Some Properties of Strictly Quasi-Fredholm Linear Relations

Hafsa Bouaniza and Maher Mnif

University of Sfax, Faculty of Sciences of Sfax, B.P. 1171, 3000 Sfax, Tunisia.

e-mail: hafsa.bouaniza@yahoo.fr, maher.mnif@gmail.com

Abstract

In this paper we first give some properties of strictly quasi-Fredholm linear relations. Next we investigate the perturbation of this class under finite rank operators.

Key words: strictly quasi-Fredholm linear relations, perturbation, finite rank operators.

Introduction

Let X and Y be two Banach spaces. A linear relation $T : D(T) \to 2^Y$ is a mapping from a subspace $D(T) \subset X$ called the domain of T, into the collection of nonempty subsets of Y such that $T(\alpha x+\beta y) = \alpha T(x)+\beta T(y)$ for all nonzero α, β scalars and $x,\ y \in D(T)$. We denote the set class of linear relations from X to Y by $LR(X,Y)$ and abbreviate $LR(X,X)$ to $LR(X)$. The graph of a relation $T \in LR(X,Y)$ is the subset $G(T)$ of $X \times Y$ defined by $G(T) = \big\{(x,y)/\ y \in Tx\big\}$. Let $T \in LR(X,Y)$. The inverse of T is the linear relation T^{-1} given by $G(T^{-1}) = \big\{(y,x)/\ (x,y) \in G(T)\big\}$. The range and kernel part of T, denoted $R(T)$ and $N(T)$ are defined respectively by $R(T) = \bigcup\limits_{x\in D(T)} Tx$ and $N(T) = T^{-1}(0)$. We say that T is injective, if $N(T) = \{0\}$, surjective if $R(T) = Y$ and bijective if T is both injective and surjective. If $M \subset X$ then the image of M under T is defined to be the set

$$T(M) = \bigcup_{x\in D(T)\cap M} Tx$$

and if $N \subset Y$, then the inverse image of N under T is defined to be the set

$$T^{-1}(N) := \big\{x \in D(T) : Tx \cap N \neq \emptyset\big\}.$$

In particular, for any $y \in R(T)$

$$T^{-1}y := \big\{x \in D(T) : y \in Tx\big\}.$$

Let M be a subspace of X' (the dual space of X). We shall adopt the following notation

$$M^{\perp} := \{x' \in X' \ :\ x'(x) = 0 \text{ for all } x \in M\}.$$

The adjoint T^* of T is defined by $G(T^*) = G(-T^{-1})^{\perp} \subset Y' \times X'$. This means that, $(y',x') \in G(T^*) \subset Y' \times X'$ if and only if, for all $(x,y) \in G(T), y'y - x'x = 0$.

For a linear relation $T \in LR(X)$, the root manifold $N^{\infty}(T)$ is defined by $N^{\infty}(T) = \bigcup_{n=1}^{\infty} N(T^n)$. Similarly, the root manifold $R^{\infty}(T)$ is defined by $R^{\infty}(T) = \bigcap_{n=1}^{\infty} R(T^n)$. The singular chain manifold of T, denoted by $R_c(T)$, is defined by $R_c(T) = N^{\infty}(T) \cap R_{\infty}(T)$ where $R_{\infty}(T) = \bigcup_{i=1}^{\infty} T^i(0)$.
For a given closed subspace E of X, let Q_E^X or simply Q_E denoted the natural quotient map from X onto X/E. We shall denote $Q_{\overline{T(0)}}^X$ by Q_T. Clearly $Q_T T$ is single valued. In fact, let $Q_T y_1$, $Q_T y_2 \in Q_T Tx$. Then $Q_T y_1 - Q_T y_2 \in Q_T Tx - Q_T Tx = Q_T T(0) = 0$. For $x \in D(T), \| Tx \| := \| Q_T Tx \|$ and the norm of T is defined by $\| T \| := \| Q_T T \|$. We note that this quantity is not a true norm since $\| T \| = 0$ does not imply $T = 0$.
Let T and $S \in LR(X)$. The linear relations $T+S$ and TS are defined respectively by $G(T+S) = \{(x, y+z) \in X \times X : (x,y) \in G(T) \text{ and } (x,z) \in G(S)\}$ and $G(TS) = \{(x,y) \in X \times X : \exists z \in X \text{ such that } (x,z) \in G(S) \text{ and } (z,y) \in G(T)\}$. We say that T commutes with S, if $TS \subseteq ST$, and T and S commute mutually if $TS = ST$. Let $T \in LR(X,Y)$. The closure of T is the relation $\overline{T}$ defined by $G(\overline{T}) = \overline{G(T)}$. The relation T is called closed if $G(T)$ is closed in $X \times Y$ or, equivalently, $\overline{T} = T$. We denote the class of all closed linear relations from X to Y by $CR(X,Y)$ and as useful we write $CR(X,X) := CR(X)$.
We say that T is continuous if for each neighborhood V in $R(T)$, $T^{-1}(V)$ is neighborhood in $D(T)$ and open if its inverse is continuous. Continuous everywhere defined linear relation on X is referred to be a bounded linear relation. We denote by $BR(X)$ the set of all bounded linear relations on X. The class of all bounded and closed linear relations on X is denoted by $BCR(X)$.

The resolvent set of $T \in CR(X)$ is the set:

$$\rho(T) = \{\lambda \in \mathbb{C} \text{ such that } (T - \lambda I) \text{ is bijective}\}.$$

The spectrum of T is defined by:

$$\sigma(T) = \mathbb{C} \setminus \rho(T).$$

The kernels and the ranges of the iterates $T^n, n \in \mathbb{N}$, of a linear relation T defined on a vector space X, form two increasing and decreasing chains, respectively.

$$N(T^0) = \{0\} \subseteq N(T) \subseteq N(T^2) \subseteq ...$$

$$R(T^0) = X \supseteq R(T) \supseteq R(T^2)...$$

If T is a bounded linear relation on a Banach space X, then, for each nonnegative integer n, T induces a linear transformation from the vector space $R(T^n)/R(T^{n+1})$ to the space $R(T^{n+1})/R(T^{n+2})$. Let $k_n(T)$ be the dimension of the null space of the induced map. $k_n(T)$ is called the difference sequence of T.
The aim goal of this paper is to give some properties of the class of strictly quasi-Fredholm linear relations on Banach space X and to study the perturbation of this class which have been introduced for the first time in [6]. Let $T \in BCR(X)$ we say that T is strictly quasi-Fredholm relation of degree $d \in \mathbb{N}$, if $k_n(T) = 0$ for all $n \geq d, k_{d-1}(T) \neq 0$ and $R(T^{d+1})$ is closed. We denote by $Sq\phi(d)(X)$, the set of all strictly quasi-Fredholm linear relations of degree d and by $Sq\phi(X)$ the set of all strictly quasi-Fredholm linear relations for some degree $d \in \mathbb{N}$. We first show that if T is a strictly quasi-Fredholm linear relation of degree d then the adjoint T^* of T is a quasi-Fredholm linear operator of degree d. After that we prove that the power of a strictly quasi-Fredholm linear relation is also strictly quasi-Fredholm linear relation. Also we show that under certain conditions the product of two linear relations is strictly quasi-Fredholm

linear relation if and only if each linear relation is strictly quasi-Fredholm.
In the literature the study of the problem of the stability of the quasi-Fredholm operators under finite rank operators was done by M. Mbekhta and V. Muller. They proved in [17] that if T is a quasi-Fredholm operator of degree $d \in \mathbb{N}$ and F is a bounded operator such that $dim(R(F)) < \infty$, then $T + F$ is also quasi-Fredholm operator of degree d.
In this work we extend the result of the stability of quasi-Fredholm operators under finite rank operator perturbations, to the case of strictly quasi-Fredholm linear relations. Hence we prove that if we consider a strictly quasi-Fredholm linear relation T of degree d with $T^{d+1}(0)$ is closed, and a finite rank operator F such that $T^n(0) \subset N(F)$ for all $n \in \mathbb{N}$, then $T + F$ is also a strictly quasi-Fredholm linear relation of the same degree d.
To make the paper easily accessible, some results from the theory of linear relations due to Cross [11] are recalled in the introduction. In Section 2 we study the adjoint and the power of strictly quasi-Fredholm linear relations. In section 3 we prove that under certain conditions the product of two linear relations is strictly quasi-Fredholm linear relation if and only if each linear relation is strictly quasi-Fredholm. Section 4 is devoted to the study of the perturbation of the class of strictly quasi-Fredholm linear relations under finite rank operators. Section 5 is dedicated to explain the main conclusions of the article.

1 Adjoint and Power for Strictly Quasi-Fredholm Linear Relations

The aim goal of this section is to prove that if T is a strictly quasi-Fredholm linear relation such that $\rho(T) \neq \emptyset$ then T^* is a quasi-Fredholm linear operator of degree d. After that we will show that the power of T is a strictly quasi-Fredholm linear relation if and only if T is a strictly quasi-Fredholm linear relation.
Let's start by recalling some definitions for the case of operators.

Definition 1.1 *([14], Definition 3.1) Let T be a closed linear operator on a Banach space X and let*

$$\Delta(T) = \{n \in \mathbb{N}, \forall m \geq n, R(T^n) \cap N(T) = R(T^m) \cap N(T)\}.$$

The degree of stable iteration $dis(T)$ of T is defined as $dis(T) = \inf \Delta(T)$, where $dis(T) = \infty$ if $\Delta(T) = \emptyset$.

Definition 1.2 *([9], Definition 1.3.4) Let Y be a Banach space. We call a subset R of the Banach space Y a range subspace if there exist a Banach space X and a bounded linear operator T from X to Y whose range is R.*

Definition 1.3 *([14], Definition 3.2) Let T be a closed linear operator on a Banach space X. T is called a quasi-Fredholm operator of degree d if there exists an integer $d \in \mathbb{N}$ such that:*

1. *$dis(T) = d$,*

2. *$R(T^n)$ is closed in X for all $n \geq d$,*

3. *$R(T) + N(T^n)$ is closed in X for all $n \geq d$.*

Now we recall the definition of the difference sequence $(k_n(T))_n$ of a linear relation and the definition of strictly quasi-Fredholm linear relation.

Definition 1.4 *If T is a bounded linear relation on a Banach space X, then, for each nonnegative integer n, T induces a linear transformation from the vector space $R(T^n)/R(T^{n+1})$ to the space $R(T^{n+1})/R(T^{n+2})$. Let $k_n(T)$ be the dimension of the null space of the induced map and $k_{-1}(T) = \infty$.*

Definition 1.5 *([6], Definition 4.1) Let X be a Banach space and $T \in BCR(X)$. We say that T is strictly quasi-Fredholm relation of degree $d \in \mathbb{N}$, if $k_n(T) = 0$ for all $n \geq d, k_{d-1}(T) \neq 0$ and $R(T^{d+1})$ is closed. We denote by $Sq\phi(d)(X)$, the set of all strictly quasi-Fredholm linear relations of degree d and by $Sq\phi(X)$ the set of all strictly quasi-Fredholm linear relations for some degree $d \in \mathbb{N}$.*

Carrently, we collect some auxiliary results which we will need repeatedly in the sequel.

Lemma 1.1 *([19], Lemma 2.1 and Lemma 2.3) Let M and N be subspaces of a vector space X. Then*
(i) $M/M \cap N \simeq (M + N)/N$.
(ii) Assume that $N \subset M$ then $\dim X/N = \dim X/M + \dim M/N$.

Lemma 1.2 *Let T be a closed linear relation in $CR(X)$. Then for each nonnegative integer n we have,*

$$R(T^n) \cap N(T)/R(T^{n+1}) \cap N(T) \simeq N(T^{n+1}) + R(T)/N(T^n) + R(T).$$

Proof

By Lemma 4.2 in [19], for all $i, k \in \mathbb{N}$ we have

$$N(T^{i+k})/\big(R(T^k) + N(T^i)\big) \cap N(T^{i+k}) \simeq R(T^i) \cap N(T^k)/R(T^{i+k}) \cap N(T^k).$$

Hence for $k = 1$ and $i = n$ we get

$$N(T^{n+1})/\big(R(T) + N(T^n)\big) \cap N(T^{n+1}) \simeq R(T^n) \cap N(T)/R(T^{n+1}) \cap N(T). \tag{1.1}$$

Now by Lemma 1.1, with $M = N(T^{n+1})$ and $N = R(T) + N(T^n)$ we have for all $n \in \mathbb{N}$,

$$N(T^{n+1})/\big(R(T) + N(T^n)\big) \cap N(T^{n+1}) \simeq N(T^{n+1}) + R(T)/R(T) + N(T^n). \tag{1.2}$$

Hence using (1.1) and (1.2) we get:

$$N(T^{n+1}) + R(T)/R(T) + N(T^n) \simeq R(T^n) \cap N(T)/R(T^{n+1}) \cap N(T).$$

□

Lemma 1.3 *If T is a strictly quasi-Fredholm linear relation of degree d such that $\rho(T) \neq \emptyset$, then for all $n \geq d$ we have*

$$R(T^n)^{\perp} + N(T)^{\perp} = (R(T^n) \cap N(T))^{\perp}.$$

Proof

As T is a strictly quasi-Fredholm linear relation of degree d then by Proposition 4.2 in [6], $R(T^n)$ is closed for all $n \geq d$. So in particular, $R(T^{n+1})$ is closed. Then $Q_T(R(T^{n+1}))$ wich is equal to $R(T^{n+1})/T(0)$ is also closed. On the other hand we have :

$$(Q_T T)^{-1}(Q_T(R(T^{n+1}))) = T^{-1}(R(T^{n+1}) + T(0)) = R(T^n) + N(T).$$

Thus, as $Q_T T$ is a bounded operator and $Q_T(R(T^{n+1}))$ is closed, we conclude that $R(T^n) + N(T)$ is closed. Now, since $N(T)$ and $R(T^n)$ are closed and $R(T^n) + N(T)$ is also closed then by Theorem III.3.9 in [11], it follows that

$$R(T^n)^\perp + N(T)^\perp = (R(T^n) \cap N(T))^\perp, \text{ for all } n \geq d.$$

□

Now, we are in the position to give the main theorem of this subsection.

Theorem 1.1 *Let $T \in BCR(X)$ be such that $\rho(T) \neq \emptyset$. If T is a strictly quasi-Fredholm linear relation of degree d then T^* is a quasi-Fredholm linear operator of degree d.*

Proof

First, we will prove that T^{*d} is a quasi-Fredholm linear operator of degree 1. First, we show that $R(T^{*nd})$ is closed for all $n \geq 1$. As T is closed and $\rho(T) \neq \emptyset$ then T^n is also closed for all $n \geq 1$. So by Proposition 4.2 in [6],we have $R(T^{nd})$ is closed for all $n \geq 1$. Therefore by Theorem 3.3.8 in [20], $R(T^{*\,nd})$ is closed.
Now, we prove that $R(T^{*d}) + N(T^{*nd})$ is closed for all $n \geq 1$. Since T is strictly quasi-Fredholm of degree d, then by Proposition 4.2 in [6], we have $N(T^d) \cap R(T^{nd})$ is closed for all $n \geq 1$. Hence $\left(N(T^d) \cap R(T^{nd})\right)^\perp$ is closed. Therefore by Lemma 1.3, we get $N(T^d)^\perp + R(T^{nd})^\perp$ is closed for all $n \geq 1$. So by Proposition III.1.4 in [11], we have $R(T^{*d}) + N(T^{*nd})$ is closed for all $n \geq 1$.
It remains now to prove that $dis(T^{*d}) = 1$. By Proposition III.1.4 in [11], we have $R(T^{nd})^\perp = N(T^{*nd})$ and $N(T^d)^\perp = R(T^{*d})$. Now, by Lemma 1.3, we get for all $n \geq 1$

$$\begin{aligned}
N(T^{*d(n+1)}) + R(T^{*d}) &= R(T^{d(n+1)})^\perp + N(T^d)^\perp \\
&= (R(T^{d(n+1)}) \cap N(T^d))^\perp \\
&= (R(T^{nd}) \cap N(T^d))^\perp, \text{ (since } k_j(T) = 0, j \geq d) \\
&= R(T^{nd})^\perp + N(T^d)^\perp \\
&= N(T^{*nd}) + R(T^{*d}).
\end{aligned}$$

Using Lemma 1.2, it follows that, for all $n \geq 1$,

$$\dim \left(R(T^{*nd}) \cap N(T^{*d})\right) / \left(R(T^{*d(n+1)}) \cap N(T^{*d})\right) = 0.$$

Therefore,

$$R(T^{*nd}) \cap N(T^{*d}) = R(T^{*d(n+1)}) \cap N(T^{*d}) \text{ for all } n \geq 1.$$

This implies that

$$dis(T^{*d}) = \inf \Delta(T^{*d}) = 1.$$

Consequently, T^{*d} is a quasi-Fredholm linear operator of degree 1.
Now, we show that T^* is quasi-Fredholm of degree d. First we prove that $dis(T^*) = d$. As T^{*d} is quasi-Fredholm of degree 1 then $dis(T^{*d}) = 1$ wish implies that

$$R(T^{*d}) \cap N(T^{*d}) = R(T^{*nd}) \cap N(T^{*d}), \text{ for all } n \geq 1.$$

Hence

$$N(T^*) \cap R(T^{*d}) \cap N(T^{*d}) = N(T^*) \cap R(T^{*nd}) \cap N(T^{*d}), \text{ for all } n \geq 1.$$

It follows that,

$$N(T^*) \cap R(T^{*d}) = N(T^*) \cap R(T^{*j}), \text{ for all } j \geq d.$$

Therefore, $dis(T^*) = d$.
On the other hand, as we have $R(T^{*jd})$ is closed for all $j \geq 1$ it follows that $R(T^{*n})$ is closed for all $n \geq d$. It remain to prove that $R(T^*) + N(T^{*n})$ is closed for all $n \geq d$. Let $(y_p + x_p) \in (R(T^*) + N(T^{*n}))$ such that $y_p + x_p \longrightarrow z$, with

$$\begin{cases} y_p \in R(T^*) \\ \text{and} \\ x_p \in N(T^{*n}) \end{cases} \text{ which implies that } \begin{cases} y_p = T^*(t_p) \\ \text{and} \\ T^{*n}(x_p) = 0. \end{cases}$$

As $\rho(T^*) \neq \emptyset$ then $\rho(T^{*d}) \neq \emptyset$ and hence there exists $\beta \in \rho(T^*)$ such that

$$(T^* - \beta I + \beta I)^{d-1} = \sum_{k=0}^{d-1} c_k (T^* - \beta I)^k$$

with $(c_k)_k$ are constants. Set that $S = (T^* - \beta I)^{-1}$, hence we have

$$\begin{aligned} T^{*d-1} S^{d-1} &= \sum_{k=0}^{d-1} c_k (T^* - \beta I)^k (T^* - \beta I)^{-d+1} \\ &= \sum_{k=0}^{d-1} c_k (T^* - \beta I)^{-(d-1-k)} \\ &= \sum_{k=0}^{d-1} c_k S^{d-1-k} \quad \text{is bounded.} \end{aligned}$$

Therefore,

$$T^{*d-1} S^{d-1} (T^*(t_p) + x_p) \longrightarrow T^{*d-1} S^{d-1}(z).$$

It's clear that,

$$\begin{cases} T^{*d-1} S^{d-1}(T^*(t_p)) \in R(T^{*d}) \\ \text{and} \\ T^{*d-1} S^{d-1}(x_p) \in N(T^*) \subset N(T^{*d}) \subset N(T^{*n}). \end{cases}$$

So

$$T^{*d-1} S^{d-1}(T^*(t_p)) + T^{*d-1} S^{d-1}(x_p) \in R(T^{*d}) + N(T^{*n}).$$

Since, $R(T^{*d}) + N(T^{*n})$ is closed therefore

$$T^{*d-1} S^{d-1}(z) \in R(T^{*d}) + N(T^{*n}).$$

So, there exist $t \in D(T^*)$ and $x \in N(T^{*n})$, such that,

$$T^{*d-1}S^{d-1}(z) = T^{*d}(t) + x.$$

Hence

$$T^{*d-1}(S^{d-1}(z) - T^*(t)) = x \in N(T^{*n}).$$

Thus,

$$S^{d-1}(z) - T^*(t) \in T^{*-(d-1)}(N(T^{*n})) = N(T^{*n+d-1}).$$

As $n \in \Delta(T^*)$ then by Proposition 3.1.1 in [15], we have $N(T^{*n+d-1}) \subset R(T^*) + N(T^{*n})$.
Therefore $S^{n-1}(z) \in R(T^*) + N(T^{*n})$. Since S is invertible then $z \in R(T^*) + N(T^{*n})$. It follows that $R(T^*) + N(T^{*n})$ is closed. Consequently T^* is a quasi-Fredholm linear operator. □

In the following we will show that the power of a linear relation T such that $\rho(T) \neq \emptyset$ is a strictly quasi-Fredholm linear relation if and only if T is a strictly quasi-Fredholm. To do this we need the following lemma.

Lemma 1.4 *([12], Theorem 2.4) Let M and N be two range subspaces of a Banach space X such that $M + N$ and $M \cap N$ are closed. Then M and N are closed.*

Theorem 1.2 *Let X be a Banach space and $T \in BCR(X)$ be such that $\rho(T) \neq \emptyset$, then for all $m \geq 1$ and $p \geq 1$, we have*

$$T \in Sq\phi(mp)(X) \text{ if and only if } T^m \in Sq\phi(p)(X).$$

Proof

Let $T \in Sq\phi(mp)(X)$. First, we show that , $k_j(T^m) = 0$ for all $j \geq p$. As $k_n(T) = 0$ for all $n \geq mp$, then

$$N(T) \cap R(T^{mp}) = N(T) \cap R(T^n), \text{ for all } n \geq mp.$$

By Lemma 2.2 in [8], we have

$$N(T^j) \cap R(T^{mp}) \subset \bigcap_{n=0}^{\infty} R(T^n) \text{ for } j \geq 1.$$

Hence for all $n \geq p$ and $j = m$, we have

$$N(T^m) \cap R(T^{mp}) \subset R(T^{mn}).$$

Then

$$N(T^m) \cap R(T^{mp}) \subset N(T^m) \cap R(T^{mn}), \text{ for all } n \geq p.$$

And as

$$N(T^m) \cap R(T^{mn}) \subset N(T^m) \cap R(T^{mp}), \text{ for all } n \geq p,$$

it follows that,

$$N(T^m) \cap R(T^{mn}) = N(T^m) \cap R(T^{mp}), \text{ for all } n \geq p.$$

Therefore, $k_j(T^m) = 0$ for all $j \geq p$. Now we show that $R((T^m)^{p+1})$ is closed. Since $\rho(T) \neq \emptyset$ so by Proposition 4.2 in [6], we have $R(T^{mp+m})$ is closed. Thus $T^m \in Sq\phi(p)(X)$.

Conversely, let $T \in BCR(X)$ be such that $T^m \in Sq\phi(p)(X)$. First we will show that $k_n(T) = 0$ for all $n \geq mp$. As $k_n(T^m) = 0$ for all $n \geq p$ it follows that,

$$N(T^m) \cap R(T^{mn}) = N(T^m) \cap R(T^{mp}), \text{ for all } n \geq p.$$

Hence,

$$N(T) \cap N(T^m) \cap R(T^{mn}) = N(T) \cap N(T^m) \cap R(T^{mp}), \text{ for all } n \geq p.$$

Thus,

$$N(T) \cap R(T^{mn}) = N(T) \cap R(T^{mp}), \text{ for all } n \geq p.$$

Therefore,

$$N(T) \cap R(T^{n}) = N(T) \cap R(T^{mp}), \text{ for all } n \geq mp.$$

Consequently, $k_n(T) = 0$ for all $n \geq mp$.
It remains to show that $R(T^{mp+1})$ is closed. First we prove that:

$$N(T^j) \cap R(T^{mp}) = N(T^j) \cap R(T^{mp+1}), \text{for all } j \in \mathbb{N}.$$

For $j = 1$ as $k_n(T) = 0, n \geq mp$ we have $N(T) \cap R(T^{mp}) = N(T) \cap R(T^{mp+1})$. Suppose that

$$N(T^j) \cap R(T^{mp}) = N(T^j) \cap R(T^{mp+1}),$$

and we will show that

$$N(T^{j+1}) \cap R(T^{mp}) = N(T^{j+1}) \cap R(T^{mp+1}).$$

As

$$N(T^{j+1}) \cap R(T^{mp+1}) \subset N(T^{j+1}) \cap R(T^{mp}),$$

it is sufficient to show that

$$N(T^{j+1}) \cap R(T^{mp}) \subset N(T^{j+1}) \cap R(T^{mp+1}).$$

Since $k_n(T) = 0$ for all $n \geq mp$, then by Lemma 2.2 in [8], we have

$$N(T^j) \cap R(T^{mp}) \subset \bigcap_{n=0}^{\infty} R(T^n) \text{ for all } j \geq 1.$$

Hence $N(T^{j+1}) \cap R(T^{mp}) \subset N(T^{j+1}) \cap R(T^{mp+1})$. Thus

$$N(T^j) \cap R(T^{mp}) = N(T^j) \cap R(T^{mp+1}) \text{ for all } j \in \mathbb{N}.$$

Then for $j = m - 1$ we have

$$N(T^{m-1}) \cap R(T^{mp}) = N(T^{m-1}) \cap R(T^{mp+1}).$$

Since $T^m \in Sq\phi(p)(X)$ it follows by Proposition 4.2 in [6], that $R(T^{mp})$ is closed and as $N(T^{m-1})$ is closed then $R(T^{mp}) \cap N(T^{m-1})$ is closed. Thus $N(T^{m-1}) \cap R(T^{mp+1})$ is also closed. On the other hand as $R(T^{mp+m})$ is closed so by Lemma 2.5 in [6], we have $R(T^{mp+1}) + N(T^{m-1})$ is also closed. Using Lemma 1.4 we get $R(T^{mp+1})$ is closed. Consequently, $T \in Sq\phi(mp)(X)$. □

2 Product of Strictly Quasi-Fredholm Linear Relations

In this section, we will show that under certain conditions the product of two linear relations is strictly quasi-Fredholm linear relation if and only if each linear relation is strictly quasi-Fredholm. We start this subsection by the following technical lemma.

Lemma 2.1 *Let F, G_1 and G_2 be three subspaces of a vector space X. Then*

$$\dim\big(G_1 + F/G_2 + F\big) \leq \dim\big(G_1/G_2\big).$$

Proof

Let $\big(\overline{x}_1, ..., \overline{x}_m\big)$ be an independent family of $G_1 + F/G_2 + F$. Then: $\overline{x_i} = x_i + G_2 + F$ with $x_i \in G_1 + F$. Then, for all $i \in \{1, ..., m\}$, $x_i = x_{i1} + x_{iF}$ with $x_{i1} \in G_1$ and $x_{iF} \in F$.
We consider the family $\big(x_{11}, ..., x_{m1}\big)$ of G_1. We prove that $\big(\widetilde{x}_{11}, ..., \widetilde{x}_{m1}\big)$ is independent in G_1/G_2. Indeed, let $\alpha_1\widetilde{x}_{11} + \cdots + \alpha_m\widetilde{x}_{m1} = \widetilde{0}$, then $\alpha_1 x_{11} + \cdots + \alpha_m x_{m1} \in G_2$. So,

$$\begin{aligned}\alpha_1\overline{x}_1 + \cdots + \alpha_m\overline{x}_m &= \overline{\alpha_1 x_1 + \cdots + \alpha_m x_m}\\ &= \alpha_1 x_1 + \cdots + \alpha_m x_m + G_2 + F\\ &= \sum_{i=1}^{m}\alpha_i x_{i1} + \sum_{i=1}^{m}\alpha_i x_{iF} + G_2 + F.\end{aligned}$$

Then, $\alpha_1\overline{x}_1 + \cdots + \alpha_m\overline{x}_m = \overline{0}$. So $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$, and hence the family $\big(\widetilde{x}_{11}, ..., \widetilde{x}_{m1}\big)$ is independent in G_1/G_2. □

Lemma 2.2 *(Proposition 3.4, [8]) Let X be a Banach space and let A, B, C and $D \in BR(X)$. Suppose that C commutes with A and B, D commutes with A, B and C, A and B commute mutually and $I \subset AC + DB$. Then*
i) If A^{-1} commutes with B, then for every n, and $R(A^nB^n) = R(A^n) \cap R(B^n)$.
ii) For every $n \in \mathbb{N}$, we have $N(A^n) \subset R(B^n)$ and $N(B^n) \subset R(A^n)$.

Lemma 2.3 *(Proposition 3.6, [8]) Let X be a Banach space and let A, B, C and $D \in BCR(X)$. Suppose that C commutes with A and B, D commutes with A, B and C, A and B commute mutually and $I \subset AC + DB$. Let $n \geq 0$.*
i) If $R(A^*)$ and $R(B^*)$ are closed, then $N(AB) = N(A) + N(B)$.
ii) Furthermore, if A^{-1} commutes with B, then $N((AB)_n) = N(A_n) + N(B_n)$.

From the previous lemmas we have the following result:

Proposition 2.1 *Let X be a Banach space and let A, B, C, $D \in BR(X)$ be such that C commutes with A and B, D commutes with A, B and C , A and B commute mutually and $I \subset AC + BD$ and A^{-1} commutes with B. Then*

$$\sup(k_n(A), k_n(B)) \leq k_n(AB) \leq k_n(A) + k_n(B).$$

Proof

We prove that $k_n(A) \leq k_n(AB)$. If $x_1, ..., x_m \in R(A^n) \cap N(A)$, where $m > k_n(AB)$, then $B^n x_i \subset R(A^n B^n)$ and $B^n x_i \subset N(A) + B^n(0)$. In fact, let $x_i \in N(A)$. Then

$$Ax_i = A(0) \Rightarrow B^n A x_i = B^n A(0) \Rightarrow AB^n x_i = AB^n(0).$$

$$\text{So } A^{-1}AB^n x_i = A^{-1}AB^n(0). \text{ It follows that } B^n x_i + N(A) = B^n(0) + N(A).$$

Thus,

$$B^n x_i \subset N(A) + B^n(0).$$

Then, $B^n x_i \subset R(A^n B^n) \cap \big(N(A) + B^n(0)\big) \subset R(A^n B^n) \cap \big(N(AB) + B^n(0)\big)$, since $N(A) \subset N(AB)$. Thus by Lemma 22.2 in [18], it follows that for $i \in \{1, ..., m\}$, $B^n x_i \subset B^n(0) + \big(R(A^n B^n) \cap N(AB)\big)$. Let, for all $i \in \{1, ..., m\}$, $y_i \in B^n x_i$. Then we have for all $i \in \{1, ..., m\}$, $y_i \in B^n(0) + \big(R(A^n B^n) \cap N(AB)\big)$. Using Lemma 2.1, we get:

$$\dim\Big(\big[B^n(0) + R(A^n B^n) \cap N(AB)\big]/\big[B^n(0) + R(A^{n+1}B^{n+1}) \cap N(AB)\big]\Big) \leq k_n(AB).$$

Since $m \geq k_n(AB)$, we can deduce that there exist $\alpha_1, ..., \alpha_m$ non trivial such that

$$\alpha_1 y_1 + \alpha_2 y_2 + ... + \alpha_m y_m \in B^n(0) + R(A^{n+1}B^{n+1}) \cap N(AB).$$

On other hand we have:

$$\begin{aligned} R(A^{n+1}B^{n+1}) &= B^n\big(BA^{n+1}(X)\big) \\ &= B^n\big(A^{n+1}B(X)\big) \subset B^n(A^{n+1}X) \\ &\subset B^n(R(A^{n+1})). \end{aligned}$$

Now, it is clear that $B^n(0) \subset B^n(R(A^{n+1}))$, so $\sum_{i=1}^{m} \alpha_i y_i \in B^n(R(A^{n+1}))$. Furthermore, since $\sum_{i=1}^{m} \alpha_i y_i \in B^n(\sum_{i=1}^{m} \alpha_i x_i)$ then $B^n(\sum_{i=1}^{m} \alpha_i x_i) \cap B^n(R(A^{n+1})) \neq \emptyset$ and so, $\sum_{i=1}^{m} \alpha_i x_i \in R(A^{n+1}) + N(B^n) \subset R(A^{n+1})$. Hence, $\sum_{i=1}^{m} \alpha_i x_i \in R(A^{n+1}) \cap N(A)$, and so $k_n(A) \leq k_n(AB)$.
Now, we will prove the second inequality.

If $k_n(A) + k_n(B) = \infty$, then there is nothing to prove. We consider the case where $k_n(A) + k_n(B)$ is finite. Let $x_1, ..., x_m \in R(A^n B^n) \cap N(AB)$, where $m > k_n(A) + k_n(B)$. We have

$$R(A^n B^n) \cap N(AB) = R(A^n) \cap N(A) + R(B^n) \cap N(B), \text{ (see Lemma 2.3)}.$$

Then there exist $y_i \in R(A^n) \cap N(A)$ and $z_i \in R(B^n) \cap N(B)$, such that $x_i = y_i + z_i$.
Consider the space $R(A^n) \cap N(A) \oplus R(B^n) \cap N(B)$. We have

$$\dim\Big(\big[R(A^n) \cap N(A) \oplus R(B^n) \cap N(B)\big]/\big[R(A^{n+1}) \cap N(A) \oplus R(B^{n+1}) \cap N(B)\big]\Big) = k_n(A) + k_n(B).$$

Then, there exists a non-trivial combination such that

$$\sum_{i=1}^{m} \alpha_i (y_i + z_i) \in R(A^{n+1}) \cap N(A) \oplus R(B^{n+1}) \cap N(B).$$

Hence, $\sum_{i=1}^{m} \alpha_i x_i \in R(A^{n+1}B^{n+1}) \cap N(AB)$ and so $k_n(AB) \leq k_n(A) + k_n(B)$. □

We are now ready to state our main theorem.

Theorem 2.1 *Let $A, B, C, D \in BCR(X)$ be such that $\rho(A) \neq \emptyset$, $\rho(B) \neq \emptyset$ and C commutes with A and B, D commutes with A, B and C, A and B commute mutually and $I \subset AC + BD$ and A^{-1} commutes with B. Then $AB \in Sq\phi(d)(X)$ if and only if $A \in Sq\phi(d)(X)$ and $B \in Sq\phi(d)(X)$.*

Proof

If $A, B \in Sq\phi(d)(X)$ then $k_n(A) = 0$ and $k_n(B) = 0$ for all $n \geq d$. Then by Proposition 2.1, we have $k_n(AB) = 0$, for all $n \geq d$. So to prove that $AB \in Sq\phi(d)(X)$, it remains to show that $R((AB)^{d+1})$ is closed. By Lemma 2.2, we have

$$R((AB)^{d+1}) = R(A^{d+1}B^{d+1}) = R(A^{d+1}) \cap R(B^{d+1}).$$

As $R(A^{d+1})$ and $R(B^{d+1})$ are closed it follows that $R((AB)^{d+1})$ is closed. Hence $AB \in Sq\phi(d)(X)$.
Now, If $AB \in Sq\phi(d)(X)$, we will prove that $A \in Sq\phi(d)(X)$ and $B \in Sq\phi(d)(X)$.
First, we show that $A \in Sq\phi(d)(X)$. As $AB \in Sq\phi(d)(X)$ so $k_n(AB) = 0$ for all $n \geq d$. Then by Proposition 2.1, we have $k_n(A) = 0$ for all $n \geq d$.
It remains to prove that $R(A^{d+1})$ is closed. We consider the sequence $(x_n)_n \in R(A^{d+1})$ such that $x_n \to x$. It suffices to show that $x \in R(A^{d+1})$. Let $\gamma_n \in B^{d+1}(x_n)$ and $\gamma \in B^{d+1}(x)$. We have $\gamma_n - \gamma \in B^{d+1}(x_n - x)$. Then,

$$d(\gamma_n - \gamma,\ B^{d+1}(0)) = \parallel B^{d+1}(x_n - x) \parallel \leq \parallel B^{d+1} \parallel \parallel x_n - x \parallel \longrightarrow 0.$$

Then there exists $t_n \in B^{d+1}(0)$ such that

$$\parallel \gamma_n - t_n - \gamma \parallel \to 0.$$

Let $\alpha_n = \gamma_n - t_n \in B^{d+1}(x_n) - B^{d+1}(0) = B^{d+1}(x_n)$ and $\alpha_n \to \gamma \in B^{d+1}(x)$ and as $\alpha_n \in R(AB)^{d+1}$ which is closed therefore $\gamma \in R(AB)^{d+1}$.
This implies that

$$x \in (B^{d+1})^{-1}(\gamma) \subset (B^{d+1})^{-1}(R(AB)^{d+1}) \subset N(B^{d+1}) + R(A^{d+1}).$$

Since by Lemma 2.2, we have $N(B^{d+1}) \subset R(A^{d+1})$ then we get $x \in R(A^{d+1})$. It follows that, $R(A^{d+1})$ is closed. Consequently, $A \in Sq\phi(d)(X)$. By the same way we prove that $B \in Sq\phi(d)(X)$. □

3 Perturbation of Strictly Quasi-Fredholm Linear Relations

In this section we will study the perturbation of strictly quasi-Fredholm linear relations under finite rank operators. First, we give these next lemmas that will be required in the proof of the main result of this section.
First, we recall Proposition 1.1.6 in [7]

Lemma 3.1 *([7], Proposition 1.1.6) Let $T \in LR(Y, X)$ and $S, R \in LR(X, Z)$. If $T(0) \subset N(S)$ or $T(0) \subset N(R)$, then*

$$(R + S)T = RT + ST.$$

Lemma 3.2 *Let $T \in BCR(X)$ and F be a bounded operator such that $R_\infty(T) \subset N(F)$. Then*

$$(T + F)^n(0) = T^n(0), \ \textit{for all}\ n \in \mathbb{N}^*. \tag{3.1}$$

Proof

We prove (3.1) by induction on n. For $n = 1$ we check that $(T + F)(0) = T(0)$. Suppose that (3.1) is true for some integer n and we prove this result for $n+1$. Since $(T + F)^n(0) = T^n(0) \subset N(F)$ then by Lemma 3.1, we have

$$(T + F)^{n+1} = (T + F)(T + F)^n = T(T + F)^n + F(T + F)^n.$$

It follows that,

$$\begin{aligned} (T + F)^{n+1}(0) &= T(T + F)^n(0) + F(T + F)^n(0) \\ &= TT^n(0) + F(T^n(0)) \\ &= T^{n+1}(0). \end{aligned}$$

Thus $(T + F)^n(0) = T^n(0)$ for all $n \in \mathbb{N}^*$. □

The next lemma is a generalization of observation 8 in [17].

Lemma 3.3 *Let $T \in BR(X)$ and F be a bounded operator such that $R_\infty(T) \subset N(F)$. Then*

$$(T + F)^n = T^n + \sum_{i=0}^{n-1} T^i F(T + F)^{n-i-1}. \tag{3.2}$$

Proof

We prove (3.2) by induction on n. For $n = 1$ the statement is trivial. Suppose that (3.2) is true for some integer n and we prove for $n+1$. By Lemma 3.2 and Proposition 3.1, we have

$$\begin{aligned} (T + F)^{n+1} &= (T + F)(T + F)^n \\ &= T(T + F)^n + F(T + F)^n \\ &= T\big(T^n + \sum_{i=0}^{n-1} T^i F(T + F)^{n-i-1}\big) + F(T + F)^n \\ &= T^{n+1} + \sum_{i=0}^{n-1} T^{i+1} F(T + F)^{n-i-1}, \text{ (by Proposition I.4.2 in [9])} \\ &= T^{n+1} + \sum_{i=0}^{n} T^i F(T + F)^{n-i}. \end{aligned}$$

□

Definition 3.1 *Let X be a Banach space. For two subspaces M and N of X, we write $M \subset^e N$ if there exists a finite-dimensional subspace F of X such that $M \subset N + F$. Obviously $M \subset^e N$ if and only if $\dim[M/(M \cap N)] < \infty$. Notice that we can assume that F is a subset of M. Similarly, we write $M =^e N$ if both $M \subset^e N$ and $N \subset^e M$.*

As a consequence of Lemma 3.3, we get the following corollary.

Corollary 3.1 *Let $T \in BR(X)$ and F be a finite rank operator such that $R_\infty(T) \subset N(F)$. Then $R(T+F)^n =^e R(T^n)$ for all $n \in \mathbb{N}^*$.*

Proof

As $\dim(R(F)) < \infty$, we can suppose without loss of generality that $\dim(R(F)) = 1$. Hence there exists $v \in X$ such that $R(F) =< v >$. By Lemma 3.3, we have

$$(T+F)^n = T^n + \sum_{i=0}^{n-1} T^i F (T+F)^{n-i-1}.$$

Hence

$$\begin{aligned}(T+F)^n(X) &\subset T^n(X) + \sum_{i=0}^{n-1} T^i F (T+F)^{n-i-1}(X)\\ &\subset T^n(X) + F(T+F)^{n-1}(X) + \sum_{i=1}^{n-1} T^i F (T+F)^{n-i-1}(X)\\ &\subset T^n(X) + < v > + \sum_{i=1}^{n-1} T^i < v > .\end{aligned}$$

Let $y_i \in T^i(< v >)$ so $T^i(< v >) =< y_i > + T^i(0)$. It follows that,

$$\begin{aligned}(T+F)^n(X) &\subset T^n(X) + < v > + \sum_{i=1}^{n-1} < y_i > + T^i(0)\\ &\subset T^n(X) + < v > + \sum_{i=1}^{n-1} < y_i > + T^{n-1}(0)\\ &\subset T^n(X) + < v > + \sum_{i=1}^{n-1} < y_i > .\end{aligned}$$

It's clear that, $\dim(< v > + \sum_{i=1}^{n-1} < y_i >) < \infty$. Hence $R(T+F)^n \subset^e R(T^n)$. Conversely by Lemma 3.2, we have $(T+F)^n(0) = T^n(0) \subset N(F) = N(-F)$ so by substituting T by $T+F$ and F by $-F$ we get $R(T^n) \subset^e R(T+F)^n$. Thus $R(T^n) =^e R(T+F)^n$ for all $n \in \mathbb{N}^*$. □

The next lemma provides the relationships between $N(T)$ and $N(T+F)$, where $T \in LR(X)$ and F a finite rank operator.

Lemma 3.4 *Let $T \in LR(X)$ and F be a finite rank operator. Then*

$$N(T+F) =^e N(T).$$

Proof

We can suppose that there exists $v \in X$ such that $R(F) =< v >$. Let $x \in N(T+F)$ then $(T+F)(x) = (T+F)(0) = T(0)$. Thus $T(x) = T(0) - F(x)$ so $x \in T^{-1}(T(0) - F(x)) = N(T) + T^{-1}(-F(x)) \subset N(T) + T^{-1}(< v >)$. Let $z \in T^{-1}(v)$ therefore

$$T^{-1}(< v >) =< z > + T^{-1}(0) =< z > + N(T).$$

It follows that, $x \in < z > + N(T)$. Hence $N(T+F) \subset^e N(T)$. Now, by substituting T by $T+F$ and F by $-F$ we get, $N(T) \subset^e N(T+F)$.
Consequently

$$N(T+F) =^e N(T).$$

□

The next lemma is elementary but essential to prove Corollary 3.2.

Lemma 3.5 *([11], Lemma 2.4) Let M_1, M_2 and N be subspaces of a linear space X and assume that $M_1 \subset M_2$. Then*

$$\dim(M_1/M_1 \cap N) \leq \dim(M_2/M_2 \cap N).$$

As a consequence of the technical lemmas 1.1 and 3.5, we have the following corollary.

Corollary 3.2 *Let $T \in BR(X)$, and F be a finite rank operator such that $R_\infty(T) \subset N(F)$. Then $R(T^n) \cap N(T) =^e R(T+F)^n \cap N(T+F)$ for all $n \in \mathbb{N}^*$.*

Proof

By Corollary 3.1, and Lemma 3.4, we have $R(T^n) =^e R(T+F)^n$ and $N(T) =^e N(T+F)$. It follows that, there exist two subspaces G_1 and G_2 with $\dim(G_1) < \infty$ and $\dim(G_2) < \infty$, such that $R(T+F)^n \subset R(T^n) + G_1$ and $N(T+F) \subset N(T) + G_2$. Hence

$$R(T+F)^n \cap N(T+F) \subset N(T+F) \subset N(T) + G_2,$$
$$R(T+F)^n \cap N(T+F) \subset R(T+F)^n \subset R(T^n) + G_1.$$

Thus, $R(T+F)^n \cap N(T+F) \subset^e N(T)$ and $R(T+F)^n \cap N(T+F) \subset^e R(T^n)$. It follows that,

$$\dim(R(T+F)^n \cap N(T+F)/R(T+F)^n \cap N(T+F) \cap N(T)) < \infty,$$
$$\dim(R(T+F)^n \cap N(T+F)/R(T+F)^n \cap N(T+F) \cap R(T^n)) < \infty.$$

By Lemma 1.1, we have

$$\dim(R(T+F)^n \cap N(T+F)/R(T+F)^n \cap N(T+F) \cap N(T) \cap R(T^n)) =$$
$$\dim(R(T+F)^n \cap N(T+F)/R(T+F)^n \cap N(T+F) \cap N(T)) +$$
$$\dim(R(T+F)^n \cap N(T+F) \cap N(T)/R(T+F)^n \cap N(T+F) \cap N(T) \cap R(T^n)).$$

Now, using Lemma 3.5, we get

$$\dim(R(T+F)^n \cap N(T+F) \cap N(T)/R(T+F)^n \cap N(T+F) \cap N(T) \cap R(T^n)) \leq$$
$$\dim(R(T+F)^n \cap N(T+F)/R(T+F)^n \cap N(T+F) \cap R(T^n)) < \infty.$$

Hence,

$$\dim(R(T+F)^n \cap N(T+F)/R(T+F)^n \cap N(T+F) \cap N(T) \cap R(T^n)) < \infty.$$

It follows that $R(T+F)^n \cap N(T+F) \subset^e N(T) \cap R(T^n)$. As by Lemma 3.2, we have $(T+F)^n(0) = T^n(0) \subset N(F) = N(-F)$. So by substituting T by $T+F$ and F by $-F$ we get, $R(T^n) \cap N(T) \subset^e N(T+F) \cap R(T+F)^n$. Consequently

$$R(T^n) \cap N(T) =^e N(T+F) \cap R(T+F)^n \text{ for all } n \in \mathbb{N}^*.$$

□

Lemma 3.6 *Let $T \in BCR(X)$ and let $F \subset X$ be a finite dimensional subspace. Suppose that $R(T) + F$ is closed. Then $R(T)$ is closed.*

Proof

Since T is bounded so $Q_T T$ is a bounded operator. Clearly

$$\begin{aligned} R(Q_T T) + Q_T(F) &= R(T)/T(0) + (F + T(0))/T(0) \\ &= (R(T) + F + T(0))/T(0) \\ &= (R(T) + F)/T(0). \end{aligned}$$

Since by hypothesis $R(T)+F$ and $T(0)$ are closed, then $(R(T)+F)/T(0)$ is closed. This implies that, $R(Q_T T)+Q_T(F)$ is closed. As $\dim(Q_T(F)) < \infty$, by applying Lemma 2 in [18], we have $R(Q_T T)$ is closed. Hence $R(T)/T(0)$ is closed. As $T(0)$ is closed then by Proposition 1.7.5 in [10], it follows that $R(T)$ is closed. □

Now, we are in the position to give the main theorem of this section.

Theorem 3.1 *Let $T \in BCR(X)$ be a strictly quasi-Fredholm of degree d with $T^{d+1}(0)$ closed and let F be a finite rank operator such that $R_\infty(T) \subset N(F)$ for all $n \in \mathbb{N}$. Then $T + F$ is also strictly quasi-Fredholm of degree d.*

Proof

As $\dim(R(F)) < \infty$, we can suppose without loss of generality that $\dim R(F) = 1$. Then there exist $z \in X$ and $\varphi \in X^*$ such that $F(x) = \varphi(x)z$ for all $x \in X$. Since $\dim R(F) < \infty$ so by Corollary 3.1, we have $R(T+F)^n =^e R(T^n)$ for all $n \in \mathbb{N}$. As T is strictly quasi-Fredholm of degree d so $k_n(T) = 0$ for all $n \geq d$ and $R(T^{d+1})$ is closed. Hence $\dim R(T^n) \cap N(T)/R(T^{n+1}) \cap N(T) = 0$ for all $n \geq d$. Let $T_1 = T_{/R(T^d)}$. Thus we have $N(T_1) = N(T) \cap R(T^d) \subset N(T) \cap R(T^n)$ for all $n \geq d$. This implies that $N(T_1) \subset R^\infty(T_1)$ and $R(T_1) = R(T^{d+1})$ is closed thus T_1 is semi-regular.
First, we claim that $N(T_1) \subset^e R^\infty(T + F)$. We distinguish two cases:
Case 1: $N^\infty(T) \subset \ker(\varphi)$.
Let $x_0 \in N(T_1)$. Since T_1 is semi-regular then there exist $x_1, x_2, ... \in R^\infty(T_1)$ such that $x_{i-1} \in Tx_i$. By the assumption we have $\varphi(x_i) = 0$, so $F(x_i) = 0$ for all i. For $n \in \mathbb{N}$ we have

$$\begin{aligned} (T+F)^n x_n &= (T+F)^{n-1}(T+F)x_n = (T+F)^{n-1}(Tx_n + Fx_n) \\ &= (T+F)^{n-1}(x_{n-1} + T(0)) \\ &= (T+F)^{n-2}(Tx_{n-1} + T^2(0)) \\ &= (T+F)^{n-3}(x_{n-3} + T^3(0)) \\ &\vdots \\ &= x_0 + T^n(0). \\ &= x_0 + (T+F)^n(0), (\text{ since }, (T+F)^n(0) = T^n(0)). \end{aligned}$$

It follows that, $x_0 \in (T + F)^n x_n \subset R(T + F)^n$. Hence, $N(T_1) \subset R(T + F)^n$ for all $n \in \mathbb{N}$ which implies that $N(T_1) \subset R^\infty(T + F)$.
Case 2: $N^\infty(T) \nsubseteq \ker(\varphi)$.
Let $k \geq 1$ such that $N(T_1^k) \nsubseteq \ker(\varphi)$. Choose the minimal k with this property so that $N(T^{k-1}) \subset \ker(\varphi)$. Hence there exists $u \in N(T_1^k)$ with $\varphi(u) = 1$. Set

$$Y = \{x \in N(T_1) : \text{there is } y \in R(T^d) \text{ with } x \in T^{k-1}y \text{ and } T^i(y) \subset \ker(\varphi) \text{ for all } 0 \leq i \leq k-1\}.$$

We claim that

$$\dim N(T_1)/Y \leq k.$$

Indeed, we have $u \in N(T_1^k)$ then there exists $v \in N(T_1)$ such that $v \in T_1^{k-1}(u)$. Since, $\varphi(u) = 1$, then we can see that v is not in Y. So, $\overline{v} \neq \overline{0}$. Let $\overline{x_1}, \ldots, \overline{x_k}$ be k nonzero vectors in $N(T_1)/Y$. Since T_1 is semi-regular, there are $y_1, \ldots, y_k \in R^\infty(T_1) \subset R(T^d)$ such that $x_j \in T^{k-1}y_j$ for all $1 \leq j \leq k$. We shall verify now that

$$T_1^i(y_j) \subset \ker(\varphi) \text{ for all } 1 \leq j \leq k \text{ and } 1 \leq i \leq k-1.$$

As $x_j \in T^{k-1}y_j$ so $y_j \in T^{-k+1}x_j \subset T^{-k}(0) = N(T^k)$. Thus, $T^i y_j \cap N(T^{k-i}) \neq \emptyset$ and hence $T^i y_j \cap \ker(\varphi) \neq \emptyset$. Let $z \in T^i y_j$ and $z_0 \in T^i y_j \cap \ker(\varphi)$. Then $z - z_0 \in T^i(0) \subset \ker(\varphi)$. Hence $z \in \ker(\varphi)$ and therefore $T_1^i(y_j) \subset \ker(\varphi)$. By the same way we can prove that

$$T^i u \subset \ker(\varphi) \text{ for all } 1 \leq i \leq k-1.$$

Furthermore, for all $1 \leq j \leq k$, we have $\varphi(y_j) \neq 0$. In fact, if $\varphi(y_j) = 0$, then by the definition of Y we get $\overline{x_j} = 0$ which is absurd.

Hence there exist $\alpha_1, \ldots, \alpha_k$ scalars such that $\varphi(\sum_{j=1}^{j=k} \alpha_j y_j) = 1$. So $\varphi(\sum_{j=1}^{j=k} \alpha_j y_j - u) = 0$. Then, $\sum_{j=1}^{j=k} \alpha_j \overline{x_j} - \overline{v} = \overline{0}$ and so, by the incomplete basis theorem we deduce that $\dim N(T_1)/Y \leq k$. Observe that $Y \subset N(T_1)$ then $N(T_1) =^e Y$. As a consequence it is sufficient to verify that $Y \subset R^\infty(T+F)$. Let $x \in Y$. We prove the following statement :

$$\forall n \in \mathbb{N},\ \exists x_n \in R(T^d) \text{ such that } x \in T^n x_n \text{ and } T^i x_n \subset \ker(\varphi) \text{ for all } 0 \leq i \leq n. \tag{3.3}$$

Since $x \in Y$ then there exists $y \in R(T^d)$ with $x \in T^{k-1}y$ and $T^i(y) \subset \ker(\varphi)$ for all $0 \leq i \leq k-1$. First consider the case $0 \leq n \leq k-1$. For the case $n = 0$ there is nothing to prove. Now let $1 \leq n \leq k-1$. We have $x \in T^{k-1}y = T^n T^{k-1-n} y$. Then there exists $x_n \in T^{k-1-n}y$ such that $x \in T^n x_n$. It remains to prove that $T^i x_n \subset \ker(\varphi)$ for all $0 \leq i \leq n$. For $0 \leq i \leq n$, taking $z_{k-n+i-1} \in T^{k-n+i-1}y \subset \ker(\varphi)$ and let $\alpha_i \in T^i x_n \subset T^{k+i-1-n}y$. We have $z_{k-n+i-1} \in \ker(\varphi)$ and $z_{k-n+i-1} - \alpha_i \in T^{k-n+i-1}(0) \subset \ker(\varphi)$. So, $\alpha_i \in \ker(\varphi)$ and therefore $T^i(x_n) \subset \ker(\varphi)$. For the case $n \geq k-1$, suppose that Eq. (3.3) is true for some $n \geq k-1$. Then, there exists $x_n \in R(T^d)$ such that $x \in T^n x_n$ and $T^i x_n \subset \ker(\varphi)$ for all $0 \leq i \leq n$. Since T_1 is semi-regular, we can find $x'_{n+1} \in R(T^d)$ such that $x_n \in T x'_{n+1}$. Set $x_{n+1} = x'_{n+1} - \varphi(x'_{n+1})u$. Then

$$\begin{aligned}
T^{n+1}x_{n+1} &= T^{n+1}x'_{n+1} - \varphi(x'_{n+1})T^{n+1}u \\
&= T^n(Tx'_{n+1}) - \varphi(x'_{n+1})T^{n+1}u \\
&= T^n(x_n + T(0)) - \varphi(x'_{n+1})T^{n+1}u \\
&= T^n(x_n) + T^{n+1}(0), (\text{ since } u \in N(T^k) \subset N(T^{n+1})) \\
&= x + T^n(0) + T^{n+1}(0) = x + T^{n+1}(0).
\end{aligned}$$

So $x \in T^{n+1}x_{n+1}$. It's clear that $\varphi(x_{n+1}) = 0$. Now, we prove that $T^i x_{n+1} \subset \ker(\varphi)$ for all $1 \leq i \leq n$. For $1 \leq i \leq k-1$.

$$\begin{aligned}
T^i x_{n+1} &= T^i x'_{n+1} - \varphi(x'_{n+1})T^i u \\
&= T^{i-1}(Tx'_{n+1}) - \varphi(x'_{n+1})T^i u \\
&= T^{i-1}(x_n + T(0)) - \varphi(x'_{n+1})T^i u \\
&= T^{i-1}(x_n) + T^i(0) - \varphi(x'_{n+1})T^i u \\
&= T^{i-1}(x_n) - \varphi(x'_{n+1})T^i u.
\end{aligned}$$

Let $z \in T^i x_{n+1}$ so there exist $\alpha \in T^{i-1}x_n$ and $\beta \in T^i u$, such that $z = \alpha - \varphi(x'_{n+1})\beta$. Since $T^i u \subset N(T^{k-1}) \subset \ker(\varphi)$ also $T^{i-1}(x_n) \subset \ker(\varphi)$. It follows that,

$$\begin{aligned}
\varphi(z) &= \varphi(\alpha - \varphi(x'_{n+1})\beta) \\
&= \varphi(\alpha) - \varphi(x'_{n+1})\varphi(\beta) \\
&= 0.
\end{aligned}$$

Therefore for all $z \in T^i x_{n+1}$ we have $\varphi(z) = 0$. Hence $T^i x_{n+1} \subset \ker(\varphi)$, for all $1 \leq i \leq k-1$.

Now, for $k \leq i \leq n$ we have $u \in N(T^k)$ so $T^i u = T^i(0)$.

$$\begin{aligned} T^i x_{n+1} &= T^i x'_{n+1} - \varphi(x'_{n+1}) T^i u \\ &= T^{i-1}(T x'_{n+1}) - T^i(0) \\ &= T^{i-1}(x_n + T(0)) - T^i(0) \\ &= T^{i-1} x_n + T^i(0). \end{aligned}$$

Let $z \in T^i x_{n+1}$ so there exist $\alpha \in T^{i-1} x_n$ and $\beta \in T^i(0)$ such that, $z = \alpha + \beta$. As $T^{i-1} x_n \subset \ker(\varphi)$ for all $j \in \mathbb{N}$ we have $T^j(0) \subset N(F)$ then $T^j(0) \subset \ker(\varphi)$ for all $j \in \mathbb{N}$ hence we have $\varphi(z) = \varphi(\alpha) + \varphi(\beta) = 0$. It follows that, for all $z \in T^i x_{n+1}$ we have $\varphi(z) = 0$. Consequently, $T^i x_{n+1} \subset \ker(\varphi)$. Thus (3.3) is true for all $n \in \mathbb{N}$.
Thus,

$$\begin{aligned} (T+F)^n x_n &= (T+F)^{n-1}(T+F)x_n = (T+F)^{n-1}(Tx_n + F(x_n)) \\ &= (T+F)^{n-1}(Tx_n) \\ &= (T+F)^{n-2}(T+F)(Tx_n) = (T+F)^{n-2}(T^2 x_n + F(T(x_n))) \\ &= (T+F)^{n-3}(T+F)(T^2 x_n) = (T+F)^{n-3}(T^3 x_n + F(T^2(x_n)) \\ &= (T+F)^{n-4}(T^4 x_n) + F(T^3(x_n)) \\ &\vdots \\ &= T^n x_n \\ &= x + T^n(0), (\text{ since }, x \in T^n x_n) \\ &= x + (T+F)^n(0), (\text{ since }, (T+F)^n(0) = T^n(0)). \end{aligned}$$

Thus $x \in R(T+F)^n$ for all $n \in \mathbb{N}$. It follows that, $Y \subset R^\infty(T+F)$. Thus $N(T_1) \subset^e R^\infty(T+F)$. Since by Corollary 3.2, we have $N(T+F) \cap R(T+F)^d =^e N(T) \cap R(T^d) = N(T_1)$, then,

$$N(T+F) \cap R(T+F)^d \subset^e R^\infty(T+F).$$

It follows that,

$$\dim N(T+F) \cap R(T+F)^d / N(T+F) \cap R(T+F)^d \cap R^\infty(T+F) < \infty.$$

Therefore,

$$\dim N(T+F) \cap R(T+F)^d / N(T+F) \cap R^\infty(T+F) < \infty.$$

So $N(T+F) \cap R(T+F)^d \subset^e R^\infty(T+F) \cap N(T+F)$. On the other hand, we have

$$N(T+F) \cap R^\infty(T+F) \subset^e N(T+F) \cap R(T+F)^d.$$

Consequently

$$R^\infty(T+F) \cap N(T+F) =^e N(T+F) \cap R(T+F)^d.$$

This implies that

$$N(T+F) \cap R(T+F)^n =^e N(T+F) \cap R(T+F)^d, \text{ for all } n \geq d.$$

Thus, $k_n(T+F) = 0$, for all $n \geq d$. Now, it remains to show that $R(T+F)^{d+1}$ is closed. As

$$R(T^{d+1}) =^e R(T+F)^{d+1}.$$

So there exist two finite rank subspaces $G_1 \subset X$ and $G_2 \subset X$ such that

$$R(T^{d+1}) + G_1 = R(T+F)^{d+1} + G_2.$$

As $R(T^{d+1})$ is closed and $\dim(G_1) < \infty$ so $R(T^{d+1}) + G_1$ is closed. This implies that $R(T+F)^{d+1} + G_2$ is closed. Since $(T+F)^{d+1}$ is continuous and $D(T+F)^{d+1} = X$ and in the other hand $(T+F)^{d+1}(0)$ is closed, it follows that $(T+F)^{d+1}$ is closed. As $\dim(G_2) < \infty$, then by applying Lemma 3.6, to $(T+F)^{d+1}$ we get $R(T+F)^{d+1}$ is closed. Consequently $T+F$ is strictly quasi-Fredholm of degree d. □

4 Conclusions

We investigate some properties of the class of strictly quasi-Fredholm linear relations previously defined in [6]. By using some notions and results from algebra and functional analysis, we prove in particular that :

◇ The adjoint of a strictly quasi-Fredholm linear relation is a quasi-Fredholm linear operator.

◇ The power of a strictly quasi-Fredholm linear relation is also a strictly quasi-Fredholm linear relation.

◇ The product of two linear relations is strictly quasi-Fredholm if and only if each of them is a strictly quasi-Fredholm linear relation.

◇ The class of strictly quasi-Fredholm linear relations is stable under perturbation by finite rank operators.

References

[1] T. Alvarez, On Regular Linear Relations, Acta Mathematica Sinica, English Series Jan, (2012) Vol. 28, $N°$ 1, 183-194.

[2] R. Arens, Operational Calculus of Linear relations, Pacific J. Math. Volume 11, Number 1 (1961), 9-23.

[3] M. Berkani, Théorème de l'application spectrale pour le spectre essentiel quasi Fredholm, Proc. Amer. Soci. 125,(1997), 763-774 .

[4] M. Berkani, Restriction of an operator to the range of its powers, Studia Mathematica 140 (2) (2000), 162-175.

[5] M. Berkani and M. Sarih, On semi B-Fredholm operators, Glasgow Math J. 43 (2001) 457-465.

[6] H. Bouaniza and M. Mnif, On strictly quasi-Fredholm linear relations and semi-B-Fredholm linear relation perturbations, Filomat vol 31, $N°$ 20, (2017), 6337-6355.

[7] E. Chafai, M. Mnif, Descent and Essential Descent Spectrum of Linear Relations, Extracta Mathematicae vol 29, Nam. 1-2, (2014), 117-139.

[8] Y. Chamkha, M. Mnif, The class of B-Fredholm linear relations, Complex Analysis and Operator Theory vol 9, issue 8,(2015),1681-1699

[9] Y. Chamkha, Spectrum of matrix linear relations and study of B-Fredholm and Quasi-Fredholm linear relations, doctoral thesis,University of Sfax-Tunisie (2014).

[10] R. Cross, On the continuous linear image of a Banach space. J. Austral. Math. Soc. (Series A) 29 (1980), 219-234.

[11] R. Cross, Multivalued theorem for the product of linear relations, Linear Algebra and its Applications 277,(1998), 127-134.

[12] F. Fakhfakh, M.Mnif, Perturbation theory of lower semi-Browder multivalued linear operators, Pub. Math. Debrecen, 78/3-4 (2011), 595-606.

[13] S. Grabiner, Uniform ascent and descent of bounded operators, J. Math. Soc. Japan 34 $N°$ 2 (1982), 317-337.

[14] B. Hamid, K. Mohammed and T. Abdelaziz, Quasi-Fredholm, Saphar Spectra for C_0 Semigroups Generators, Ital. of pure and appl math. $N°$ 36, (2016), (359, 366).

[15] J. PH. Labrousse, Les opérateurs Quasi Fredholm: une généralisation des opérateurs semi-Fredholm, Rend. Circ. Math, Palermo. 29, (1980), 161-258.

[16] J. PH. Labrousse, The Kato decomposition of quasi-Fredholm relations, Operators and Matrices, 4 (2010), 1-51.

[17] M. Mbekhta, V. Muller, On the axiomatic of spectrum II, Studia mathematica 119 (2) (1996), 128-147.

[18] V. Muller, Spectral theory of linear operators and spectral systems in Banach algebras, Operator Theory: Advances and Applications Vol.139 Berlin (2007).

[19] A. Sandovici, H. DE Snoo, H. Winkler: Ascent, nullity, defect, and related notions for linear relations in linear spaces, Linear Algebra and its Applications 423, (2007) 456-497.

[20] D. Wilcox, Multivalued semi-Fredholm Operators in Normed Linear Spaces, doctoral thesis, University of Cape Town (2002).