Some results on Generalized Sasakian space forms

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Abstract: In this paper we study the $W_3 \cdot R = 0$, $\tilde{C} \cdot W_3 = 0$, $P \cdot W_3 = 0$, $W_3 \cdot Q = 0$, $Q \cdot W_3 = 0$, where $\tilde{C}$ is the Concircular curvature tensor, P is the projective curvature tensor, W_3 curvature tensor in generalized Sasakian space forms.

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1. Introduction

In 2004, P. Alegre, D. E. Blair and A. Carriazo [1] introduced the concept of generalized Sasakian space forms. The generalized Sasakian space form is defined as follows:

A generalized Sasakian space form in a Sasakian manifold (M, ϕ, ξ, η, g) whose Riemannian curvature tensor is given by

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\ &+ 2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ &- g(Y,Z)\eta(X)\xi\}, \end{aligned}$$

where f_1, f_2, f_3 are differentiable functions on M and X, Y, Z are vector fields on M. In such a case, we shall write generalized Sasakian space form as $M(f_1, f_2, f_3)$. This type of manifold appears as a natural generalization of the well known Sasakian space form $M(c)$, which can be obtained as a particular case of generalized Sasakian space form by taking $f_1 = \frac{c+3}{4}$, $f_2 = \frac{c-1}{4}$ and $f_3 = \frac{c-1}{4}$, where c denotes constant ϕ-sectional curvature.

The generalized Sasakian space forms have been extensively studied by many authors like [2, 3, 4, 6, 10, 11, 12, 14, 15, 18]. Based on the above work in this paper we study the $W_3(\xi, X) \cdot R = 0$, $\tilde{C} \cdot W_3 = 0$, $P \cdot W_3 = 0$, $W_3 \cdot Q = 0$, $Q \cdot W_3 = 0$, where $\tilde{C}$ is the Concircular curvature tensor, P is the projective curvature tensor, W_3 curvature tensor in generalized Sasakian space forms.

2. Preliminaries

An n-dimensional Riemannian manifold M is called an almost contact metric manifold if there exist a $(1,1)$ tensor field ϕ, a vector field ξ and a 1-form η such that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi \cdot \xi = 0, \quad \eta(\phi X) = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(X,\xi) = \eta(X), \tag{2.2}$$

$$g(\phi X, Y) = -g(X, \phi Y). \tag{2.3}$$

For an n-dimensional generalized Sasakian space form, we have

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\ &+ 2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ &- g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{2.4}$$

$$S(X,Y) = [(n-1)f_1 + 3f_2 - f_3]g(X,Y) + [-3f_2 - (n-2)f_3]\eta(X)\eta(Y), \tag{2.5}$$

$$r = (n-1)\{nf_1 + 3nf_2 - 2f_3\}. \tag{2.6}$$

For an n-dimensional almost contact metric manifold the Concircular curvature tensor $\tilde{C}$, Projective curvature tensor P and W_3 curvature tensor is given by:

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{2.7}$$

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}[S(Y,Z)X - S(X,Z)Y], \tag{2.8}$$

$$W_3(X,Y)Z = R(X,Y)Z + \frac{1}{(n-1)}[g(Y,Z)QX - S(X,Z)Y], \tag{2.9}$$

where R is the curvature tensor, Q is the Ricci operator and S is the Ricci tensor i.e., $g(QX,Y) = S(X,Y)$. From (2.4) for a generalized Sasakian space forms, we have

$$\eta(R(X,Y)Z) = (f_1 - f_3)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}, \tag{2.10}$$

$$R(X,Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\}, \tag{2.11}$$

$$R(\xi,X)Y = (f_1 - f_3)\{g(X,Y)\xi - \eta(Y)X\}, \tag{2.12}$$

From (2.5), we get

$$S(X,\xi) = (n-1)(f_1 - f_3)\eta(X). \tag{2.13}$$

3. Generalized Sasakian space form satisfying $W_3(\xi,X) \cdot R = 0$

This section deals with the study of generalized Sasakian-space form satisfying $W_3(\xi,X) \cdot R = 0$, where W_3 is a curvature tensor. In [17], Pokhariyal introduced the notion of a new curvature tensor, denoted by W_3 and studied its relativistic significance.

Let $M(f_1, f_2, f_3)$ be an n-dimensional generalized Sasakian-space form satisfying $W_3(\xi, X) \cdot R = 0$. Then, we have

$$\begin{aligned} & W_3(\xi, X)R(U, V)W - R(W_3(\xi, X)U, V)W \\ & -R(U, W_3(\xi, X)V)W - R(U, V)W_3(\xi, X)W = 0. \end{aligned} \tag{3.1}$$

Putting $W = \xi$ in (3.1) and by virtue of (2.11), we obtain

$$\begin{aligned} & (f_1 - f_3)\eta(W_3(\xi, X)U)V - (f_1 - f_3)\eta(W_3(\xi, X)V)U \\ & -2(f_1 - f_3)\eta(X)R(U, V)\xi + 2(f_1 - f_3)R(U, V)X = 0. \end{aligned} \tag{3.2}$$

By virtue of (2.9) in (3.2), we get

$$(f_1 - f_3)^2[g(X, U)V + S(X, U)V - g(X, V)U - S(X, V)U] + 2(f_1 - f_3)R(U, V)X = 0. \tag{3.3}$$

Taking inner product with ξ in (3.2), then we have

$$(f_1 - f_3)^2[g(X, U)\eta(V) + S(X, U)\eta(V) - g(X, V)\eta(U) - S(X, V)\eta(U)] + 2(f_1 - f_3)g(R(U, V)X, \xi) = 0. \tag{3.4}$$

Putting $U = \xi$ in (3.4) and by virtue of (2.12), (2.13), we get

$$S(X, V) = g(X, V) + [(n - 1)(f_1 - f_3) - 1]\eta(X)\eta(V). \tag{3.5}$$

On Contracting (3.6), we obtain

$$r = (n - 1)[1 + (f_1 - f_3)]. \tag{3.6}$$

Hence, we state the following:

Theorem 3.1. *An n-dimensional generalized Sasakian space form M satisfying the condition $W_3(\xi, X) \cdot R = 0$ is an η-Einstein manifold with scalar curvature $r = (n - 1)[1 + (f_1 - f_3)]$.*

4. Generalized Sasakian space form satisfying $\tilde{C} \cdot W_3 = 0$

This section deals with the study of generalized Sasakian-space form satisfying $\tilde{C} \cdot W_3 = 0$, where $\tilde{C}$ is a Concircular curvature tensor. A transformation of an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation . The interesting invariant of a concircular transformation is the concircular curavture tensor $\tilde{C}$, which is defined by [19].

Let $M(f_1, f_2, f_3)$ be an n-dimensional generalized Sasakian-space form satisfying $\tilde{C} \cdot W_3 = 0$. Then, we have

$$\begin{aligned} & \tilde{C}(X, Y)W_3(Z, U)V - W_3(\tilde{C}(X, Y)Z, U)V \\ & -W_3(Z, \tilde{C}(X, Y)U)V - W_3(Z, U)\tilde{C}(X, Y)V = 0. \end{aligned} \tag{4.1}$$

Putting $X = \xi$ in (4.1) and by virtue of (2.11), we obtain

$$\begin{aligned} & \tilde{C}(\xi, Y)W_3(Z, U)V - W_3(\tilde{C}(\xi, Y)Z, U)V \\ & -W_3(Z, \tilde{C}(\xi, Y)U)V - W_3(Z, U)\tilde{C}(\xi, Y)V = 0. \end{aligned} \tag{4.2}$$

By virtue of (2.7) in (4.2), we get

$$\left[(f_1 - f_3) - \frac{r}{n(n-1)}\right][g(Y, W_3(Z,U)V)\xi - \eta(W_3(Z,U)V)Y - g(Y,Z)W_3(\xi,U)V$$
$$+\eta(Z)W_3(Y,U)V - g(Y,U)W_3(Z,\xi)V + \eta(U)W_3(Z,Y)V$$
$$-g(Y,V)W_3(Z,U)\xi + \eta(V)W_3(Z,U)Y] = 0. \tag{4.3}$$

Taking inner product with ξ in (4.2), we get

$$\left[(f_1 - f_3) - \frac{r}{n(n-1)}\right][g(Y, W_3(Z,U)V) - \eta(W_3(Z,U)V)\eta(Y) - g(Y,Z)\eta(W_3(\xi,U)V)$$
$$+\eta(Z)\eta(W_3(Y,U)V) - g(Y,U)\eta(W_3(Z,\xi)V) + \eta(U)\eta(W_3(Z,Y)V)$$
$$-g(Y,V)\eta(W_3(Z,U)\xi) + \eta(V)\eta(W_3(Z,U)Y)] = 0. \tag{4.4}$$

Putting $U = \xi$ in (4.4) and by virtue of (2.9), (2.12), (2.13) and on simplification, we obtain

$$\left[(f_1 - f_3) - \frac{r}{n(n-1)}\right][(f_1 - f_3)g(Y,V)\eta(Z) - \frac{1}{(n-1)}S(Y,V)\eta(Z)]. \tag{4.5}$$

From (4.6), either $[(f_1 - f_3) - \frac{r}{n(n-1)}] = 0$ or

$$S(Y,V)\eta(Z) = (n-1)(f_1 - f_3)g(Y,V)\eta(Z). \tag{4.6}$$

Putting $Z = \xi$ in (4.6), we have

$$S(Y,V) = (n-1)(f_1 - f_3)g(Y,V). \tag{4.7}$$

On contracting (5.6), we obtain

$$r = n(n-1)(f_1 - f_3). \tag{4.8}$$

Hence, we state the following:

Theorem 4.2. *An n-dimensional generalized Sasakian space form M satisfying the condition $\tilde{C} \cdot W_3 = 0$ is an Einstein manifold with scalar curvature $r = n(n-1)(f_1 - f_3)$.*

5. Generalized Sasakian space form satisfying $P \cdot W_3 = 0$

Let $M(f_1, f_2, f_3)$ be an n-dimensional generalized Sasakian-space form satisfying $P \cdot W_3 = 0$. Then, we have

$$P(X,Y)W_3(Z,U)V - W_3(P(X,Y)Z,U)V$$
$$-W_3(Z,P(X,Y)U)V - W_3(Z,U)P(X,Y)V = 0. \tag{5.1}$$

Putting $X = \xi$ in (5.1), we obtain

$$P(\xi,Y)W_3(Z,U)V - W_3(P(\xi,Y)Z,U)V$$
$$-W_3(Z,P(\xi,Y)U)V - W_3(Z,U)P(\xi,Y)V = 0. \tag{5.2}$$

By virtue of (2.8) in (5.2), we get

$$\begin{aligned}&(f_1-f_3)[g(Y,W_3(Z,U)V)\xi-g(Y,Z)W_3(\xi,U)V-g(Y,U)W_3(Z,\xi)V-g(Y,V)W_3(Z,U)\xi]\\ &+\frac{1}{(n-1)}[-S(Y,W_3(Z,U)V)\xi+S(Y,Z)W_3(\xi,U)V+S(Y,U)W_3(Z,\xi)V+S(Y,V)W_3(Z,U)\xi]=0. \quad (5.3)\end{aligned}$$

Taking inner product with ξ in (5.2), we get

$$\begin{aligned}&(f_1-f_3)[g(Y,W_3(Z,U)V)-g(Y,Z)\eta(W_3(\xi,U)V)-g(Y,U)\eta(W_3(Z,\xi)V)-g(Y,V)\eta(W_3(Z,U)\xi)]\\ &+\frac{1}{(n-1)}[-S(Y,W_3(Z,U)V)+S(Y,Z)\eta(W_3(\xi,U)V)+S(Y,U)\eta(W_3(Z,\xi)V)\\ &+S(Y,V)\eta(W_3(Z,U)\xi)]=0. \quad (5.4)\end{aligned}$$

Putting $U=\xi$ in (5.4) and by virtue of (2.9), (2.12), (2.13) and on simplification, we obtain

$$S(Y,QZ)\eta(V)=(n-1)^2(f_1-f_3)^2g(Y,Z)\eta(V). \quad (5.5)$$

Putting $V=\xi$ in (5.5), we have

$$S^2(Y,Z)=(n-1)^2(f_1-f_3)^2g(Y,Z). \quad (5.6)$$

Hence, we state the following:

Theorem 5.3. *An n-dimensional generalized Sasakian space form M satisfying the condition $P\cdot W_3=0$, then the square of the Ricci tensor S^2 is equal to the $(n-1)^2(f_1-f_3)^2$ times of the metric tensor g.*

6. Generalized Sasakian space form satisfying $W_3\cdot Q=0$

Let $M(f_1,f_2,f_3)$ be an n-dimensional generalized Sasakian-space form satisfying $W_3\cdot Q=0$. Then, we have

$$W_3(X,Y)QZ-Q(W_3(X,Y)Z)=0. \quad (6.1)$$

Putting $Y=\xi$ in (6.1), we obtain

$$W_3(X,\xi)QZ-Q(W_3(X,\xi)Z)=0. \quad (6.2)$$

By virtue of (2.9) in (6.2), we get

$$\begin{aligned}&(n-1)(f_1-f_3)^2\eta(Z)X-\frac{1}{(n-1)}S(X,QZ)\xi\\ &+(n-1)(f_1-f_3)^2g(Z,X)\xi-\frac{1}{(n-1)}\eta(Z)Q^2X=0. \quad (6.3)\end{aligned}$$

Taking inner product with ξ in (6.3) and on simplification, we have

$$(n-1)(f_1-f_3)^2g(X,Z)-\frac{1}{(n-1)}S(X,QZ)=0. \quad (6.4)$$

On simplifying (6.4), we obtain

$$S(X,QZ)=(n-1)^2(f_1-f_3)^2g(X,Z). \quad (6.5)$$

which implies,

$$S^2(X,Z)=(n-1)^2(f_1-f_3)^2g(X,Z). \quad (6.6)$$

Hence, we state the following:

Theorem 6.4. *An n-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ satisfying the condition $W_3 \cdot Q = 0$, then the square of the Ricci tensor S^2 is equal to the $(n-1)^2(f_1 - f_3)^2$ of the metric tensor g.*

7. Generalized Sasakian space form satisfying $Q \cdot W_3 = 0$

Let $M(f_1, f_2, f_3)$ be an n-dimensional generalized Sasakian-space form satisfying $Q \cdot W_3 = 0$. Then, we have

$$Q(W_3(X,Y)Z) - W_3(QX,Y)Z - W_3(X,QY)Z - W_3(X,Y)QZ = 0. \tag{7.1}$$

Putting $Y = \xi$ in (7.1), we obtain

$$Q(W_3(X,\xi)Z) - W_3(QX,\xi)Z - W_3(X,Q\xi)Z - W_3(X,\xi)QZ = 0. \tag{7.2}$$

By virtue of (2.9) in (7.2), we get

$$\frac{2}{(n-1)} S(QX,Z)\xi + 2(f_1 - f_3)S(Z,X)\xi - 2(f_1 - f_3)\eta(Z)QX - 2(f_1 - f_3)^2\eta(Z)X = 0. \tag{7.3}$$

Taking inner product with ξ in (7.3) and on simplification, we have

$$S(QX,Z) = -(n-1)(f_1 - f_3)S(Z,X) + 2(n-1)^2(f_1 - f_3)^2\eta(Z)\eta(X). \tag{7.4}$$

which implies,

$$S^2(X,Z) = -(n-1)(f_1 - f_3)S(Z,X) + 2(n-1)^2(f_1 - f_3)^2\eta(Z)\eta(X). \tag{7.5}$$

Hence, we state the following:

Theorem 7.5. *An n-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ satisfying the condition $Q \cdot W_3 = 0$, then the square of the Ricci tensor S^2 is equal to the linear combination of the $(n-1)(f_1 - f_3)$ times of the Ricci tensor S and $2(n-1)^2(f_1 - f_3)^2$ times of $\eta \otimes \eta$.*

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