)} \int_a^x (x-t)^{n-\alpha-1} y(t)\, dt, \\ {}_x^{RL} D_b^{\alpha} y(x) &= \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx}\right)^{(n)} \int_x^b (t-x)^{n-\alpha-1} y(t)\, dt, \end{aligned} \tag{2}$$

for $n-1 < \alpha < n$, $n \in \mathbb{N} - \{0\}$.

Definition 3 *The left and right Caputo fractional derivative operators of order* $\alpha > 0$ *for a function* $y : [a,b] \to \mathbb{R}$, *which is at least* $n-$*times differentiable can be defined respectively, as:*

$$ {}_a^C D_x^{\alpha} y(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} y^{(n)}(t)\, dt, \tag{3}$$

and

$$ {}_{x}^{C}D_{b}^{\alpha}y(x) = \frac{1}{\Gamma(n-\alpha)}\int_{x}^{b}(t-x)^{n-\alpha-1}(-1)^{n}y^{(n)}(t)\,dt, \tag{4} $$

for $n-1<\alpha<n,\ n\in\mathbb{N}-\{0\}$.

Also, we list some well known results in fractional variational calculus theory, [12, 20].

(i) : Assume that $n-1<\alpha<n$, $n\in\mathbb{N}-\{0\}$ and f is of class C^{n} on $[a,b]$. Then its left and right Caputo derivatives of order α are continuous on $[a,b]$.

(ii) : Let $f:[a,b]\to\mathbb{R}$ be a continuous function such that:

$$ \int_{a}^{b}f(x)\,\delta(x)\,dx = 0, \tag{5} $$

holds for every $\delta\in C^{n}([a,b])$, $n\geq 0$, satisfying $\delta(a)=\delta(b)=0$. Then, $f(x)=0$ on $[a,b]$.

(iii) : Let $0<\alpha<1$, and $f,\ g:[a,b]\to\mathbb{R}$ be C^{1} functions. Then,

$$ \int_{a}^{b}g(x)\ {}_{a}^{C}D_{x}^{\alpha}f(x)\,dx = \int_{a}^{b}f(x)\ {}_{x}^{RL}D_{b}^{\alpha}g(x)\,dx+\left[{}_{x}I_{b}^{1-\alpha}g(x)\,f(x)\right]_{a}^{b}, \tag{6} $$

and

$$ \int_{a}^{b}g(x)\ {}_{x}^{C}D_{b}^{\alpha}f(x)\,dx = \int_{a}^{b}f(x)\ {}_{a}^{RL}D_{x}^{\alpha}g(x)\,dx-\left[{}_{a}I_{x}^{1-\alpha}g(x)\,f(x)\right]_{a}^{b}. \tag{7} $$

Remark 4 *If* $f(a)=f(b)=0$, *then*

$$ \int_{a}^{b}g(x)\ {}_{a}^{C}D_{x}^{\alpha}f(x)\,dx = \int_{a}^{b}f(x)\ {}_{x}^{RL}D_{b}^{\alpha}f(x)\,dx, \tag{8} $$

and

$$ \int_{a}^{b}g(x)\ {}_{x}^{C}D_{b}^{\alpha}f(x)\,dx = \int_{a}^{b}f(x)\ {}_{a}^{RL}D_{x}^{\alpha}f(x)\,dx. \tag{9} $$

3 Fractional Euler-Lagrange Equations

In this section, we prove optimality conditions of Euler-Lagrange type for variational problems with functional that contains m unknown functions with their Caputo fractional derivatives.

For $(\alpha_j)_{j=1,2,...,m}$, $(\beta_j)_{j=1,2,...,m} \in (0,1)$, we consider the following problem: find functions $(y_j)_{j=1,2,...,m} \in C^1([a,b])$ that maximize or minimize the functional

$$J(y_1,...,y_m) = \int_a^b L\big(x, y_1(x), ..., y_m(x), {}_a^C D_x^{\alpha_1} y_1(x), \quad ..., {}_a^C D_x^{\alpha_m} y_m(x), {}_x^C D_b^{\beta_1} y_1(x), ..., {}_x^C D_b^{\beta_m} y_m(x)\big) dx, \tag{10}$$

subject to the boundary conditions:

$$y_j(a) = y_{ja}, \; y_j(b) = y_{jb}. \tag{11}$$

For the sake of convenience, we denote by $\partial_j L$, $j = 1, ..., 3m+1$, the partial derivative of the function $L : [a,b] \times \mathbb{R}^{3m} \to \mathbb{R}$ with respect to its jth argument. Also, we assume that:

(H_1) : The function L is of class C^1 on all its arguments.

(H_2) : $(\partial_j L)_{j=m+2,...,2m+1}$ has continuous right Riemann-Liouville fractional derivative of order α_i, $i = 1, 2, ..., m$, respectively.

(H_3) : $(\partial_j L)_{j=2m+2,...,3m+1}$ has continuous left Riemann-Liouville fractional derivative of order β_i, $i = 1, 2, ..., m$, respectively.

Theorem 5 *If $(y_j)_{j=1,2,...,m}$ is an extremum of the functional given by (10)&(11), then for all $x \in [a,b]$, the functions $(y_j)_{j=1,2,...,m}$ satisfy the fractional Euler–Lagrange equations:*

$$\partial_j L + {}_x^{RL} D_b^{\alpha_{j-1}} \partial_{m+j} L + {}_a^{RL} D_x^{\beta_{j-1}} \partial_{2m+j} L = 0, \quad j = 2, ..., m+1. \tag{12}$$

Proof. Let $\epsilon_j > 0$ and define S by:

$$S := \left\{ \delta_j \in C^1([a,b]) : \delta_j(a) = \delta_j(b) = 0 \right\}.$$

Assume that $(y_j)_{j=1,2,...,m}$ is solution of (10)&(11) and $y_j + \epsilon_j \delta_j$ is the variation of y_j. Then,

$$\begin{aligned} J(\epsilon_1, ..., \epsilon_m) &= J(y_1 + \epsilon_1\delta_1, ..., y_m + \epsilon_m\delta_m) \\ &= \int_a^b L\big(x, (y_1 + \epsilon_1\delta_1)(x), ..., (y_m + \epsilon_m\delta_m)(x), \\ &\quad {}_a^C D_x^{\alpha_1} (y_1 + \epsilon_1\delta_1)(x), ..., {}_a^C D_x^{\alpha_m} (y_m + \epsilon_m\delta_m)(x), \\ &\quad {}_x^C D_b^{\beta_1} (y_1 + \epsilon_1\delta_1)(x), ..., {}_x^C D_b^{\beta_m} (y_m + \epsilon_m\delta_m)(x)\big) dx. \end{aligned} \tag{13}$$

Since $J(\epsilon_1, ..., \epsilon_m)$ is a extremum at $(\epsilon_1, ..., \epsilon_m) = (0, ..., 0)$, the necessary optimality condition is:

$$\left(\frac{\partial J}{\partial \epsilon_i}\right)(\epsilon_i = 0) = 0, \quad i = 1, 2, ..., m. \tag{14}$$

On the other hand, the linearity of the operators ${}_a^C D_x^{\alpha_j}$ and ${}_x^C D_b^{\beta_j}$, implies that

$${}_a^C D_x^{\alpha_j}\left(y_j+\epsilon_j\delta_j\right)(x)= {}_a^C D_x^{\alpha_j} y_j+\epsilon_j\, {}_a^C D_x^{\alpha_j}\delta_j\left(x\right),$$

and

$${}_x^C D_b^{\beta_j}\left(y_j+\epsilon_j\delta_j\right)(x)= {}_x^C D_b^{\beta_j} y_j+\epsilon_j\, {}_x^C D_b^{\beta_j}\delta_j\left(x\right).$$

Therefore, for each $j=2,...,m+1$, we obtain:

$$\int_a^b \left[\partial_j L\delta_{j-1}(x)+\partial_{m+j}L\, {}_a^C D_x^{\alpha_{j-1}}\delta_{j-1}(x)+\partial_{2m+j}L\, {}_x^C D_b^{\beta_{j-1}}\delta_{j-1}(x)\right]dx = 0. \tag{15}$$

By Eq. (8) and Eq. (9), we can state that

$$\int_a^b \left[\partial_j L\delta_{j-1}(x)+\delta_{j-1}(x)\ {}_x^{RL} D_b^{\alpha_{j-1}}\partial_{m+j}L+\delta_{j-1}(x)\ {}_a^{RL} D_x^{\beta_{j-1}}\partial_{2m+j}L\right]dx = 0, \tag{16}$$

where $j=2,...,m+1$.

Consequently,

$$\int_a^b \left[\left(\partial_j L+ {}_x^{RL} D_b^{\alpha_{j-1}}\partial_{m+j}L+ {}_a^{RL} D_x^{\beta_{j-1}}\partial_{2m+j}L\right)\delta_{j-1}(x)\right]dx = 0, \quad j=2,...,m+1. \tag{17}$$

Then, thanks to (ii), we conclude that

$$\partial_j L+ {}_x{}^{RL} D_b^{\alpha_{j-1}}\partial_{m+j}L+ {}_a^{RL} D_x^{\beta_{j-1}}\partial_{2m+j}L = 0, \quad j=2,...,m+1. \tag{18}$$

■

Remark 6 *For $\alpha_j=\beta_j=1$, the problem (10) reduces to:*

$$J\left(y_1,...,y_m\right) = \int_a^b L\left(x,y_1(x),...,y_m(x),y_1'(x),...,y_m'(x)\right)dx. \tag{19}$$

Then, the fractional Euler-Lagrange equations given by Eq. (12) reduce to to the classical Euler–Lagrange equations:

$$\partial_j L-\frac{d}{dx}\left(\partial_{m+j}L\right) = 0, \quad j=2,...,m+1, \tag{20}$$

where $\left(\partial_j L\right),\quad j=2,...,m+1$, are the partial derivatives of the function L with respect to its jth argument.

Example 7 *Consider the following fractional variational problem:*

$$\begin{aligned}J\left(y_1,y_2,y_3\right) &= \int_{-1}^1 \Big(2x+y_1^3+y_2^3+y_3^3+2\left({}_{-1}^C D_x^{\frac{4}{5}}y_1\right)^2+3\left({}_{-1}^C D_x^{\frac{5}{6}}y_2\right)^2 \\ &+\left({}_{-1}^C D_x^{\frac{6}{7}}y_3\right)^2- {}_x^C D_1^{\frac{3}{4}}y_1- {}_x^C D_1^{\frac{2}{3}}y_2- {}_x^C D_1^{\frac{1}{2}}y_3\Big)dx,\end{aligned}$$

with the boundary conditions:

$$y_1(-1) = 1, \quad y_1(1) = -1, \quad y_2(-1) = 2\sqrt{2}, \quad y_2(1) = 1, \quad y_3(-1) = 0, \quad y_3(1) = \frac{3}{4}.$$

We have: $m = 3$, $a = -1$, $b = 1$, $y_{1-1} = 1$, $y_{11} = -1$, $y_{2-1} = 2\sqrt{2}$, $y_{21} = 1$, $y_{3-1} = 0$, $y_{31} = \frac{3}{4}$, $\alpha_1 = \frac{4}{5}$, $\alpha_2 = \frac{5}{6}$, $\alpha_3 = \frac{6}{7}$, $\beta_1 = \frac{3}{4}$, $\beta_2 = \frac{2}{3}$, $\beta_3 = \frac{1}{2}$,

and

$$\begin{aligned} L &= 2x + y_1^3 + y_2^3 + y_3^3 + 2\left({}_{-1}^{C}D_x^{\frac{4}{5}} y_1\right)^2 + 3\left({}_{-1}^{C}D_x^{\frac{5}{6}} y_2\right)^2 \\ &\quad + \left({}_{-1}^{C}D_x^{\frac{6}{7}} y_3\right)^2 - {}_{x}^{C}D_1^{\frac{3}{4}} y_1 - {}_{x}^{C}D_1^{\frac{2}{3}} y_2 - {}_{x}^{C}D_1^{\frac{1}{2}} y_3. \end{aligned}$$

Then, the associated fractional Euler–Lagrange equations are:

$$\begin{cases} \partial_2 L + {}_{x}^{RL}D_1^{\frac{4}{5}} \partial_5 L + {}_{-1}^{RL}D_x^{\frac{3}{4}} \partial_8 L = 0, \\ \partial_3 L + {}_{x}^{RL}D_1^{\frac{5}{6}} \partial_6 L + {}_{-1}^{RL}D_x^{\frac{2}{3}} \partial_9 L = 0, \\ \partial_4 L + {}_{x}^{RL}D_1^{\frac{6}{7}} \partial_7 L + {}_{-1}^{RL}D_x^{\frac{1}{2}} \partial_{10} L = 0. \end{cases}$$

Hence,

$$\begin{cases} 3y_1^2 + 4\, {}_{x}^{RL}D_1^{\frac{4}{5}} \left({}_{-1}^{C}D_x^{\frac{4}{5}} y_1\right) - {}_{-1}^{RL}D_x^{\frac{3}{4}}(1) = 0, \\ 3y_2^2 + 6\, {}_{x}^{RL}D_1^{\frac{5}{6}} \left({}_{-1}^{C}D_x^{\frac{5}{6}} y_2\right) - {}_{-1}^{RL}D_x^{\frac{2}{3}}(1) = 0, \\ 3y_3^2 + 2\, {}_{x}^{RL}D_1^{\frac{6}{7}} \left({}_{-1}^{C}D_x^{\frac{6}{7}} y_3\right) - {}_{-1}^{RL}D_x^{\frac{1}{2}}(1) = 0. \end{cases}$$

4 Fractional Iso-perimetric Problem

In this section, we study the extremum of the functional given by (10)&(11), subject to the following fractional integral constraint:

$$\begin{aligned} I(y_1, ..., y_m) &= \int_a^b F\big(x, y_1(x), ..., y_m(x), {}_{a}^{C}D_x^{\alpha_1} y_1(x), ..., {}_{a}^{C}D_x^{\alpha_m} y_m(x), \\ &\quad {}_{x}^{C}D_b^{\beta_1} y_1(x), ..., {}_{x}^{C}D_b^{\beta_m} y_m(x)\big)dx \\ &= r \end{aligned} \tag{21}$$

where $r \in \mathbb{R}$.

As before, we denote by $\partial_j F$, $j = 1, ..., 3m+1$ the partial derivative of the function $F : [a, b] \times \mathbb{R}^{3m} \to \mathbb{R}$ with respect to its jth argument.

We also assume that:

(H_4) : The function F is of class C^1 on all its arguments.

$(H_5) : (\partial_j F)_{j=m+2,...,2m+1}$ has continuous right Riemann-Liouville fractional derivative of order α_i, $i = 1, 2, ..., m$, respectively.

$(H_6) : (\partial_j F)_{j=2m+2,...,3m+1}$ has continuous left Riemann-Liouville fractional derivative of order β_i, $i = 1, 2, ..., m$, respectively.

Theorem 8 *Assume that $(y_j)_{j=1,2,...,m}$ is an extremum of the functional given by (10)&(11), such that $I(y_1, ..., y_m) = r$. If $(y_j)_{j=1,2,...,m}$ is not an extremum of I, then there exists a constant μ satisfaying*

$$\partial_j E + {}_x^{RL}D_b^{\alpha_{j-1}} \partial_{m+j} E + {}_a^{RL}D_x^{\beta_{j-1}} \partial_{2m+j} E = 0, \quad j = 2, ..., m+1. \tag{22}$$

for all $x \in [a, b]$, where $E = L + \mu F$.

Proof. Assume that $(y_j)_{j=1,2,...,m}$ is a solution of (10)&(11), such that $I(y_1, ..., y_m) = r$, and $(y_j)_{j=1,2,...,m}$ is not an extremum of I.

Let $\epsilon_j^1, \epsilon_j^2 > 0$, $\delta_j^1, \delta_j^2 \in S$, and $y_j + \epsilon_j^1 \delta_j^1 + \epsilon_j^2 \delta_j^2$ be the variation of y_j, such taht $\delta_j^1, \delta_j^2 \in S$. Then,

$$\begin{aligned} J^*\left(\epsilon_1^1, ..., \epsilon_m^1, \epsilon_1^2, ..., \epsilon_m^2\right) &= J\left(y_1 + \epsilon_1^1\delta_1^1 + \epsilon_1^2\delta_1^2, ..., y_m + \epsilon_m^1\delta_m^1 + \epsilon_m^2\delta_m^2\right) \\ &= \int_a^b L\big(x, \left(y_1 + \epsilon_1^1\delta_1^1 + \epsilon_1^2\delta_1^2\right)(x), ..., \\ &\quad \left(y_m + \epsilon_m^1\delta_m^1 + \epsilon_m^2\delta_m^2\right)(x), {}_a^C D_x^{\alpha_1}\left(y_1 + \epsilon_1^1\delta_1^1 + \epsilon_1^2\delta_1^2\right)(x), \\ &\quad ..., {}_a^C D_x^{\alpha_m}\left(y_m + \epsilon_m^1\delta_m^1 + \epsilon_m^2\delta_m^2\right)(x), {}_x^C D_b^{\beta_1}\left(y_1 + \epsilon_1^1\delta_1^1 + \epsilon_1^2\delta_1^2\right)(x), \\ &\quad ..., {}_x^C D_b^{\beta_m}\left(y_m + \epsilon_m^1\delta_m^1 + \epsilon_m^2\delta_m^2\right)(x)\big)dx, \end{aligned} \tag{23}$$

and

$$\begin{aligned} I^*\left(\epsilon_1^1, ..., \epsilon_m^1, \epsilon_1^2, ..., \epsilon_m^2\right) &= I\left(y_1 + \epsilon_1^1\delta_1^1 + \epsilon_1^2\delta_1^2, ..., y_m + \epsilon_m^1\delta_m^1 + \epsilon_m^2\delta_m^2\right) \\ &= \int_a^b F\big(x, \left(y_1 + \epsilon_1^1\delta_1^1 + \epsilon_1^2\delta_1^2\right)(x), ..., \\ &\quad \left(y_m + \epsilon_m^1\delta_m^1 + \epsilon_m^2\delta_m^2\right)(x), {}_a^C D_x^{\alpha_1}\left(y_1 + \epsilon_1^1\delta_1^1 + \epsilon_1^2\delta_1^2\right)(x), \\ &\quad ..., {}_a^C D_x^{\alpha_m}\left(y_m + \epsilon_m^1\delta_m^1 + \epsilon_m^2\delta_m^2\right)(x), {}_x^C D_b^{\beta_1}\left(y_1 + \epsilon_1^1\delta_1^1 + \epsilon_1^2\delta_1^2\right)(x), \\ &\quad ..., {}_x^C D_b^{\beta_m}\left(y_m + \epsilon_m^1\delta_m^1 + \epsilon_m^2\delta_m^2\right)(x)\big)dx - r. \end{aligned} \tag{24}$$

The formula (24) implies that for $\epsilon_{j-1}^1 = \epsilon_{j-1}^2 = 0$, we get $I^*(0, ..., 0) = 0$ and

$$\begin{aligned} \left(\frac{\partial I^*}{\partial \epsilon_{j-1}^2}\right)\left(\epsilon_{j-1}^2 = 0\right) &= \int_a^b \begin{bmatrix} \partial_j F \delta_{j-1}^2(x) + (\partial_{m+j}F)\ {}_a^C D_x^{\alpha_{j-1}} \delta_{j-1}^2(x) \\ + (\partial_{2m+j}F)\ {}_x^C D_b^{\beta_{j-1}} \delta_{j-1}^2(x) \end{bmatrix} dx \\ &= \int_a^b \left[\partial_j F \delta_{j-1}^2(x) + \delta_{j-1}^2(x)\ {}_x^{RL}D_b^{\alpha_{j-1}} \partial_{m+j} F + \delta_{j-1}^2(x)\ {}_a^{RL}D_x^{\beta_{j-1}} \partial_{2m+j} F\right] dx \\ &= \int_a^b \left[\partial_j F + {}_x^{RL}D_b^{\alpha_{j-1}} \partial_{m+j} F + {}_a^{RL}D_x^{\beta_{j-1}} \partial_{2m+j} F\right] \delta_{j-1}^2(x)\, dx, \end{aligned} \tag{25}$$

for all $j = 2, ..., m+1$.

Then, taking into account that $(y_j)_{j=1,2,...,m}$ is not an extremum of I, there exists a family of function $\left(\delta_j^2\right)_{j=1,2,...,m}$, where

$$\left(\frac{\partial I^*}{\partial \epsilon_j^2}\right)\left(\epsilon_1^1 = ... = \epsilon_m^1 = 0, \epsilon_1^2 = ... = \epsilon_m^2 = 0\right) \neq 0, \quad j = 1, 2, ..., m. \tag{26}$$

Since $I^*(0, ..., 0) = 0$, applying implicit function theorem, there exists a family of function $\left(\epsilon_j^2\right)(\cdot)$, $j = 1, 2, ..., m$ defined in a neigh-borhood of zero, where

$$I^*\left(\epsilon_j^1, \epsilon_j^2\left(\epsilon_j^1\right)\right) = 0. \tag{27}$$

Using the Lagrange multiplier rule, there exists a constant μ, where

$$\nabla\left(J^*(0, ..., 0) + \mu I^*(0, ..., 0)\right) = (0, .., 0). \tag{28}$$

It is easy to show that for all $j = 2, ..., m+1$,

$$\left(\frac{\partial J^*}{\partial \epsilon_{j-1}^1}\right)(0, ..., 0) = \int_a^b \left(\partial_j L + {}_x^{RL}D_b^{\alpha_{j-1}}\partial_{m+j}L + {}_a^{RL}D_x^{\beta_{j-1}}\partial_{2m+j}L\right)\delta_{j-1}^1 dx, \tag{29}$$

and

$$\left(\frac{\partial I^*}{\partial \epsilon_{j-1}^1}\right)(0, ..., 0) = \int_a^b \left(\partial_j F + {}_x^{RL}D_b^{\alpha_{j-1}}\partial_{m+j}F + {}_a^{RL}D_x^{\beta_{j-1}}\partial_{2m+j}F\right)\delta_{j-1}^1 dx. \tag{30}$$

Taking $E = L + \mu F$, we obtain

$$\int_a^b \left(\partial_j E + {}_x^{RL}D_b^{\alpha_{j-1}}\partial_{m+j}E + {}_a^{RL}D_x^{\beta_{j-1}}\partial_{2m+j}E\right)\delta_{j-1}^1 dx = 0. \tag{31}$$

By (ii), we can obtain:

$$\partial_j E + {}_x^{RL}D_b^{\alpha_{j-1}}\partial_{m+j}E + {}_a^{RL}D_x^{\beta_{j-1}}\partial_{2m+j}E = 0, \quad j = 2, ..., m+1.$$

■

Now, we present an example to illustrate our theorem.

Example 9 *Consider the following fractional iso-perimetric problem:*

$$J(y_1, y_2) = \int_{-1}^1 \left(x^2 + y_1^2 + y_2^2 + \left({}_{-1}^{C}D_x^{\frac{3}{4}}y_1\right)^2 + \left({}_{-1}^{C}D_x^{\frac{7}{8}}y_2\right)^2 + \left({}_x^{C}D_1^{\frac{4}{5}}y_1\right)^2 + \left({}_x^{C}D_1^{\frac{5}{7}}y_2\right)^2\right)dx,$$

subject to the boundary conditions:

$$y_1(-1) = 0, \quad y_1(1) = \sqrt{2}, \quad y_2(-1) = 1, \quad y_2(1) = \frac{3}{2},$$

and the fractional integral constraint:

$$\begin{aligned} I(y_1, y_2) &= \int_{-1}^{1} \big(x - 2y_1 - y_2 + {}_{-1}^{C}D_x^{\frac{3}{4}} y_1 \\ &\quad + {}_{-1}^{C}D_x^{\frac{7}{8}} y_2 + {}_{x}^{C}D_1^{\frac{4}{5}} y_1 + {}_{x}^{C}D_1^{\frac{5}{7}} y_2\big) dx \\ &= 3. \end{aligned}$$

Here, $m = 2, \quad a = -1, \quad b = 1, \quad y_{1-1} = 0, \quad y_{11} = \sqrt{2}, \quad y_{2-1} = 1, \quad y_{21} = \frac{3}{2}, \quad \alpha_1 = \frac{3}{4}, \quad \alpha_2 = \frac{7}{8}, \quad \beta_1 = \frac{4}{5}, \quad \beta_2 = \frac{5}{7}, \quad r = 3,$

and

$$\begin{aligned} E &= x^2 + y_1^2 + y_2^2 \\ &\quad + \left({}_{-1}^{C}D_x^{\frac{3}{4}} y_1\right)^2 + \left({}_{-1}^{C}D_x^{\frac{7}{8}} y_2\right)^2 + \left({}_{x}^{C}D_1^{\frac{4}{5}} y_1\right)^2 + \left({}_{x}^{C}D_1^{\frac{5}{7}} y_2\right)^2 \\ &\quad + \mu \left(x - 2y_1 - y_2 + {}_{-1}^{C}D_x^{\frac{3}{4}} y_1 + {}_{-1}^{C}D_x^{\frac{7}{8}} y_2 + {}_{x}^{C}D_1^{\frac{4}{5}} y_1 + {}_{x}^{C}D_1^{\frac{5}{7}} y_2\right). \end{aligned}$$

For this problem, the fractional Euler–Lagrange equations (22) are given by:

$$\begin{cases} \partial_2 E + {}_{x}^{RL}D_1^{\frac{3}{4}} \partial_4 E + {}_{-1}^{RL}D_x^{\frac{4}{5}} \partial_6 E = 0, \\ \\ \partial_3 E + {}_{x}^{RL}D_1^{\frac{7}{8}} \partial_5 E + {}_{-1}^{RL}D_x^{\frac{5}{7}} \partial_7 E = 0. \end{cases}$$

Thus, μ *should satisfy the fractional Euler–Lagrange equations:*

$$\begin{cases} 2y_1 - 2\mu + {}_{x}^{RL}D_1^{\frac{3}{4}} \left(2\, {}_{-1}^{C}D_x^{\frac{3}{4}} y_1 + \mu\right) + {}_{-1}^{RL}D_x^{\frac{4}{5}} \left(2\, {}_{x}^{C}D_1^{\frac{4}{5}} y_1 + \mu\right) = 0, \\ \\ 2y_2 - \mu + {}_{x}^{RL}D_1^{\frac{7}{8}} \left(2\, {}_{-1}^{C}D_x^{\frac{7}{8}} y_2 + \mu\right) + {}_{-1}^{RL}D_x^{\frac{5}{7}} \left(2 {}_{x}^{C}D_1^{\frac{5}{7}} y_2 + \mu\right) = 0. \end{cases}$$

5 An Optimal Time Problem

Here, we are interested to find the optimal time T as well as the functions $(y_j)_{j=1,2,...,m}$ that maximize or minimize the functional for the following variational problem:

$$\begin{aligned} J'(y_1, ..., y_m, T) = \int_a^T L^*\big(&x, y_1(x), ..., y_m(x), {}_a^C D_x^{\alpha_1} y_1(x), \\ &..., {}_a^C D_x^{\alpha_m} y_m(x)\big) dx, \end{aligned} \tag{32}$$

where

$$\left((y_j)_{j=1,...,m}, T\right) \in \left\{\left(C^1([a,b]), [a,b]\right) : y_j(a) = y_{ja}\right\}. \tag{33}$$

Theorem 10 *Assume that* $\left((y_j)_{j=1,...,m}, T\right)$ *is an extremum of* $J^{'}$ *defined by* (32)&(33)*. Then, for all* $x \in [a, T]$,

$$\partial_j L^* + {}_x^{RL}D_T^{\alpha_{j-1}} \partial_{m+j} L^* = 0, \quad j = 2, ..., m+1, \tag{34}$$

and the following transversality conditions are satisfied

1. $L^*\left(T, y_1(T), ..., y_m(T), {}_x^C D_T^{\alpha_1} y_1(T), ..., {}_x^C D_T^{\alpha_m} y_m(T)\right) = 0,$

2. ${}_x I_T^{1-\alpha_{j-1}} (\partial_{m+j} L^*)(T) = 0, \quad j = 2, ..., m+1.$

Proof. Let $y_j + \epsilon\delta_j$ be variations of y_j and $T + \epsilon\Delta T$ variation of T, such that $\epsilon > 0, \quad \Delta T \in \mathbb{R}$ and

$$S^* \ : \ = \left\{\delta_j \in C^1([a, b]) : \delta_j(a) = 0, \quad j = 1, ..., m\right\}.$$

Define

$$J^{'}(\epsilon) = J^{'}(y_1 + \epsilon\delta_1, ..., y_m + \epsilon\delta_m, T + \epsilon\Delta T). \tag{35}$$

Let $\left((y_j)_{j=1,...,m}, T\right)$ be an extremum of $J^{'}$. Then, we get

$$\left(\frac{\partial J^{'}}{\partial \epsilon}\right)(\epsilon = 0) = 0, \tag{36}$$

which implies that

$$\begin{aligned} & \int_a^T \left[\partial_j L^* \delta_{j-1}(x) + \partial_{m+j} L^* \ {}_a^C D_T^{\alpha_{j-1}} \delta_{j-1}(x)\right] dx \\ & + \Delta T L^*\left(T, y_1(T), ..., y_m(T), {}_x^C D_T^{\alpha_1} y_1(T), ..., {}_x^C D_T^{\alpha_m} y_m(T)\right) \\ = \ & 0. \end{aligned} \tag{37}$$

where $j = 2, ..., m+1$.

Using Eq. (6) given in (iii), we obtain

$$\begin{aligned} & \int_a^T \left[\partial_j L^* \delta_{j-1}(x) + \delta_{j-1}(x) \ {}_x^{RL}D_T^{\alpha_{j-1}} \partial_{m+j} L^*\right] dx + {}_x I_T^{1-\alpha_{j-1}} (\partial_{m+j} L^*)(T) \delta_{j-1}(T) \\ & + \Delta T L^*\left(T, y_1(T), ..., y_m(T), {}_x^C D_T^{\alpha_1} y_1(T), ..., {}_x^C D_T^{\alpha_m} y_m(T)\right) \\ = \ & \int_a^T \left[\partial_j L^* + {}_x^{RL}D_T^{\alpha_{j-1}} \partial_{m+j} L^*\right] \delta_{j-1}(x) dx + {}_x I_T^{1-\alpha_{j-1}} (\partial_{m+j} L^*)(T) \delta_{j-1}(T) \\ & + \Delta T L^*\left(T, y_1(T), ..., y_m(T), {}_x^C D_T^{\alpha_1} y_1(T), ..., {}_x^C D_T^{\alpha_m} y_m(T)\right) \\ = \ & 0. \end{aligned} \tag{38}$$

By fixing $\delta_{j-1} \equiv 0$, $j = 2, ..., m+1$, and by the arbitrariness of ΔT, we obtain the first transversality condition:

$$L^* \left(T, y_1(T), ..., y_m(T), {}_x^C D_T^{\alpha_1} y_1(T), ..., {}_x^C D_T^{\alpha_m} y_m(T)\right) = 0.$$

Then, the second transversality condition is proved by choosing $\delta_{j-1}$is zero on $[a, T)$ and $\delta_{j-1}(T) \neq 0$, $\quad j = 2, ..., m+1$. If $\delta_{j-1}$is free on $[a, T)$ and $\delta_{j-1}(T) = 0$, $\quad j = 2, ..., m+1$, we obtain the Euler-Lagrange equation (34). This completes the proof. ■

Example 11 *Let us consider the following optimal time problem:*

$$J'(y_1, y_2) = \int_{-2}^{T} (2x^3 - \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 - 2\left({}_{-2}^C D_x^{\frac{2}{3}} y_1\right)^2 + 2\left({}_{-2}^C D_x^{\frac{3}{4}} y_2\right)^2) dx,$$

with the boundary conditions:

$$y_1(-2) = 1, \quad y_2(-2) = -1.$$

We have: $m = 2, \quad a = -2, \quad b = 2, \quad y_{1-2} = 1, \quad y_{2-2} = -1, \quad \alpha_1 = \frac{2}{3}, \quad \alpha_2 = \frac{3}{4},$

and

$$L^* = 2x^3 - \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 - 2\left({}_{-2}^C D_x^{\frac{2}{3}} y_1\right)^2 + 2\left({}_{-2}^C D_x^{\frac{3}{4}} y_2\right)^2.$$

Then, the fractional Euler–Lagrange equations associated are:

$$\begin{cases} \partial_2 L^* + {}_x^{RL} D_T^{\frac{2}{3}} \partial_4 L^* = 0, \\ \\ \partial_3 L^* + {}_x^{RL} D_T^{\frac{3}{4}} \partial_5 L^* = 0, \end{cases}$$

which implies that

$$\begin{cases} -y_1 - 4\, {}_x^{RL} D_T^{\frac{2}{3}} \left({}_{-2}^C D_x^{\frac{2}{3}} y_1\right) = 0, \\ \\ y_2 + 4\, {}_x^{RL} D_T^{\frac{3}{4}} \left({}_{-2}^C D_x^{\frac{3}{4}} y_2\right) = 0. \end{cases}$$

And the the transversality conditions are:

$$\begin{cases} 2T^3 - \frac{1}{2}y_1^2(T) + \frac{1}{2}y_2^2(T) - 2\left({}_{-2}^C D_T^{\frac{2}{3}} y_1\right)^2(T) + 2\left({}_{-2}^C D_T^{\frac{3}{4}} y_2\right)^2(T) = 0, \\ -4\, {}_x I_T^{\frac{1}{3}} \left({}_{-2}^C D_x^{\frac{2}{3}} y_1\right)(T) = 0, \\ 4\, {}_x I_T^{\frac{1}{4}} \left({}_{-2}^C D_x^{\frac{3}{4}} y_2\right)(T) = 0. \end{cases}$$

6 Conclusion

We consider the following problem: find functions $(y_j)_{j=1,2,...,m} \in C^1([a, b])$ that maximize or minimize the functional

$$J(y_1, ..., y_m) = \int_a^b L\big(x, y_1(x), ..., y_m(x), {}_a^C D_x^{\alpha_1} y_1(x), \\ ..., {}_a^C D_x^{\alpha_m} y_m(x), {}_x^C D_b^{\beta_1} y_1(x), ..., {}_x^C D_b^{\beta_m} y_m(x)\big) dx,$$

subject to the boundary conditions:

$$y_j(a) = y_{ja}, \; y_j(b) = y_{jb}.$$

First, we conclude that the solution $(y_j)_{j=1,2,...,m}$ of the given fractional variational problem, satisfy the fractional Euler–Lagrange equations:

$$\partial_j L + {}_x^{RL}D_b^{\alpha_{j-1}}\partial_{m+j}L + {}_a^{RL}D_x^{\beta_{j-1}}\partial_{2m+j}L = 0, \quad j = 2, ..., m+1.$$

Then, we consider the fractional iso-perimetric problem that depends on maximizing or minimizing the above functional subject to the given boundary conditions and the fractional integral constraint:

$$\begin{aligned} I(y_1, ..., y_m) &= \int_a^b F\Big(x, y_1(x), ..., y_m(x), {}_a^C D_x^{\alpha_1} y_1(x), ..., {}_a^C D_x^{\alpha_m} y_m(x), \\ &\quad {}_x^C D_b^{\beta_1} y_1(x), ..., {}_x^C D_b^{\beta_m} y_m(x)\Big)dx \\ &= r. \end{aligned}$$

Second, we conclude that if the solution $(y_j)_{j=1,2,...,m}$ is not an extremum of I, then there exists a constant μ satisfaying

$$\partial_j E + {}_x^{RL}D_b^{\alpha_{j-1}}\partial_{m+j}E + {}_a^{RL}D_x^{\beta_{j-1}}\partial_{2m+j}E = 0, \quad j = 2, ..., m+1.$$

for all $x \in [a, b]$, where $E = L + \mu F$.

Finally, we state that the optimal solution pair functions-time $\Big((y_j)_{j=1,...,m}, T\Big) \in \big\{\big(C^1([a,b]), [a,b]\big) : y_j(a) = y_{ja}\big\}$ of

$$J'(y_1, ..., y_m, T) = \int_a^T L^*\Big(x, y_1(x), ..., y_m(x), {}_a^C D_x^{\alpha_1} y_1(x), ..., {}_a^C D_x^{\alpha_m} y_m(x)\Big)dx.$$

satisfy, for all $x \in [a, T]$,

$$\partial_j L^* + {}_x^{RL}D_T^{\alpha_{j-1}}\partial_{m+j}L^* = 0, \quad j = 2, ..., m+1,$$

and the following transversality conditions:

1. $L^*\big(T, y_1(T), ..., y_m(T), {}_x^C D_T^{\alpha_1} y_1(T), ..., {}_x^C D_T^{\alpha_m} y_m(T)\big) = 0,$

2. ${}_x I_T^{1-\alpha_{j-1}}(\partial_{m+j}L^*)(T) = 0, \quad j = 2, ..., m+1.$

For the future, we will study an isoperimetric problem with multiple constraints.

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