# Some new results on the curvatures of the spherical indicatrices of the involutes of a spacelike curve with a spacelike binormal in Minkowski 3-space

Mustafa Bilici[1*], Mustafa Çalışkan[2]

[1*]Ondokuz Mayıs University, Educational Faculty, Mathematics and Science Education, Samsun, Turkey

[2]Gazi University, Faculty of Sciences, Department of Mathematics, Ankara, Turkey

mbilici@omu.edu.tr, mustafacaliskan@gazi.edu.tr

**Abstract**

In the present paper, we dealt with the spherical indicatrices of involutes of a given spacelike curve with spacelike binormal. Then it was calculated relationships between arc lengths and geodesic curvatures of the these indicatrices in Minkowski 3-space. In addition, some interesting results were achieved in the event that the evolute curve was a helix.

**Keywords:** Minkowski space, involute curve, spherical indicatrices, geodesic curvature, arc length

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## 1. Introduction

For many years, the subject of curves will continue to draw attention as an interesting subject of differential geometry. Undoubtedly, one of them is the involute-evolute curve couple, which is well known both in the Euclidean and Minkowski space. (see [1-9]). Bükcü and Karacan [6] investigated the involutes of the spacelike curve with a spacelike binormal $\alpha$ in Minkowski 3-space. It was seen that the involute curve $\beta$ must be a time like curve. Let the Frenet-Serret frames of the curves $\alpha$ and $\beta$ be {$T$, $N$, $B$} and {$T^*$, $N^*$, $B^*$}, respectively. More specifically, the causal characteristics of the Frenet frames of the curves $\alpha$ and $\beta$ are { $T$ spacelike, $N$ time like, $B$ spacelike } and { $T^*$ time like, $N^*$ spacelike, $B^*$ spacelike }. Also transformation matrix between the Frenet frames of the curve couple $(\beta,\alpha)$ have been found as depend on curvatures of the evolute curve $\alpha$. On the other hand, In [5], Bilici and Çalışkan have obtained the relationships between the Frenet frames of the spacelike curve $\alpha$ and the time like involute curve $\beta$ as depend on Lorentzian spacelike angle $\theta$ $(0<\theta<\pi)$ between the unit binormal vector and the Darboux vector of spacelike curve $\alpha$ in the Minkowski 3-space. Furthermore, some new characterizations with relation to the involute-evolute curve couple have been given. Recently Bilici and Çalışkan [4] have computed the curvatures of the spherical indicatrices of the involutes of a given time like curve in Minkowski 3-space.

In this study, we carry tangents of the time like involute with a spacelike binormal to the center of the unit pseudosphere $H_0^2$ and we obtain a spacelike curve $\left(T^*\right)$ with equation $\beta_{T^*}=T^*$ on $H_0^2$. This curve is called the first spherical indicatrix or tangent indicatrix of the involute curve. Similarly one consider the principal normal indicatrix $\left(N^*\right)$ and the binormal indicatrix $\left(B^*\right)$ on the unit pseudosphere $S_1^2$. Thus, some preliminary results are expressed by calculating the arc lengths and geodesic curvatures of the involute-evolute curve couple.

## Preliminaries

Lorentzian inner product in $IR^3$ can be written as

$$g(X,X) = -x_1^2 + x_2^2 + x_3^2,$$

where $X = (x_1, x_2, x_3) \in IR^3$. A vector $X$ is said to be time like if $g(X,X) < 0$, space like if $g(X,X) > 0$ and light like (or null) if $g(X,X) = 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in $IR_1^3$ where $s$ is a pseudo-arclength parameter, can locally be time like spacelike or null (light like), if all of its velocity vectors $\alpha(s)$ are respectively time like, space like or null. The norm of a vector $X$ is defined by

$$\|X\| = \sqrt{|g(X,X)|}$$

and two vectors $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3) \in IR_1^3$ are orthogonal if and only if $g(X,Y) = 0$.

Now let $X$ and $Y$ be two vectors in $IR_1^3$, then the Lorentzian cross product is given by

$$X \times Y = \begin{vmatrix} -e_1 & -e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_3y_2 - x_2y_3, x_1y_3 - x_3y_1, x_1y_2 - x_2y_1), [10].$$

We denote by {*T*, *N*, *B*} the moving Frenet frame along the curve $\alpha$. Then *T*, *N* and *B* are the tangent, the principal normal and the binormal vector of the curve $\alpha$, respectively. Depending on the causal character of the curve $\alpha$, we have the following Frenet formulas and instantaneous rotation vector:

Let $\alpha$ be a unit speed spacelike space curve with a spacelike binormal. In this trihedron, we assume that *T* and *B* spacelike vectors and *N* time like vector. For these vectors, we can write

$$T \times N = -B \ , \ N \times B = -T \ , \ B \times T = N.$$

Depending on the causal character of the curve $\alpha$, the following Frenet-Serret formulas are given in [11].

$$\begin{cases} T' = \kappa N, \quad N' = \kappa T + \tau B, \quad B' = \tau N \\ g(T,T) = g(B,B) = 1, \ g(N,N) = -1, \ g(T,N) = g(N,B) = g(B,T) = 0 \end{cases}$$

The space like Darboux vector for the space like curve $\alpha$ is given by

$$\omega = -\tau T + \kappa B, [12].$$

For the curve $\alpha$, $\theta$ being a Lorentzian space like angle between the B and the $\omega$, we can write

$$\begin{cases} \kappa = \|\omega\| \cos\theta \\ \tau = \|\omega\| \sin\theta \end{cases} , \quad g(\omega,\omega) = \|\omega\|^2 = \kappa^2 + \tau^2.$$

and the unit vector $C$ of direction $\omega$ is $C = -\sin\theta T + \cos\theta B$

**Remark 1.** We can easily see from above equation that $\frac{\tau}{\kappa} = \tan\theta$. In that case, if $\theta = \text{constant}$ then $\alpha$ is a general helix.

For the arc length of the spherical indicatrix of $(C)$ we get

$$s_C = \int_0^s \left\| \frac{dC}{ds} \right\| ds = \int_0^s |\theta'|\, ds,$$

After some calculations, we have for the arc lengths of the spherical indicatrices $(T),(N),(B)$ measured from the points corresponding to $s=0$

$$s_T = \int_0^s |\kappa| ds,\ s_N = \int_0^s \|\omega\| ds,\ s_B = \int_0^s |\tau| ds$$

for their geodesic curvatures with respect to $IR_1^3$

$$k_T = \frac{1}{\cos\theta},\ k_N = \frac{1}{\|\omega\|}\sqrt{\left|\theta'^2 - \|\omega\|^2\right|},\ k_B = \frac{1}{\sin\theta},\ k_C = \sqrt{\left|1 - \left(\frac{\|\omega\|}{\theta'}\right)^2\right|}$$

and for their geodesic curvatures with respect to $S_1^2$ or $H_0^2$

$$\gamma_T = \left\| \bar{\nabla}_{t_T} t_T \right\| = \tan\theta,\ \gamma_N = \left\| \bar{\bar{\nabla}}_{t_N} t_N \right\| = \frac{\theta'}{\|\omega\|},\ \gamma_B = \left\| \bar{\nabla}_{t_B} t_B \right\| = \cot\theta,\ \gamma_C = \left\| \bar{\nabla}_{t_C} t_C \right\| = \frac{\|\omega\|}{\theta'}$$ , [2].

Note that $\bar{\nabla}$ and $\bar{\bar{\nabla}}$ are Levi-Civita connections on $S_1^2$ and $H_0^2$, respectively. Then Gauss equations are

given by the followings.

$$\nabla_X Y = \bar{\nabla}_X Y + \varepsilon g(S(X),Y)\xi,\ \nabla_X Y = \bar{\bar{\nabla}}_X Y + \varepsilon g(S(X),Y)\xi,\ \varepsilon = g(\xi,\xi),$$

where $\xi$ is a unit normal vector field and $S$ is the shape operator of $S_1^2$ (or $H_0^2$).

The unit pseudosphere and pseudohyperbolic space of radius 1 and center 0 in $IR_1^3$ are given by

$$S_1^2 = \left\{ X = (x_1, x_2, x_3) \in IR_1^3 : g(X,X) = 1 \right\}$$

and

$$H_0^2 = \left\{ X = (x_1, x_2, x_3) \in IR_1^3 : g(X,X) = -1 \right\}$$

respectively, [10].

**Definition 1.** Let $\alpha = \alpha(s)$, $\beta = \beta(s^*) \subset IR_1^3$ be two curves. Let Frenet frames of $\alpha$ and $\beta$ be $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$, respectively. $\beta$ is called the involute of $\alpha$ ($\alpha$ is called the evolute of $\beta$) if

$$g(T, T^*) = 0,$$ [2].

It should be noted that according to reference [6], if the evolute $\alpha$ is a unit speed spacelike curve with spacelike binormal then the involute curve $\beta$ is a time like curve in $IR_1^3$.

**Lemma 1.** Let $(\alpha,\beta)$ be the involute-evolute curve couple. The Frenet vectors of the curve couple as follow:

$$\begin{bmatrix} T^* \\ N^* \\ B^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\cos\theta & 0 & -\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \text{ [5].}$$

**3. Some new results on the curvatures of the spherical indicatrices of the involutes of a spacelike curve with a spacelike binormal**

In this section, we compute the arc-lengths of the spherical indicatrix curves $(T^*),(N^*),(B^*)$ and then we calculate the geodesic curvatures of these curves in $IR_1^3$ and $S_1^2$ (or $H_0^2$).

Firstly, for the arc-length $s_{T^*}$ of tagent indicatrix $(T^*)$ of the involute curve $\beta$, we can write

$$s_{T^*} = \int_0^{s^*} \left\| \frac{dT^*}{ds^*} \right\| ds^* = \int_0^{s} \left\| \frac{dN}{ds} \right\| ds,$$

$$s_{T^*} = \int_0^{s} \sqrt{\kappa^2 + \tau^2}\, ds$$

$$s_{T^*} = \int_0^{s} \|\omega\|\, ds.$$

If the arc length for the principal normal indicatrix $(N^*)$ is $s_{N^*}$ it is

$$s_{N^*} = \int_0^{s} \left\| \frac{dN^*}{ds} \right\| ds = \int_0^{s} \left\| \frac{d\left(-\cos\theta\, T + \sin\theta B\right)}{ds} \right\| ds,$$

$$s_{N^*} = \int_0^{s} \sqrt{\left|\theta'^2 - \|\omega\|^2\right|}\, ds.$$

If the arc length for the binormal indicatrix $(B^*)$ is $s_{B^*}$ it is

$$s_{B^*} = \int_0^{s} \left\| \frac{dB^*}{ds} \right\| ds = \int_0^{s} \left\| \frac{d\left(-\sin\theta\, T + \cos\theta\, B\right)}{ds} \right\| ds,$$

$$s_{B^*} = \int_0^{s} |\theta'|\, ds.$$

Thus we can give the following corollaries:

***Corollary 3.1.*** *For the arc length of the tagent indicatrix* $(T^*)$ *of the involute of a time like curve, it is obvious that*

$$s_{T^*} = s_N .$$

***Corollary 3.2.*** *If the evolute curve* $\alpha$ *is a helix then for the arc-length of the principal normal indicatrix* $(N^*)$*, we can write*

$$s_{N^*} = s_N .$$

***Corollary 3.3.*** *For the arc length of the binormal indicatrix* $(B^*)$ *of the involute of a time like curve, we have*

$$s_{B^*} = s_C .$$

Now let us compute the geodesic curvatures of the spherical indicatrices $(T^*),(N^*),(B^*)$ with respect to $IR_1^3$. For the geodesic curvature $k_{T^*}$ of the tangent indicatrix $(T^*)$ of the curve $\beta$, we can write

$$k_{T^*} = \left\| \nabla_{t_{T^*}} t_{T^*} \right\|. \tag{1}$$

Differentiating the curve $\beta_{T^*}(s_{T^*}) = T^*(s)$ with the respect to $s_{T^*}$ and normalizing, we obtain

$$t_{T^*} = \cos\theta T + \sin\theta B. \tag{2}$$

By taking derivative of the last equation we have

$$\nabla_{t_{T^*}} t_{T^*} = \left(-\theta' \sin\theta T + \|\omega\| N + \theta' \cos\theta B\right)\frac{1}{\|\omega\|}. \tag{3}$$

By substituting (3) into the Eq. (1) we get

$$k_{T^*} = \frac{1}{\|\omega\|}\sqrt{\left|\theta'^2 - \|\omega\|^2\right|}. \tag{4}$$

From $k_T = \dfrac{1}{\cos\theta}$ we have $\theta' = \dfrac{k_T'}{k_T\sqrt{k_T^2 - 1}}$. If we set $\theta'$ in the Eq. (4) then we have

$$k_{T^*} = \sqrt{\left|\frac{k_T'^2}{k_T^2\left(k_T^2 - 1\right)\|\omega\|^2} - 1\right|}. \tag{5}$$

**Corollary 3.4.** If the evolute curve $\alpha$ is a helix then we have for the geodesic curvature of the tangent indicatrix $(T^*)$ of the involute curve $\beta$

$$k_{T^*} = 1.$$

Similarly, by differentiating the curve $\beta_{N^*}(s_{N^*}) = N^*(s)$ with the respect to $s_{N^*}$ and by normalizing we obtain

$$t_{N^*} = -\sigma \sin\theta T + \frac{1}{k_N} N + \sigma \cos\theta B, \quad \left(\sigma = \frac{\gamma_N}{k_N}\right). \tag{6}$$

By taking derivative of the last equation and using the definition of geodesic curvature, we have

$$\nabla_{t_{N^*}} t_{N^*} = \left[\left(-\sigma' \sin\theta - \theta'\sigma \cos\theta + \frac{\kappa}{k_N}\right)T - \left(\frac{k_N'}{k_N^{\,2}}\right)N + \left(\sigma' \cos\theta - \varphi'\sigma \sin\theta + \frac{\tau}{k_N}\right)B\right]\frac{1}{\|\omega\| k_N}. \tag{7}$$

$$k_{N^*} = \frac{1}{\|\omega\| k_N}\sqrt{\sigma'^2 + \left(\theta'\sigma - \frac{\|\omega\|}{k_N}\right)^2 - \frac{k_N'^2}{k_N^{\,4}}}. \tag{8}$$

**Corollary 3.5.** If the evolute curve $\alpha$ is a helix then we have for the geodesic curvature of the principal normal indicatrix $(N^*)$ of the involute curve $\beta$

$$k_{N^*} = 1.$$

By differentiating the curve $\beta_{B^*}(s_{B^*}) = B^*(s)$ with the respect to $s_{B^*}$ and by normalizing we obtain

$$t_{B^*} = -\cos\theta\, T - \sin\theta\, B. \tag{9}$$

By taking derivative of the last equation

$$\nabla_{t_{B^*}} t_{B^*} = \sin\theta\, T - \frac{\|\omega\|}{\theta'} N - \cos\theta\, B, \tag{10}$$

and by taking the norm of the last equation, we obtain

$$k_{B^*} = \sqrt{1 - \left(\frac{\|\omega\|}{\theta'}\right)^2}. \tag{11}$$

From $k_B = \frac{1}{\sin\theta}$ we have $\theta' = -\frac{k_B'}{k_B\sqrt{k_B^{\,2} - 1}}$. If we set $\theta'$ in the Eq. (11) then we have

$$k_{B^*} = \sqrt{1 - \frac{\|\omega\|^2 k_B^{\,2}\left(k_B^{\,2} - 1\right)}{k_B'^2}}. \tag{12}$$

**Corollary 3.6.** If the evolute curve $\alpha$ is a helix then the geodesic curvature $k_{B^*}$ of the binormal indicatrix $(B^*)$ of the involute curve $\beta$ is undefined.

Now let us compute the geodesic curvatures of the spherical indicatrices $\left(T^*\right),\left(N^*\right),\left(B^*\right)$ with respect $S_1^2$ or $H_0^2$

For the geodesic curvature $\gamma_{T^*}$ of the tangent indicatrix $\left(T^*\right)$ of the curve $\beta$ with respect to $H_0^2$, we can write

$$\gamma_{T^*} = \left\| \overline{\overline{\nabla}}_{t_{T^*}} t_{T^*} \right\|. \tag{13}$$

From the Gauss equation we can write

$$\nabla_{t_{T^*}} t_{T^*} = \overline{\overline{\nabla}}_{t_{T^*}} t_{T^*} + \varepsilon\, g\left(S\left(t_{T^*}\right), t_{T^*}\right) T^*, \tag{14}$$

where $\varepsilon = g\left(T^*, T^*\right) = -1$, $S\left(t_{T^*}\right) = -t_{T^*}$ and $g\left(S\left(t_{T^*}\right), t_{T^*}\right) = -1$. From the Eq. (3) and (14), it follows that

$$\overline{\overline{\nabla}}_{t_{T^*}} t_{T^*} = -\frac{\theta'}{\|\omega\|} \sin\theta\, T + \frac{\theta'}{\|\omega\|} \cos\theta\, B. \tag{15}$$

Substituting (15) in the Eq. (13), we obtain

$$\gamma_{T^*} = \frac{\theta'}{\|\omega\|}. \tag{16}$$

By using $\gamma_T = \tan\theta$, we obtain following relationship between $\gamma_T$ and $\gamma_{T^*}$:

$$\gamma_{T^*} = \frac{1}{\|\omega\|}\left(\frac{\gamma_T{}'}{1+\gamma_T{}^2}\right). \tag{17}$$

**Corollary 3.7.** If the evolute curve $\alpha$ is a helix then we have for the geodesic curvature of the tangent indicatrix $\left(T^*\right)$ of the involute curve $\beta$

$$\gamma_{T^*} = 0$$

For the geodesic curvature $\gamma_{N^*}$ of the principal normal indicatrix $\left(N^*\right)$ of the curve $\beta$ with respect to $S_1^2$, we can write

$$\gamma_{N^*} = \left\| \overline{\nabla}_{t_{N^*}} t_{N^*} \right\|. \tag{18}$$

Using the Gauss equation and the Eq. (7), we can write

$$\bar{\nabla}_{t_{N^*}} t_{N^*} = \left[\left(-\sigma' \sin\theta - \theta'\sigma\cos\theta + \frac{\kappa}{k_N} + \|\omega\| k_N \cos\theta\right)T + \left(-\frac{k_N'}{k_N{}^2}\right)N + \left(\sigma'\cos\theta - \theta'\sigma\sin\theta + \frac{\tau}{k_N} + \|\omega\| k_N \sin\theta\right)B\right]\frac{1}{\|\omega\| k_N}. \quad (19)$$

By taking the norm of the last equation we obtain

$$\gamma_{N^*} = \frac{1}{\|\omega\| k_N}\sqrt{\left|\sigma'^2 + \left[\frac{\|\omega\|}{k_N} + \left(\|\omega\| k_N - \theta'\sigma\right)\right]^2 - \frac{k_N'^2}{k_N{}^4}\right|}. \quad (20)$$

By using $\gamma_N = \frac{\theta'}{\|\omega\|}$, we get

$$\gamma_{N^*} = \frac{1}{\|\omega\| k_N}\sqrt{\left|\sigma'^2 + \left[\frac{\|\omega\|}{k_N}\left(1 + k_N{}^2 - \gamma_N{}^2\right)\right]^2 - \frac{k_N'^2}{k_N{}^4}\right|}. \quad (21)$$

**Corollary 3.8.** If the evolute curve $\alpha$ is a helix then we have for the geodesic curvature of the principal normal indicatrix $(N^*)$ of the involute curve $\beta$

$$\gamma_{N^*} = 2.$$

For the geodesic curvature $\gamma_{B^*}$ of the principal normal indicatrix $(B^*)$ of the curve $\beta$ with respect to $S_1^2$, we can write

$$\gamma_{B^*} = \left\|\bar{\nabla}_{t_{B^*}} t_{B^*}\right\|. \quad (22)$$

Using the Gauss equation and the Eq. (10), we can write

$$\bar{\nabla}_{t_{B^*}} t_{B^*} = \sin\theta T - \frac{\|\omega\|}{\theta'}N - \cos\theta B + B^* \quad (23)$$

.

By using $B^* = -\sin\theta T + \cos\theta B$ given in the Lemma 1. we obtain

$$\bar{\nabla}_{t_{B^*}} t_{B^*} = -\frac{\|\omega\|}{\theta'}N. \quad (24)$$

By taking the norm of the last equation we obtain

$$\gamma_{B^*} = \frac{\|\omega\|}{\theta'}. \quad (25)$$

By using $\gamma_B = \cot\theta$, we obtain following relationship between $\gamma_B$ and $\gamma_{B^*}$:

$$\gamma_{B^*} = -\frac{\|\omega\|\left(1+\gamma_B{}^2\right)}{\gamma_B{}'}. \tag{26}$$

Thus we can give the following corollary:

**Corollary 3.9.** If the evolute curve $\alpha$ is a helix then the geodesic curvature $\gamma_{B^*}$ of the binormal indicatrix $\left(B^*\right)$ of the involute curve $\beta$ is undefined.

**Conclusions**

In this research, we transfer the spherical indicatrix concept to the involutes of a given space like curve with space like binormal in the Minkowski 3-space. Then, some interesting results are obtained relationships between arc lenghts and geodesic curvatures of the involute-evolute curve couple. It is thought that similar studies can be planned in different spaces for many other curve couple. It is also hoped that this study will provide the impetus for new studies in this area.

**Conflicts of Interest**

Author declares that there is no conflict of interest.

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