Structure of New Solitary Solutions for The Schwarzian Korteweg De Vries Equation And (2+1)-Ablowitz-Kaup-Newell-Segur Equation

Raghda A.M. Attia[1], Dianchen Lu[2]and Mostafa M.A. Khater[3]

[1, 2, 3] Department of Mathematics, Faculty of Science, Jiangsu University, China.

rony_695@yahoo.com &[2] dclu@ujs.edu.cn & [3] mostafa.khater2024@yahoo.com

Abstract

In this research, we introduce and represent the modified Khater method on two basic models in the optical fiber. These two models describe the dynamics of the wave movement in the optical fiber. It is a new modification of new recent method which developed by Mostafa M. A. Khater in 2017. We implement this new modified technique on Schwarzian Korteweg de Vries equation and (2+1)-Ablowitz-Kaup-Newell-Segur equation. This modification of Khater method produces more closed solutions than many other methods. Schwarzian Korteweg de Vries (SKdV) equation has a closed relationship with (2+1)-Ablowitz- Kaup-Newell-Segur equation. Schwarzian Korteweg de Vries equation prescribes the location in a micro-segment of space and motion of the isolated waves in varied fields which localized in a tiny portion of space. It is a great and basic system in fluid mechanics, nonlinear optics, plasma physics, and quantum field theory.

Keywords: Schwarzian Korteweg de Vries equation; (2+1)-Ablowitz-Kaup-Newell-Segur equation; Modified Khater method; Optical traveling wave solutions; Exact, solitary and approximate solutions.

1. Introduction

Partial differential equations (PDEs) that's the important part of the math that praises his names to many strands of science and that's because of his potential and abilities to characterize many cosmic and natural phenomena like physics and chemistry and biology, fluid mechanics, hydrodynamics, optics, plasma physics and other strands of science and knowledge. Especially, when Zabusky & Kruskal (1965) introduced the mean of the soliton. This is because of its analytical and descriptive capabilities for these different models where many recent techniques have emerged and scientists have developed ways to access the exact and individual solutions to these models. For example of these methods:

The $\left(\frac{G\prime}{G}\right)$ –expansion method, the $e^{-\phi(\xi)}$–expansion method, modified Kudryashov methods, modified $\left(\frac{G\prime}{G}\right)$ –expansion method, the $\left(\frac{G\prime}{G},\frac{1}{G}\right)$ –expansion method, extended $e^{-\phi(\xi)}$–expansion method, the extended tanh-function method, the Kudryashov, Novel $\left(\frac{G\prime}{G}\right)$ –expansion method, the improved $\tan\left(\frac{\phi}{2}\right)$-expansion method, modified simple equation method, Khater method , Adomian decomposing method and so [1]- [24].

Khater method is looked like one of the latest methods in this zone as it just detected from just one year and also it features the results of some techniques so that, these techniques can be examined as a particular condition of Khater technique.

Schwarzian Korteweg de Vries equation was displayed by Krichever and Novikov in the following form: see [25]

$$\frac{\Psi_t}{\Psi_x}+\left(\frac{\Psi_{x\,x}}{\Psi_x}\right)_x-\frac{1}{2}\left(\frac{\Psi_{x\,x}}{\Psi_x}\right)^2=0. \tag{1.1}$$

where Ψ satisfies Newton's equation of motion in a cubic potential. This equation has also another form as below:

$$S_t + \frac{1}{4}\, S_{x\,x\,z} - \frac{S_x\, S_{x\,z}}{2\,S} - \frac{S_{x\,x}\, S_z}{4\,S} + \frac{S_x^2\, S_z}{2\,S^2} - \frac{S_x}{8} \int \left(\frac{S_x^2}{S^2}\right)_z dx = 0. \tag{1.2}$$

This equation plays an important and vital role in a nonlocal form and a right-moving soliton. Schwarzian Korteweg de Vries is so closed to (2+1) Ablowitz–Kaup–Newell–Segur (AKNS) equation.

In this research, we implement a modified Khater method on these two modules. We demonstrate the basic steps of this new method. Through this study, the reader observes the extent of rapprochement between both methods but the only difference between the two methods is how much convergence solutions are given using modified Khater method for approximate solutions which speeds up modified Khater method is the successful extension of Khater method.

The vestiges of this paper are regulated as below: In section **2**, we give the structure of modified Khater method. In section 3, we implement the modified Khater method to gain new structure of the exact and solitary traveling wave solutions of the two mentioned models indicated above. In section **4**, conclusions are given.

2. Structure of modified Khater method:

Examination the nonlinear partial differential equation be in the below:

$$Q\big(u, u_x, u_y, u_z, u_t, u_{x\,x}, u_{x\,y}, u_{x\,z}, u_{x\,t}, \dots\big) = 0, \tag{2.1}$$

where Q is the polynomial function of u. Utilizing the wave transformation:

$$u(x, y, z, t) = u(\xi), \xi = x + y + z - ct. \tag{2.2}$$

We convert a nonlinear partial differential equation (NLPDE) into the nonlinear ordinary differential equation (NLODE) to be in the following form:

$$\Psi(u, u', u'', u''', \dots) = 0, \tag{2.3}$$

where Ψ is the function of $u(\xi)$. Balancing the highest order derivative term and nonla inear term which involved in Eq. (2.3).

Step 1. According to the modified Khater method, the general exact solution of an ordinary differential equation in the below:

$$u(\xi) = \sum_{i=0}^{N} a_i\, a^{i\, f(\xi)}. \tag{2.4}$$

Such that $f(\xi)$ depends on the following auxiliary equation:

$$f'(\xi) = \frac{1}{\ln(a)}\left[\alpha\, a^{-f(\xi)} + \beta + \sigma\, a^{f(\xi)}\right], \tag{2.5}$$

where (α, β, σ) are arbitrary constant will identify later.

Step 2. Evaluate the positive integer N in Eq. (2.4) and that by using the balance technique.

$$D\left[\frac{d^{\epsilon}u(\xi)}{d\,\xi^{\epsilon}}\right] = N + \epsilon,\ D\left[u^{\epsilon}\left(\frac{d^{\epsilon}u(\xi)}{d\,\xi^{\epsilon}}\right)^{s}\right] = \epsilon\,N + s(N+\epsilon).$$

Step 3. Replacement Eq. (2.4) into Eq. (2.4) and collecting all the terms of the same power of $\left[a^{i\,f(\xi)}, i = 0,1,2,\dots\right]$. We obtain a system of algebraic equations. Solving this system by utilization any computer program to obtain all previous parameters.

3. Application:

In this section, we implement modified Khater method for these two models (the Schwarzian Korteweg de Vries equation and (2+1)-Ablowitz-Kaup-Newell-Segur equation) and also we show the exact traveling wave solutions and solitary wave solutions of each one of these models.

3.1. The Schwarzian Korteweg de Vries equation:

Consider the Schwarzian Korteweg de Vries equations [26]- [30] in the following form:

$$\begin{cases} 4\,u^2\,v_x - 4\,u\,u_x v + u^2\,u_{x\,x\,z} - u\,u_{x\,x}u_z - 3\,u\,u_x u_{x\,z} + 3\,u_x^2 u_z - u^4\,u_z = 0, \\ u_t - v_t = 0. \end{cases} \tag{3.1.1}$$

Using the wave transformation $[u(x,z,t) = u(\xi), v(x,z,t)\xi = v(\xi), \xi = x + z - c\,t]$ and integration the second equation with zero constant of integration in the system then substituting the obtained equation into the first equation of the system, we obtain:

$$u^2\,u''' - 4\,u\,u'u'' + 3\,u'^2 - u^4\,u' = 0. \tag{3.1.2}$$

Balancing the highest order derivative term and nonlinear term $[u^2\,u'''\&u^4\,u'] \Rightarrow (N = 1)$. According to the general solution of the suggested method (modified Khater method), we get the general exact solution of Eq. (3.1.2) in the following form:

$$u(\xi) = a_0 + a_1\,a^{f(\xi)}. \tag{3.1.3}$$

Substituting Eq. (3.1.3) and its derivatives into Eq. (3.1.2). Collecting the coefficient of the same power of $\left[a^{i\,f(\xi)}, i = 0,1,2,\dots\right]$ and equating the result equations with zero. We get the system of algebraic equation. Solving this system by any computer program or even manually, we obtain:

Case 1.

$$\sigma = \frac{a_0\,(\beta - a_0)}{\alpha},\ a_0 = a_0\,, a_1 = \frac{a_0\,(\beta - a_0)}{\alpha}.$$

Consequently, we obtain the exact traveling solution in the form:

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\ a^{f(\xi)}\right]. \tag{3.1.4}$$

Therefore, the solitary traveling wave solutions are in the following form:

When $(\beta^2 - 4\alpha\sigma < 0\ \&\ \sigma \neq 0)$:

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\left(\frac{-\beta}{2\sigma} + \frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2\sigma}\, tan\left(\frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2}\,\xi\right)\right)\right], \tag{3.1.5}$$

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\left(\frac{-\beta}{2\sigma} + \frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2\sigma}\, cot\left(\frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2}\,\xi\right)\right)\right]. \tag{3.1.6}$$

When ($\beta^2 - 4\alpha\sigma > 0$ & $\sigma \neq 0$):

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\left(\frac{-\beta}{2\sigma} - \frac{\sqrt{(\beta^2 - 4\alpha\sigma)}}{2\sigma}\, tanh\left(\frac{\sqrt{(\beta^2 - 4\alpha\sigma)}}{2}\,\xi\right)\right)\right], \tag{3.1.7}$$

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\left(\frac{-\beta}{2\sigma} - \frac{\sqrt{(\beta^2 - 4\alpha\sigma)}}{2\sigma}\, coth\left(\frac{\sqrt{(\beta^2 - 4\alpha\sigma)}}{2}\,\xi\right)\right)\right]. \tag{3.1.8}$$

When ($\beta^2 + 4\alpha^2 > 0$ & $\sigma \neq 0$ & $\sigma = -\alpha$):

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\left(\frac{\beta}{2\alpha} + \frac{\sqrt{\beta^2 + 4\alpha^2}}{2\alpha}\, tanh\left(\frac{\sqrt{\beta^2 + 4\alpha^2}}{2}\,\xi\right)\right)\right], \tag{3.1.9}$$

$$u(\xi) = a_0\left[1\frac{(\beta - a_0)}{\alpha}\left(\frac{\beta}{2\alpha} + \frac{\sqrt{\beta^2 + 4\alpha^2}}{2\alpha}\, tanh\left(\frac{\sqrt{\beta^2 + 4\alpha^2}}{2}\,\xi\right)\right)\right]. \tag{3.1.10}$$

When ($\beta^2 + 4\alpha^2 < 0$ & $\sigma \neq 0$ & $\sigma = -\alpha$):

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\left(\frac{\beta}{2\alpha} - \frac{\sqrt{-(\beta^2 + \alpha^2)}}{2\alpha}\, tan\left(\frac{\sqrt{-(\beta^2 + \alpha^2)}}{2}\,\xi\right)\right)\right], \tag{3.1.11}$$

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\left(\frac{\beta}{2\alpha} - \frac{\sqrt{-(\beta^2 + \alpha^2)}}{2\alpha}\, cot\left(\frac{\sqrt{-(\beta^2 + \alpha^2)}}{2}\,\xi\right)\right)\right]. \tag{3.1.12}$$

When ($\beta^2 - 4\alpha^2 > 0$ & $\sigma \neq 0$ & $\sigma = \alpha$):

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\left(-\frac{\beta}{2\alpha} + \frac{\sqrt{-(\beta^2 - 4\alpha^2)}}{2\alpha}\, tan\left(\frac{\sqrt{-(\beta^2 - 4\alpha^2)}}{2}\,\xi\right)\right)\right], \tag{3.1.13}$$

$$(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\left(-\frac{\beta}{2\alpha} + \frac{\sqrt{-(\beta^2 - 4\alpha^2)}}{2\alpha}\, cot\left(\frac{\sqrt{-(\beta^2 - 4\alpha^2)}}{2}\,\xi\right)\right)\right]. \tag{3.1.14}$$

When ($\beta^2 - 4\alpha^2 > 0$ & $\sigma \neq 0$ & $\sigma = \alpha$):

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\left(-\frac{\beta}{2\alpha} - \frac{\sqrt{(\beta^2 - 4\alpha^2)}}{2\alpha}\,tanh\left(\frac{\sqrt{(\beta^2 - 4\alpha^2)}}{2}\,\xi\right)\right)\right], \tag{3.1.15}$$

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\left(-\frac{\beta}{2\alpha} - \frac{\sqrt{(\beta^2 - 4\alpha^2)}}{2\alpha}\,coth\left(\frac{\sqrt{(\beta^2 - 4\alpha^2)}}{2}\,\xi\right)\right)\right]. \tag{3.1.16}$$

When $(\alpha\sigma < 0\ \&\ \sigma \neq 0\ \&\ \beta = 0)$:

$$u(\xi) = a_0\left[1 - \frac{(\beta - a_0)}{\alpha}\,\frac{\sqrt{-\alpha\,\sigma}}{\sigma}\,tanh\left(\frac{\sqrt{-\alpha\,\sigma}}{2}\,\xi\right)\right], \tag{3.1.17}$$

$$u(\xi) = a_0\left[1 - \frac{(\beta - a_0)}{\alpha}\,\frac{\sqrt{-\alpha\,\sigma}}{\sigma}\,coth\left(\frac{\sqrt{-\alpha\,\sigma}}{2}\,\xi\right)\right]. \tag{3.1.18}$$

When $(\alpha\sigma > 0\ \&\ \sigma \neq 0\ \&\ \beta = 0)$:

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\,\frac{\sqrt{\alpha\,\sigma}}{\sigma}\,tan\left(\sqrt{\alpha\,\sigma}\,\xi\right)\right], \tag{3.1.19}$$

$$u(\xi) = a_0\left[1 - \frac{(\beta - a_0)}{\alpha}\,\sqrt{\frac{\alpha}{\sigma}}\,cot\left(\sqrt{\alpha\,\sigma}\,\xi\right)\right]. \tag{3.1.20}$$

When $(\beta = 0\ \&\alpha = -\sigma)$:

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\left(\frac{e^{2\,\alpha\,\xi} + 1}{e^{2\,\alpha\,\xi} - 1}\right)\right]. \tag{3.1.21}$$

When $(\beta = k\ \&\alpha = 2\,k\&\sigma = 0)$:

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\,\left(e^{k\,\xi} - 2\right)\right]. \tag{3.1.22}$$

When $(\beta = 0\ \&\alpha = \sigma)$:

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\,[tan(\alpha\,\xi + C)]\right]. \tag{3.1.23}$$

When $(\sigma = 0)$:

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\,\left(e^{\beta\,\xi} - \frac{\alpha}{\beta}\right)\right]. \tag{3.1.24}$$

Where $(C\ \&K)$ are arbitrary constants.

Case 2.

$$\sigma = -\frac{a_0\,(\beta + a_0)}{\alpha},\ a_0 = a_0\,, a_1 = \frac{a_0\,(\beta + a_0)}{\alpha}.$$

Where $[C\ \& K]$ are arbitrary constants.

Consequently, we obtain the exact traveling solution in the form:

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\ a^{f(\xi)}\right]. \tag{3.1.25}$$

Therefore, the solitary traveling wave solutions are in the following form:

When $(\beta^2 - 4\alpha\sigma < 0\ \&\ \sigma \neq 0)$:

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\left(\frac{-\beta}{2\sigma} + \frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2\sigma}\ tan\left(\frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2}\ \xi\right)\right)\right], \tag{3.1.26}$$

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\left(\frac{-\beta}{2\sigma} + \frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2\sigma}\ cot\left(\frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2}\ \xi\right)\right)\right]. \tag{3.1.27}$$

When $(\beta^2 - 4\alpha\sigma > 0\ \&\ \sigma \neq 0)$:

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\left(\frac{-\beta}{2\sigma} - \frac{\sqrt{(\beta^2 - 4\alpha\sigma)}}{2\sigma}\ tanh\left(\frac{\sqrt{(\beta^2 - 4\alpha\sigma)}}{2}\ \xi\right)\right)\right], \tag{3.1.28}$$

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\left(\frac{-\beta}{2\sigma} - \frac{\sqrt{(\beta^2 - 4\alpha\sigma)}}{2\sigma}\ coth\left(\frac{\sqrt{(\beta^2 - 4\alpha\sigma)}}{2}\ \xi\right)\right)\right]. \tag{3.1.29}$$

When $(\beta^2 + 4\alpha^2 > 0\ \&\ \sigma \neq 0\ \&\ \sigma = -\alpha)$:

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\left(\frac{\beta}{2\alpha} + \frac{\sqrt{\beta^2 + 4\alpha^2}}{2\alpha}\ tanh\left(\frac{\sqrt{\beta^2 + 4\alpha^2}}{2}\ \xi\right)\right)\right], \tag{3.1.30}$$

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\left(\frac{\beta}{2\alpha} + \frac{\sqrt{\beta^2 + 4\alpha^2}}{2\alpha}\ tanh\left(\frac{\sqrt{\beta^2 + 4\alpha^2}}{2}\ \xi\right)\right)\right]. \tag{3.1.31}$$

When $(\beta^2 + 4\alpha^2 < 0\ \&\ \sigma \neq 0\ \&\ \sigma = -\alpha)$:

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\left(\frac{\beta}{2\alpha} - \frac{\sqrt{-(\beta^2 + \alpha^2)}}{2\alpha}\, tan\left(\frac{\sqrt{-(\beta^2 + \alpha^2)}}{2}\,\xi\right)\right)\right], \tag{3.1.32}$$

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\left(\frac{\beta}{2\alpha} - \frac{\sqrt{-(\beta^2 + \alpha^2)}}{2\alpha}\, cot\left(\frac{\sqrt{-(\beta^2 + \alpha^2)}}{2}\,\xi\right)\right)\right]. \tag{3.1.33}$$

When $(\beta^2 - 4\alpha^2 > 0\ \&\ \sigma \neq 0\ \&\ \sigma = \alpha)$:

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\left(-\frac{\beta}{2\alpha} + \frac{\sqrt{-(\beta^2 - 4\alpha^2)}}{2\alpha}\, tan\left(\frac{\sqrt{-(\beta^2 - 4\alpha^2)}}{2}\,\xi\right)\right)\right], \tag{3.1. 34}$$

$$(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\left(-\frac{\beta}{2\alpha} + \frac{\sqrt{-(\beta^2 - 4\alpha^2)}}{2\alpha}\, cot\left(\frac{\sqrt{-(\beta^2 - 4\alpha^2)}}{2}\,\xi\right)\right)\right]. \tag{3.1.35}$$

When $(\beta^2 - 4\alpha^2 > 0\ \&\ \sigma \neq 0\ \&\ \sigma = \alpha)$:

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\left(-\frac{\beta}{2\alpha} - \frac{\sqrt{(\beta^2 - 4\alpha^2)}}{2\alpha}\, tanh\left(\frac{\sqrt{(\beta^2 - 4\alpha^2)}}{2}\,\xi\right)\right)\right], \tag{3.1.36}$$

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\left(-\frac{\beta}{2\alpha} - \frac{\sqrt{(\beta^2 - 4\alpha^2)}}{2\alpha}\, coth\left(\frac{\sqrt{(\beta^2 - 4\alpha^2)}}{2}\,\xi\right)\right)\right]. \tag{3.1.37}$$

When $(\alpha\sigma < 0\ \&\ \sigma \neq 0\ \&\ \beta = 0)$:

$$u(\xi) = a_0\left[1 - \frac{(\beta + a_0)}{\alpha}\,\frac{\sqrt{-\alpha\,\sigma}}{\sigma}\, tanh\left(\frac{\sqrt{-\alpha\,\sigma}}{2}\,\xi\right)\right], \tag{3.1.38}$$

$$u(\xi) = a_0\left[1 - \frac{(\beta + a_0)}{\alpha}\,\frac{\sqrt{-\alpha\,\sigma}}{\sigma}\, coth\left(\frac{\sqrt{-\alpha\,\sigma}}{2}\,\xi\right)\right]. \tag{3.1.39}$$

When $(\alpha\sigma > 0\ \&\ \sigma \neq 0\ \&\ \beta = 0)$:

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\,\frac{\sqrt{\alpha\,\sigma}}{\sigma}\, tan\left(\sqrt{\alpha\,\sigma}\,\xi\right)\right], \tag{3.1.40}$$

$$u(\xi) = a_0\left[1 - \frac{(\beta + a_0)}{\alpha}\,\sqrt{\frac{\alpha}{\sigma}}\, cot\left(\sqrt{\alpha\,\sigma}\,\xi\right)\right]. \tag{3.1.41}$$

When ($\beta = 0 \,\&\alpha = -\sigma$):

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\ \left(\frac{e^{2\,\alpha\,\xi} + 1}{e^{2\,\alpha\,\xi} - 1}\right)\right]. \tag{3.1.42}$$

When ($\beta = k \,\&\alpha = 2\,k\&\sigma = 0$):

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\ \left(e^{k\,\xi} - 2\right)\right]. \tag{3.1.43}$$

When ($\beta = 0 \,\&\alpha = \sigma$):

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\ [tan(\alpha\,\xi + C)]\right]. \tag{3.1.44}$$

When ($\sigma = 0$):

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\ \left(e^{\beta\,\xi} - \frac{\alpha}{\beta}\right)\right]. \tag{3.1.45}$$

Where ($C \,\&K$) are arbitrary constants.

3.2. The (2+1)-Ablowitz-Kaup-Newell-Segur equation:

Examine the (2+1)-Ablowitz-Kaup-Newell-Segue equation [26] and [31] - [34] in the following form:

$$4\,u_{x\,t} + u_{x\,x\,x\,z} + 8\,u_{x\,z}u_x + 4\,u_z\,u_{x\,x} = 0. \tag{3.2.1}$$

Using the wave transformation $[u(x,z,t) = u(\xi), v(x,z,t)\xi = v(\xi), \xi = x + z - c\,t]$, we get:

$$-4\,c\,u' + u''' + 6\,u'^2 = 0. \tag{3.2.2}$$

Balancing the highest order derivative term and nonlinear term $[\,u'''\&\,u'^2] \Rightarrow\ (N = 1)$. According to the general solution of the suggested method (modified Khater method), we get the general exact solution of Eq. (3.2.2) is the same to general exact solution of Eq. (3.1.2) . Substituting Eq. (3.1.3) and its derivatives into Eq. (3.2.2). Collecting the coefficient of the same power of $\left[a^{i\,f(\xi)}, i = 0,1,2,\dots\right]$ and equating the result equations with zero. We get the system of algebraic equation. Solving this system by any computer program or even manually, we obtain:

$$\alpha = \frac{\beta^2 - 4\,c}{4\,\sigma}, a_0 = a_0, a_1 = -\sigma.$$

Consequently, the exact traveling wave solution is in below:

$$u(\xi) = a_0 - \sigma\,a^{f(\xi)}. \tag{3.2.3}$$

Therefore, the solitary traveling wave solutions are in the following form:

When ($\beta^2 - 4\alpha\sigma < 0 \,\&\, \sigma \neq 0$):

$$u(\xi) = a_0 - \sigma\left[\frac{-\beta}{2\sigma} + \frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2\sigma}\tan\left(\frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2}\,\xi\right)\right], \tag{3.2.4}$$

$$u(\xi) = a_0 - \sigma\left[\frac{-\beta}{2\sigma} + \frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2\sigma}\cot\left(\frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2}\,\xi\right)\right]. \tag{3.2.5}$$

When ($\beta^2 - 4\alpha\sigma > 0$ & $\sigma \neq 0$):

$$u(\xi) = a_0 - \sigma\left[\frac{-\beta}{2\sigma} - \frac{\sqrt{(\beta^2 - 4\alpha\sigma)}}{2\sigma}\tanh\left(\frac{\sqrt{(\beta^2 - 4\alpha\sigma)}}{2}\,\xi\right)\right], \tag{3.2.6}$$

$$u(\xi) = a_0 - \sigma\left[\frac{-\beta}{2\sigma} - \frac{\sqrt{(\beta^2 - 4\alpha\sigma)}}{2\sigma}\coth\left(\frac{\sqrt{(\beta^2 - 4\alpha\sigma)}}{2}\,\xi\right)\right]. \tag{3.2.7}$$

When ($\beta^2 + 4\alpha^2 > 0$ & $\sigma \neq 0$ & $\sigma = -\alpha$):

$$u(\xi) = a_0 - \sigma\left[\frac{\beta}{2\alpha} + \frac{\sqrt{\beta^2 + 4\alpha^2}}{2\alpha}\tanh\left(\frac{\sqrt{\beta^2 + 4\alpha^2}}{2}\,\xi\right)\right], \tag{3.2.8}$$

$$u(\xi) = a_0 - \sigma\left[\frac{\beta}{2\alpha} + \frac{\sqrt{\beta^2 + 4\alpha^2}}{2\alpha}\coth\left(\frac{\sqrt{\beta^2 + 4\alpha^2}}{2}\,\xi\right)\right]. \tag{3.2.9}$$

When ($\beta^2 + 4\alpha^2 < 0$ & $\sigma \neq 0$ & $\sigma = -\alpha$):

$$u(\xi) = a_0 - \sigma\left[\frac{\beta}{2\alpha} - \frac{\sqrt{-(\beta^2 + 4\alpha^2)}}{2\alpha}\tan\left(\frac{\sqrt{-(\beta^2 + 4\alpha^2)}}{2}\,\xi\right)\right], \tag{3.2.10}$$

$$u(\xi) = a_0 - \sigma\left[\frac{\beta}{2\alpha} - \frac{\sqrt{-(\beta^2 + 4\alpha^2)}}{2\alpha}\cot\left(\frac{\sqrt{-(\beta^2 + 4\alpha^2)}}{2}\,\xi\right)\right]. \tag{3.2.11}$$

When ($\beta^2 - 4\alpha^2 < 0$ & $\sigma \neq 0$ & $\sigma = \alpha$):

$$u(\xi) = a_0 - \sigma\left[-\frac{\beta}{2\alpha} + \frac{\sqrt{-(\beta^2 - 4\alpha^2)}}{2\alpha}\tan\left(\frac{\sqrt{-(\beta^2 - 4\alpha^2)}}{2}\,\xi\right)\right], \tag{3.2.12}$$

$$u(\xi) = a_0 - \sigma\left[-\frac{\beta}{2\alpha} + \frac{\sqrt{-(\beta^2 - 4\alpha^2)}}{2\alpha}\cot\left(\frac{\sqrt{-(\beta^2 - 4\alpha^2)}}{2}\,\xi\right)\right]. \tag{3.2.13}$$

When ($\beta^2 - \alpha^2 > 0$ & $\sigma \neq 0$ & $\sigma = \alpha$):

$$u(\xi) = a_0 - \sigma\left[-\frac{\beta}{2\alpha} - \frac{\sqrt{(\beta^2 - 4\alpha^2)}}{2\alpha}\tanh\left(\frac{\sqrt{(\beta^2 - 4\alpha^2)}}{2}\,\xi\right)\right], \quad (3.2.14)$$

$$u(\xi) = a_0 - \sigma\left[-\frac{\beta}{2\alpha} - \frac{\sqrt{(\beta^2 - 4\alpha^2)}}{2\alpha}\coth\left(\frac{\sqrt{(\beta^2 - 4\alpha^2)}}{2}\,\xi\right)\right]. \quad (3.2.15)$$

When ($\alpha\sigma < 0$ & $\sigma \neq 0$ & $\beta = 0$):

$$u(\xi) = a_0 + \sqrt{-\alpha\,\sigma}\tanh\left(\frac{\sqrt{-\alpha\,\sigma}}{2}\,\xi\right), \quad (3.2.16)$$

$$u(\xi) = a_0 + \sqrt{-\alpha\,\sigma}\coth\left(\frac{\sqrt{-\alpha\,\sigma}}{2}\,\xi\right), \quad (3.2.17)$$

When ($\alpha\sigma > 0$ & $\sigma \neq 0$ & $\beta = 0$):

$$u(\xi) = a_0 - \sqrt{\alpha\,\sigma}\,tan\left(\sqrt{\alpha\,\sigma}\,\xi\right), \quad (3.2.18)$$

$$u(\xi) = a_0 + \sqrt{\alpha\,\sigma}\,tan\left(\sqrt{\alpha\,\sigma}\,\xi\right). \quad (3.2.19)$$

When ($\beta = 0$ &$\alpha = -\sigma$):

$$u(\xi) = a_0 - \sigma\left[\frac{e^{2\,\alpha\,\xi} + 1}{e^{2\,\alpha\,\xi} - 1}\right]. \quad (3.2.20)$$

When ($\alpha = \sigma = 0$):

$$u(\xi) = a_0 - \sigma e^{\,\beta\,\xi}. \quad (3.2.21)$$

When ($\beta^2 = \alpha\,\sigma$):

$$u(\xi) = a_0 + \frac{-2\alpha\sigma\,(\beta\,\xi + 2)}{\beta^2\,\xi}. \quad (3.2.22)$$

When ($\beta = k$ &$\alpha = 2\,k$&$\sigma = 0$):

$$u(\xi) = a_0 - \sigma\left[e^{k\,\xi} - 2\right]. \quad (3.2.23)$$

When ($\beta = k$ &$\sigma = 2\,k$&$\alpha = 0$):

$$u(\xi) = a_0 - \frac{\sigma e^{k\,\xi}}{1 - e^{k\,\xi}}. \quad (3.2.24)$$

When ($\alpha = 0$):

$$u(\xi) = a_0 - \frac{\sigma\beta\,e^{\beta\,\xi}}{2 - \sigma\,e^{\beta\,\xi}}. \quad (3.2.25)$$

When ($\beta = \sigma = 0$):

$$u(\xi) = a_0 - \sigma\,\alpha\,\xi. \tag{3.2.26}$$

When $(\beta = \alpha = 0)$:

$$u(\xi) = a_0 + \frac{\sigma}{\sigma\,\xi}. \tag{3.2.27}$$

When $(\beta = 0\ \&\alpha = \sigma)$:

$$u(\xi) = a_0 - \sigma tan(\alpha\,\xi + C). \tag{3.2.28}$$

When $(\sigma = 0)$:

$$u(\xi) = a_0 - \sigma\left[e^{\beta\,\xi} - \frac{\alpha}{\beta}\right]. \tag{3.2.29}$$

Where k, C are arbitrary constant.

4. Figures:

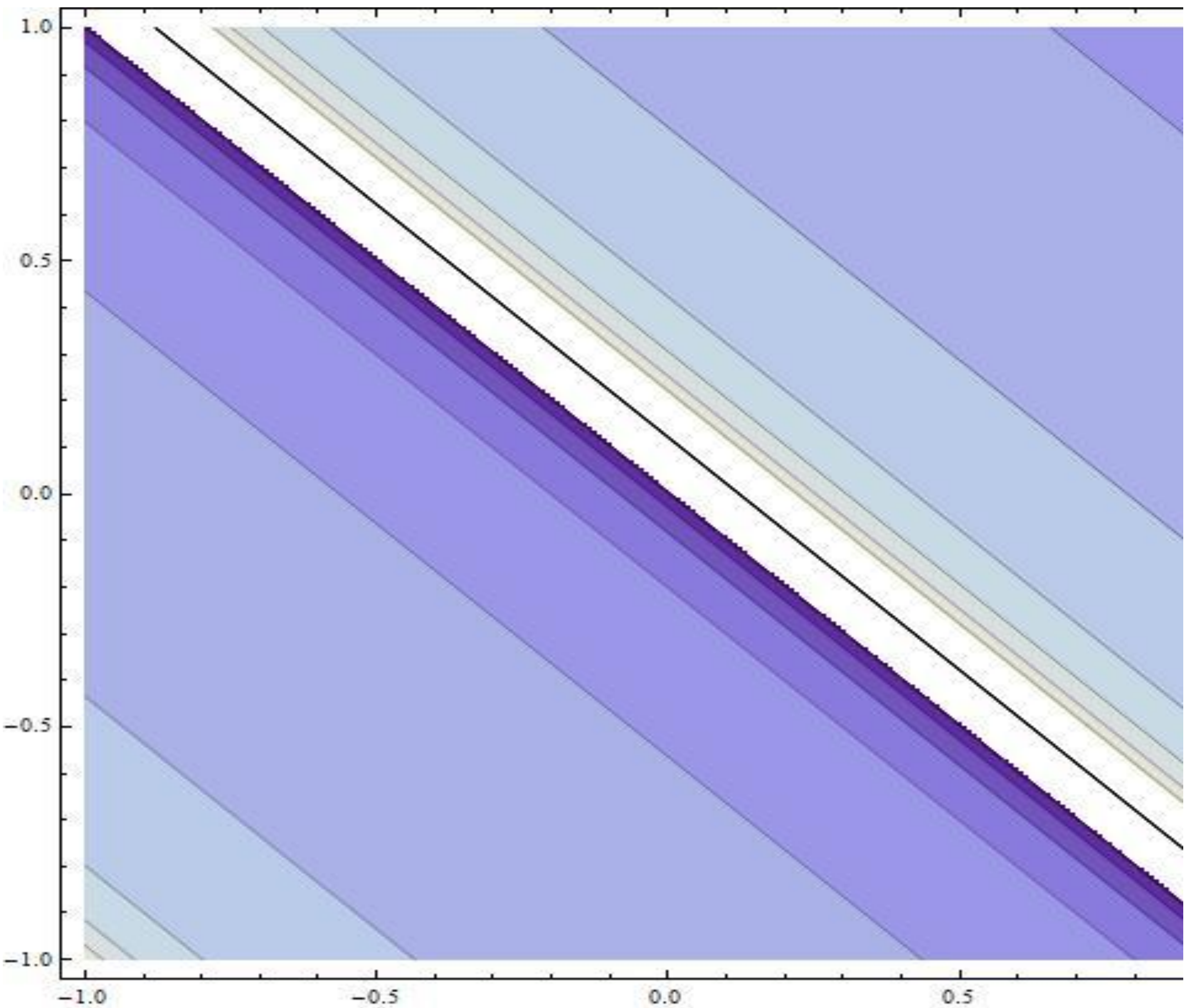

Fig.1 Contour plot of Eq. (3.1.5) when ($\alpha = 3, \beta = 2, \sigma = 4, a_0 = 5, z = 1, c = -1$).

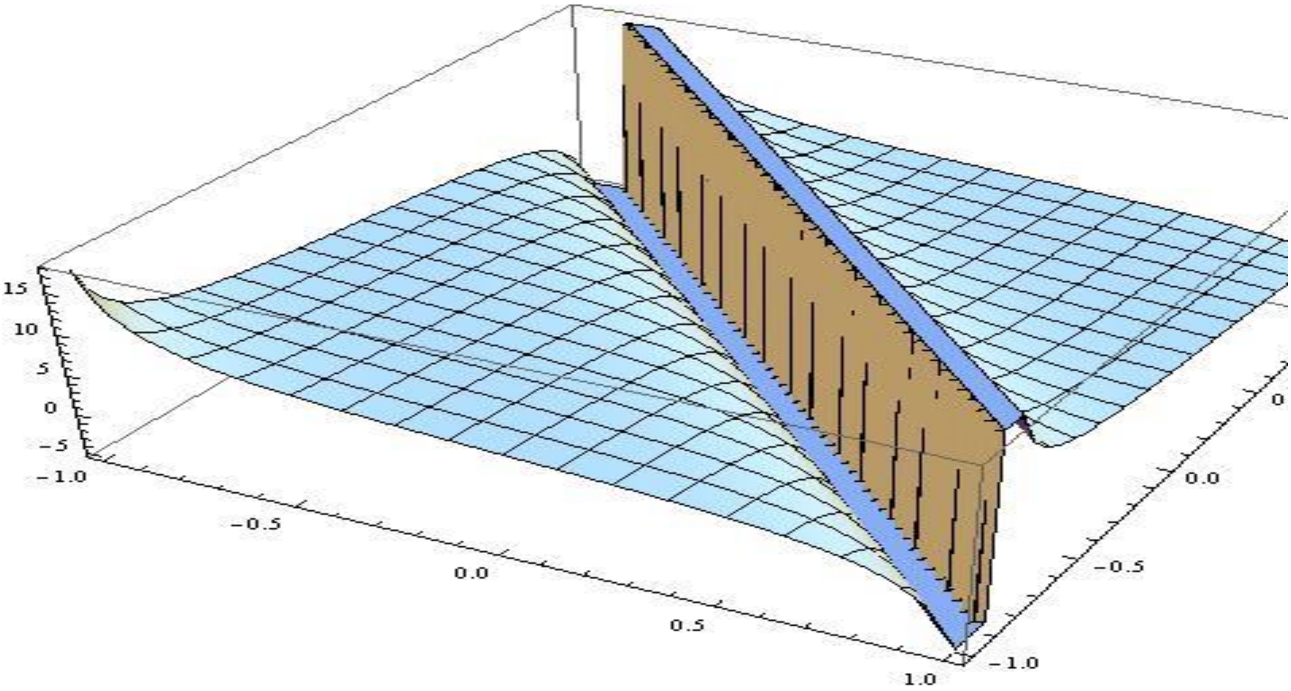

Fig.2 3D-plot of Eq. (3.1.5) when ($\alpha = 3, \beta = 2, \sigma = 4, a_0 = 5, z = 1, c = -1$).

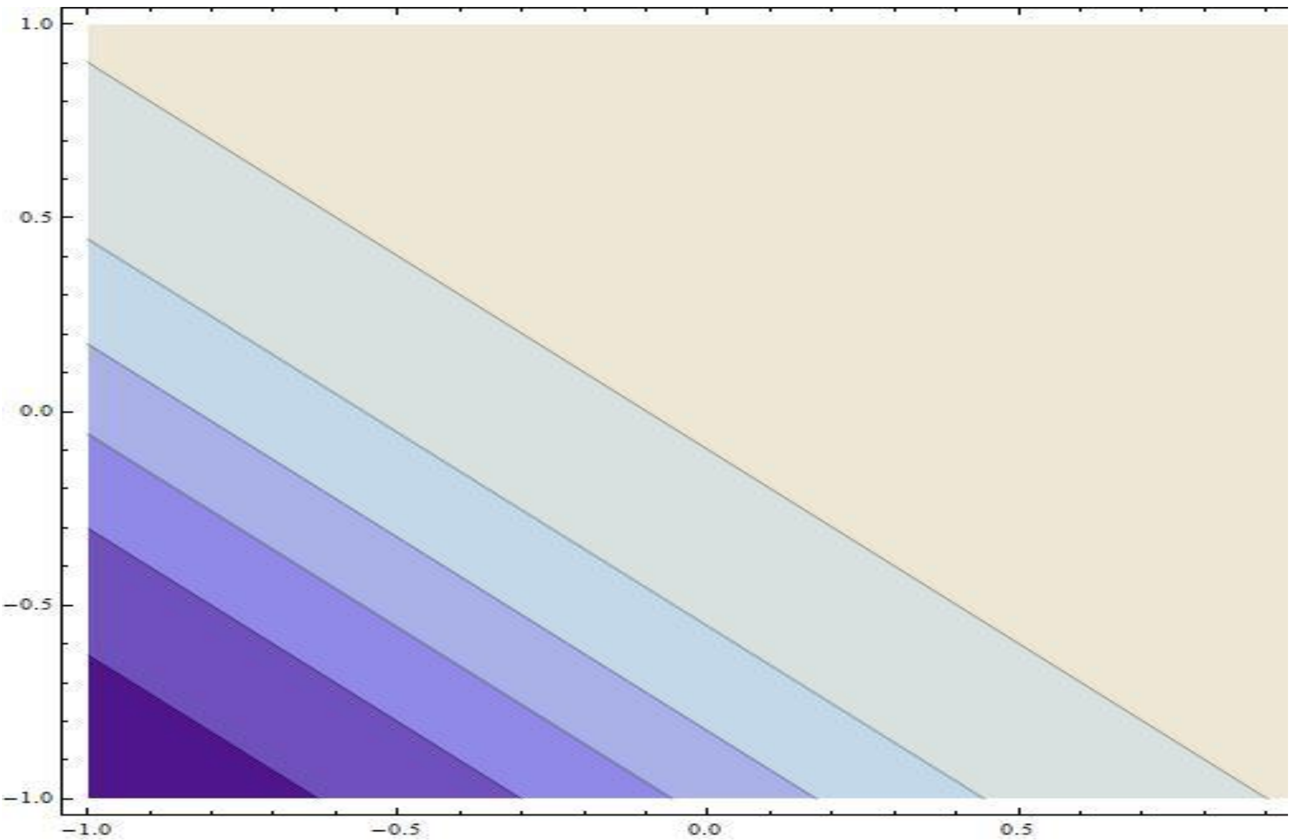

Fig.3 Contour plot of Eq. (3.1.6) when ($\alpha = 1, \beta = 3, \sigma = 2, a_0 = 5, z = 1, c = -1$).

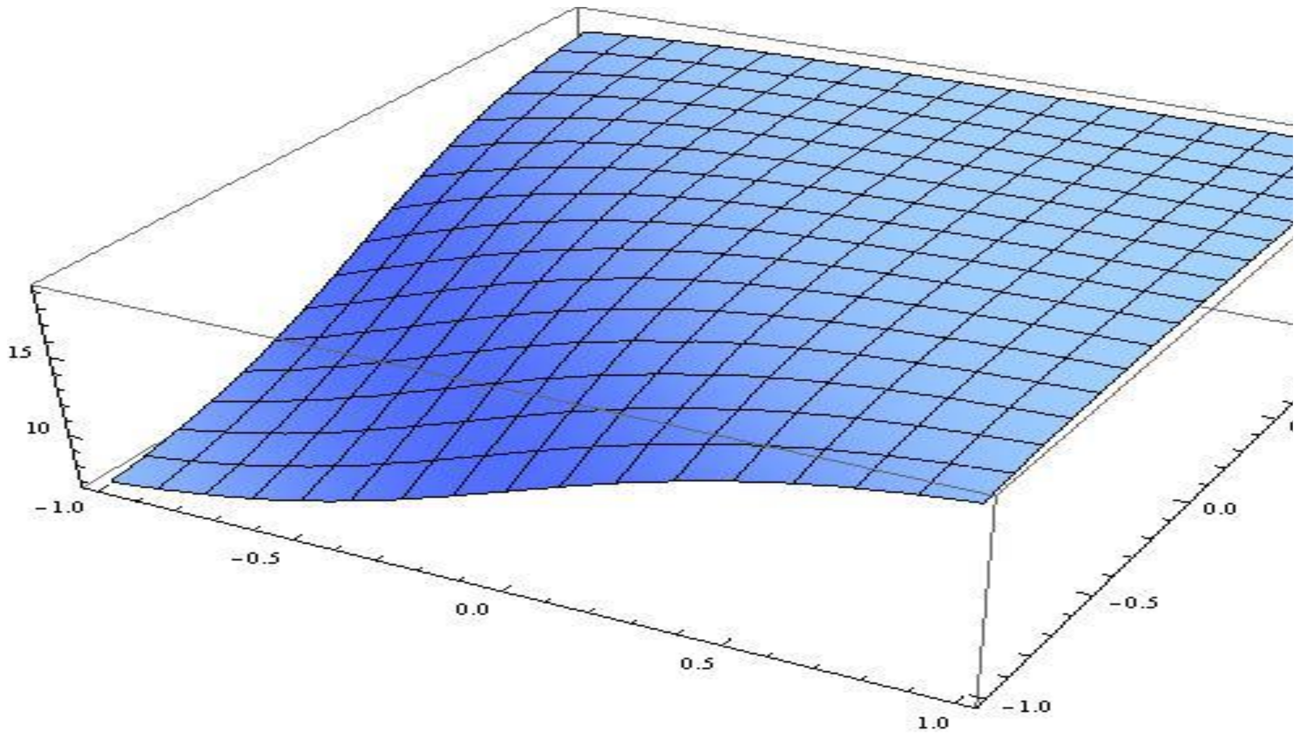

Fig.4 3D-plot of Eq. (3.1.6) when ($\alpha = 1, \beta = 3, \sigma = 2, a_0 = 5, z = 1, c = -1$).

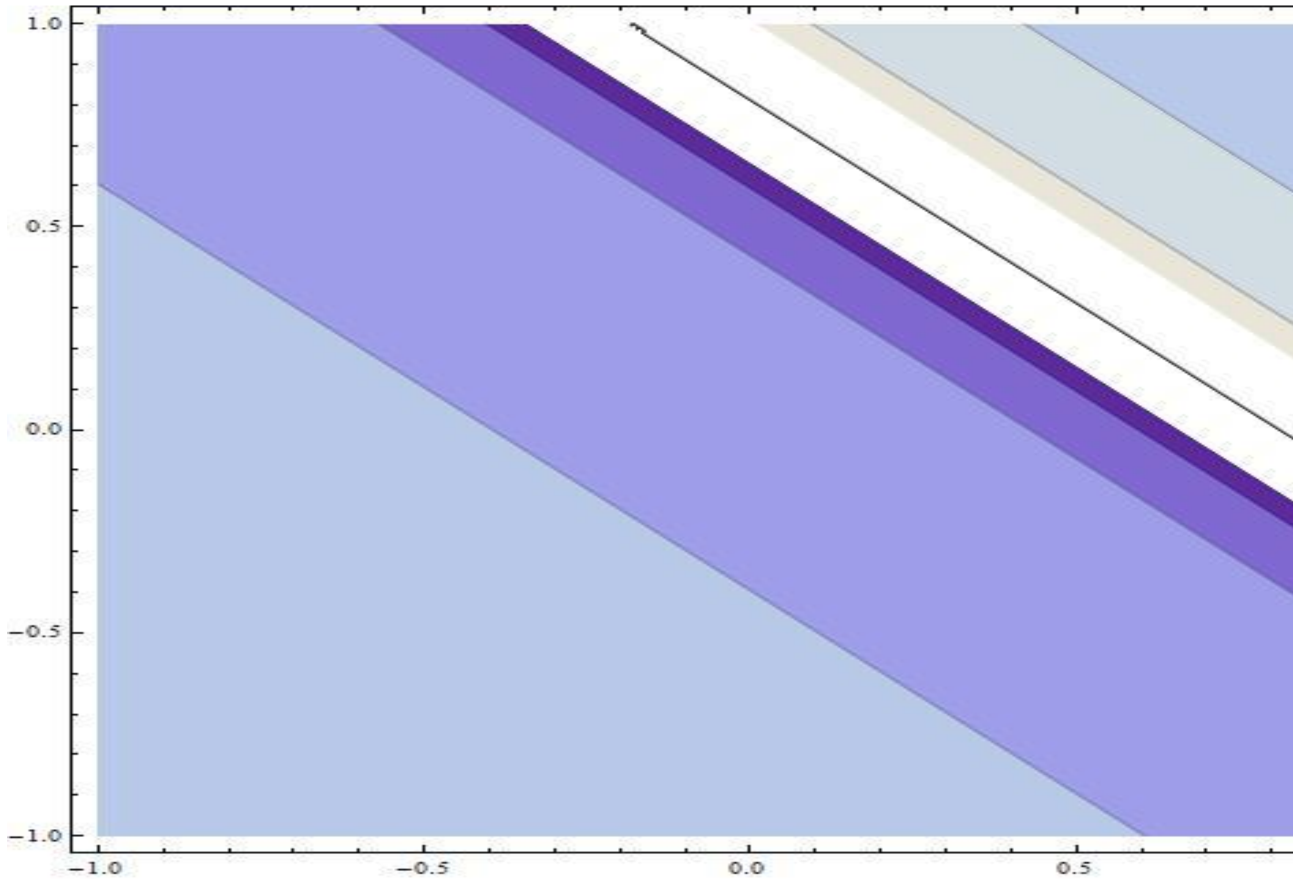

Fig.5 Contour plot of Eq. (3.1.11) when ($\alpha = 2, \beta = 1, \sigma = -2, a_0 = 5, z = 1, c = -1$).

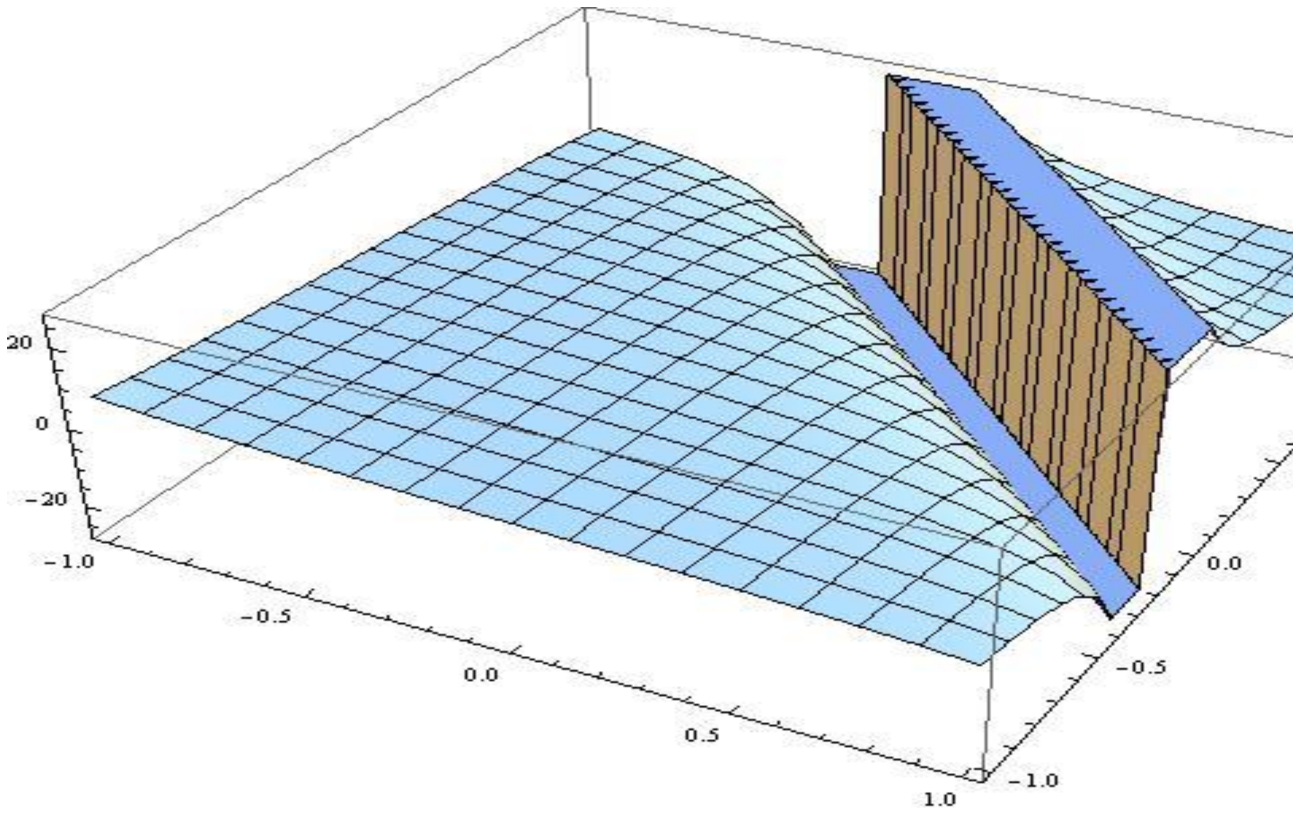

Fig.6 3D-plot of Eq. (3.1.11) when ($\alpha = 2, \beta = 1, \sigma = -2, a_0 = 5, z = 1, c = -1$).

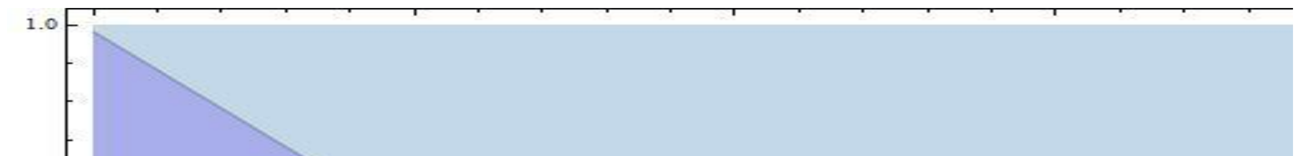

Fig.7 Contour plot of Eq. (3.1.21) when ($\alpha = -2, \beta = 0, \sigma = 2, a_0 = 5, z = 1, c = -1$).

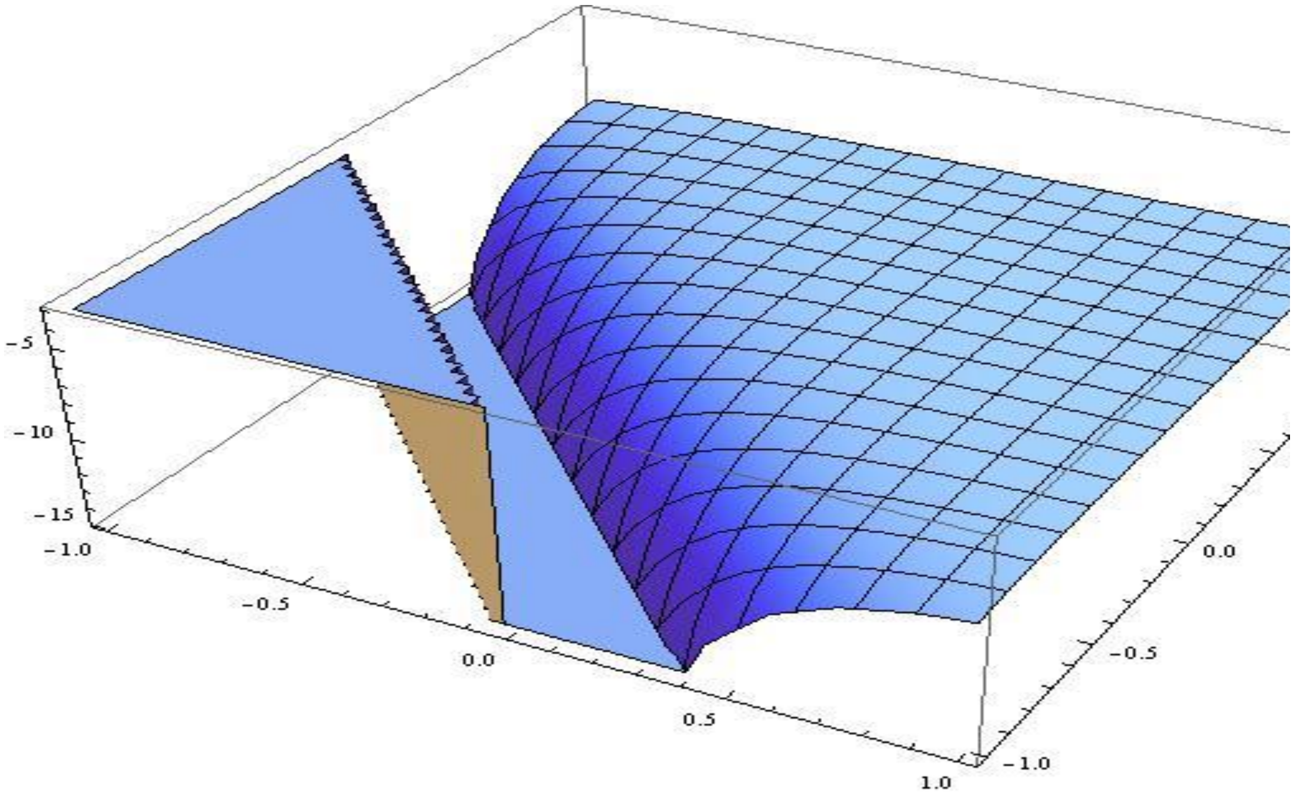

Fig.8 3D-plot of Eq. (3.1.21) when ($\alpha = -2, \beta = 0, \sigma = 2, a_0 = 5, z = 1, c = -1$).

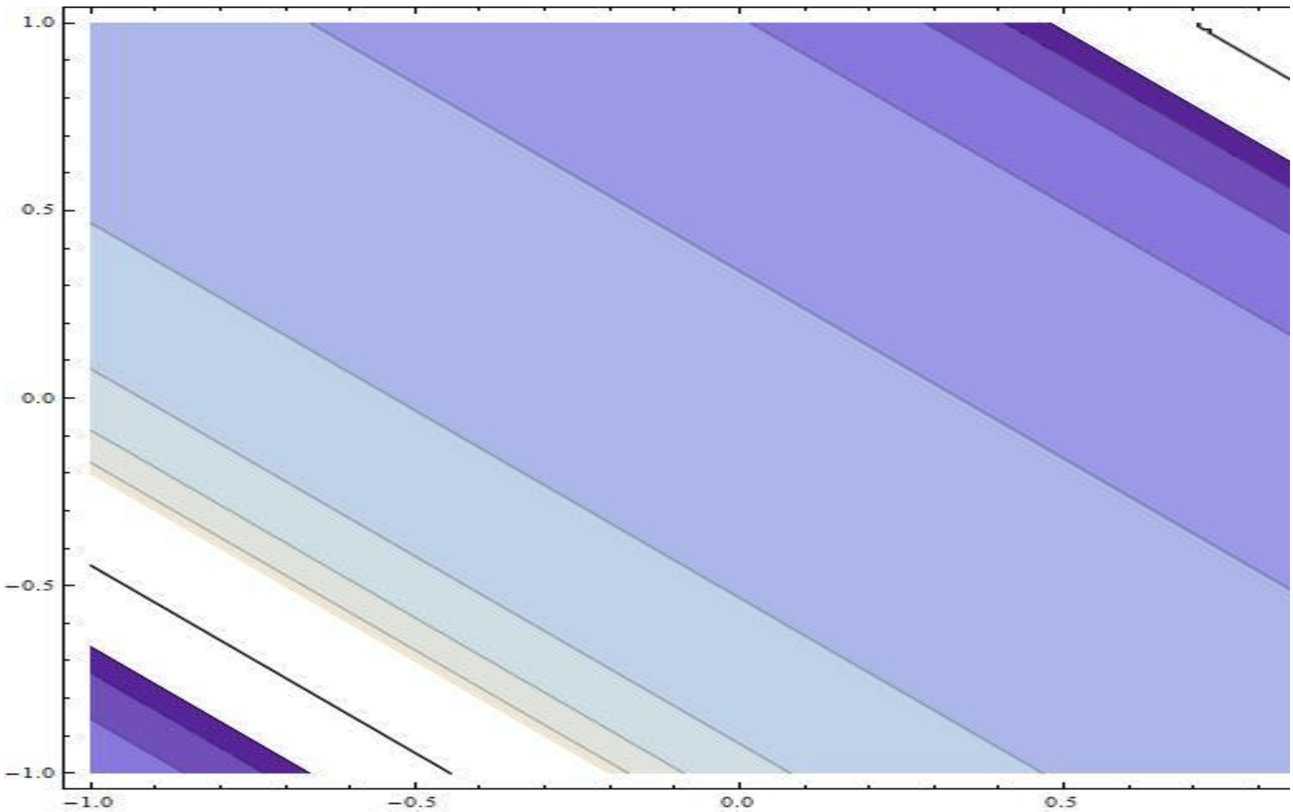

Fig.9 Contour plot of Eq. (3.1.25) when ($\alpha = 1, \beta = 0, \sigma = 1, a_0 = 5, z = 1, c = -1$).

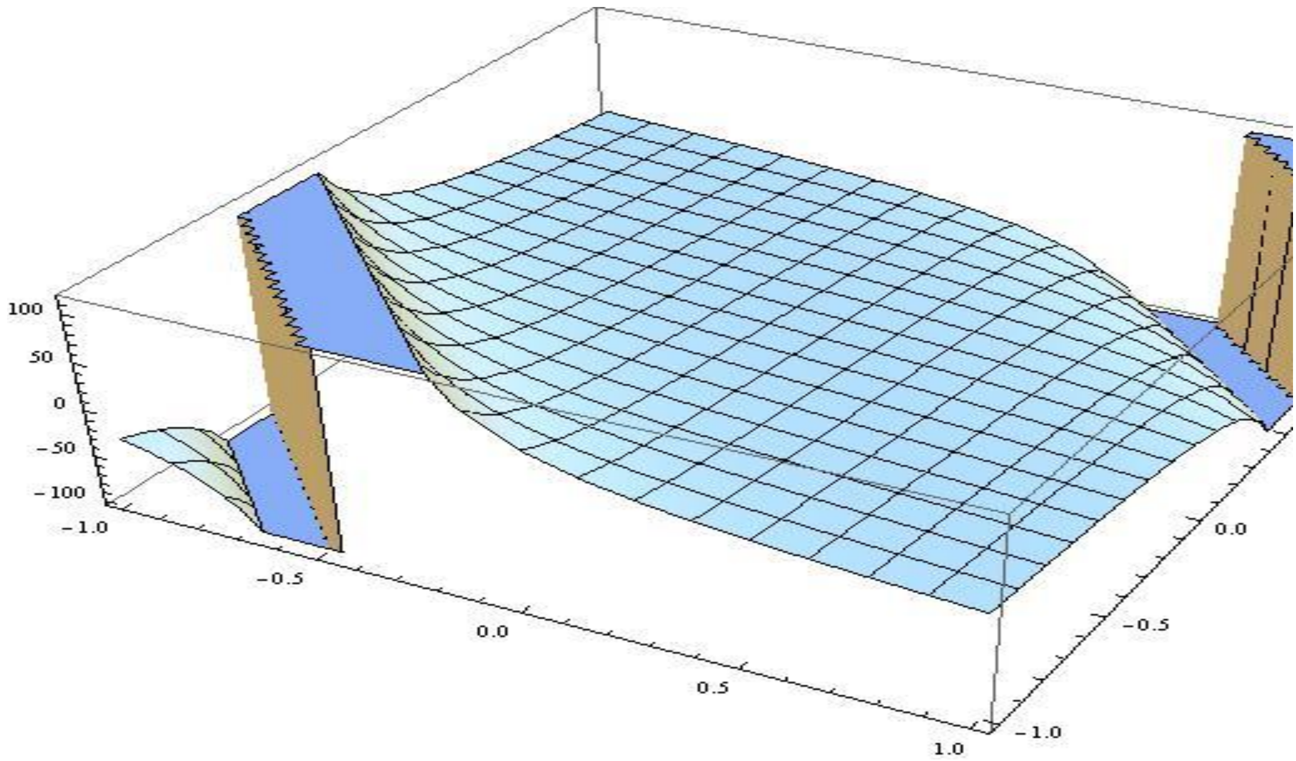

Fig.10 3D-plot of Eq. (3.1.25) when ($\alpha = 1, \beta = 0, \sigma = 1, a_0 = 5, z = 1, c = -1$).

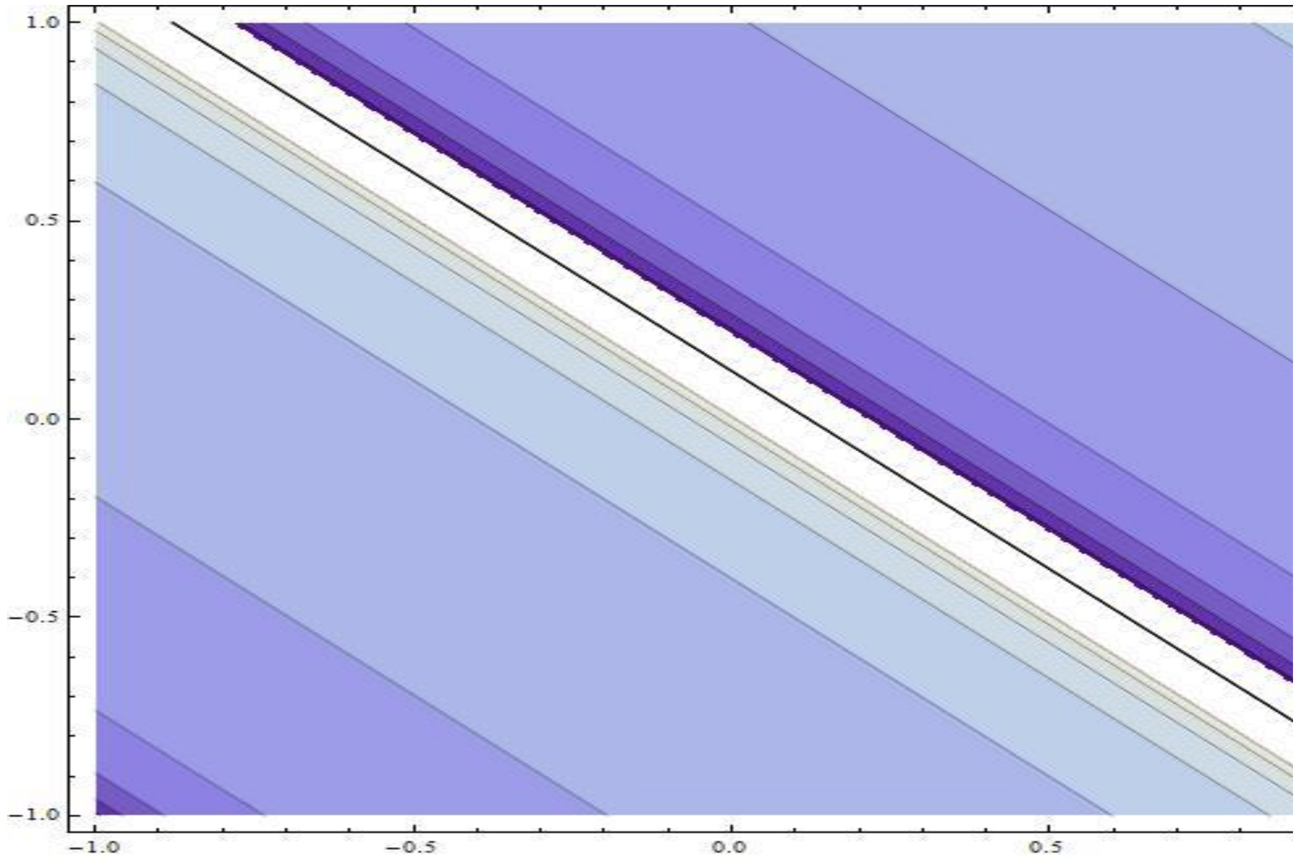

Fig.11 Contour plot of Eq. (3.1.42) when ($\alpha = 2, \beta = 0, \sigma = 1, a_0 = 5, z = 1, c = -1$).

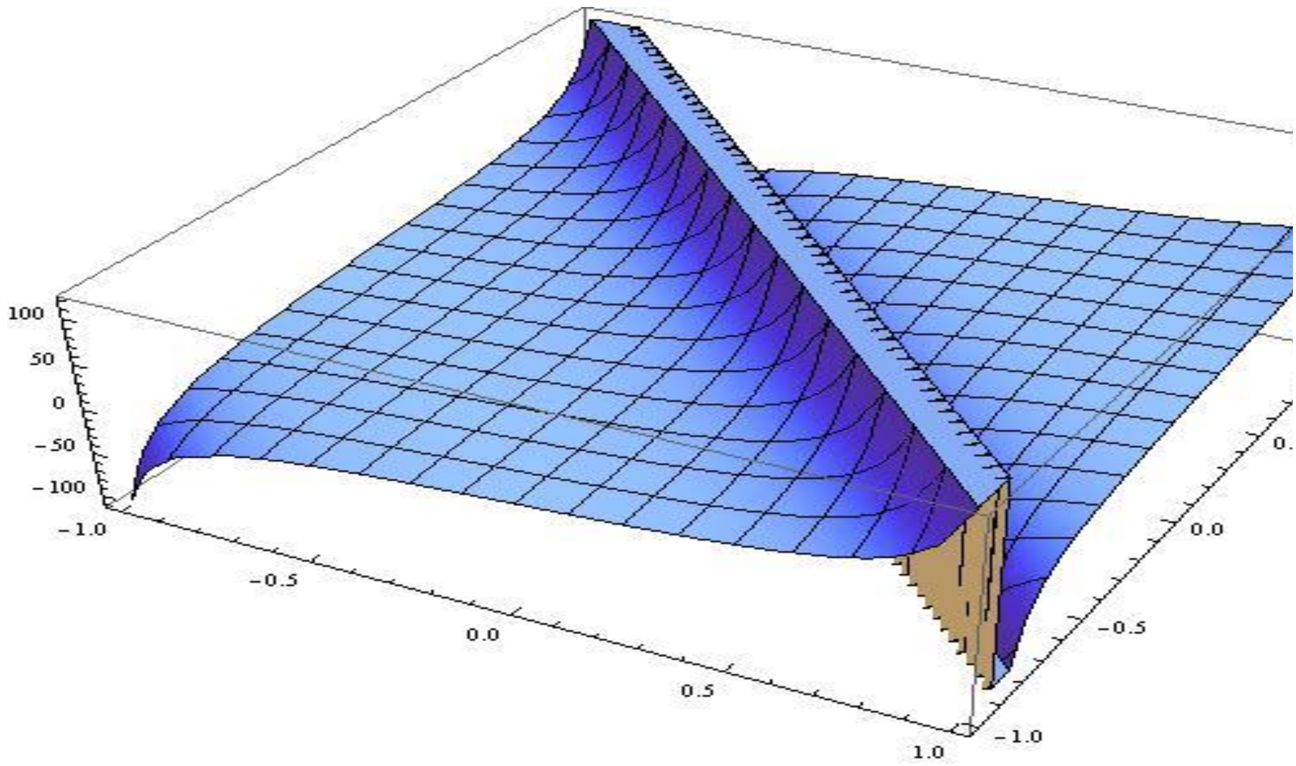

Fig.12 3D-plot of Eq. (3.1.42) when ($\alpha = 2, \beta = 0, \sigma = 1, a_0 = 5, z = 1, c = -1$).

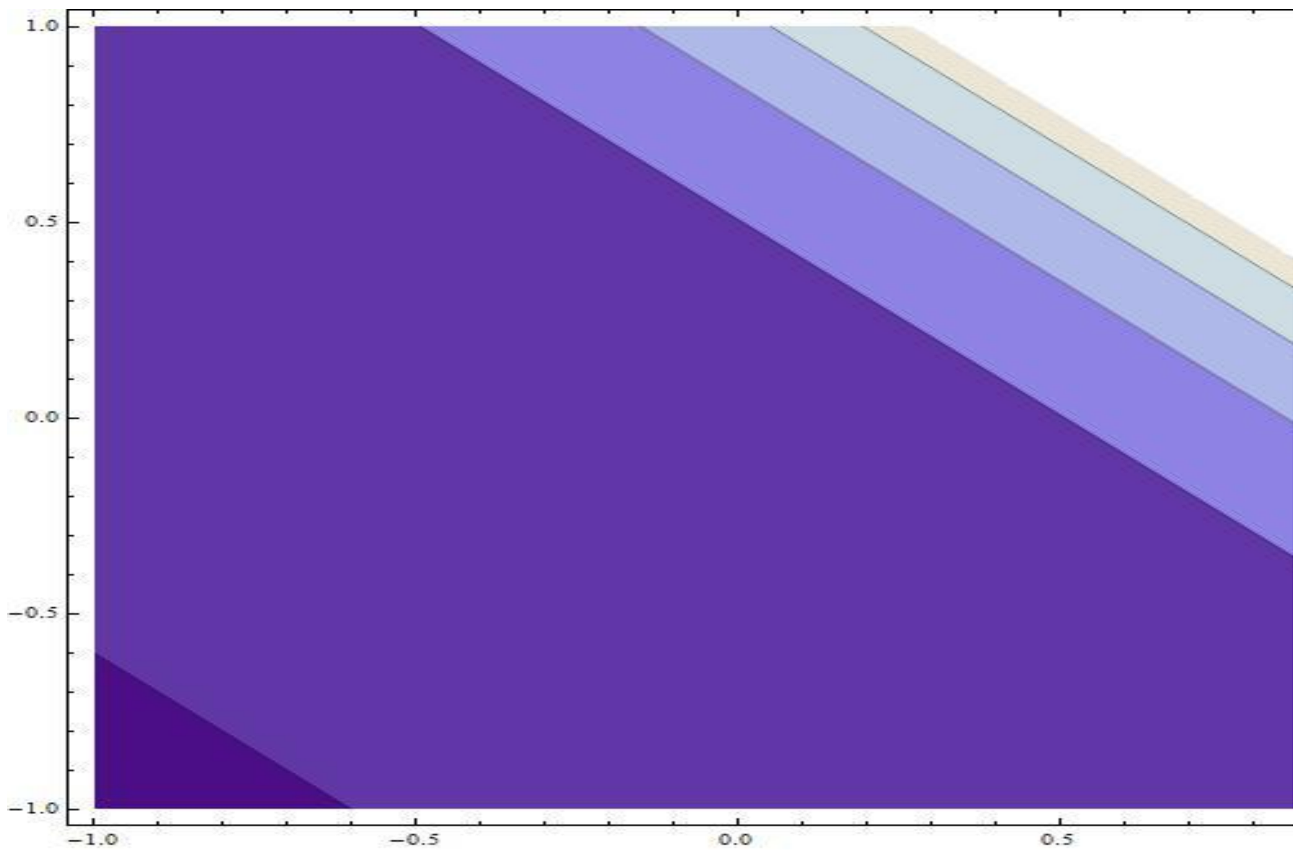

Fig.13 Contour plot of Eq. (3.1.51) when ($\alpha = 1, \beta = 2, \sigma = 0, a_0 = 5, z = 1, c = -1$).

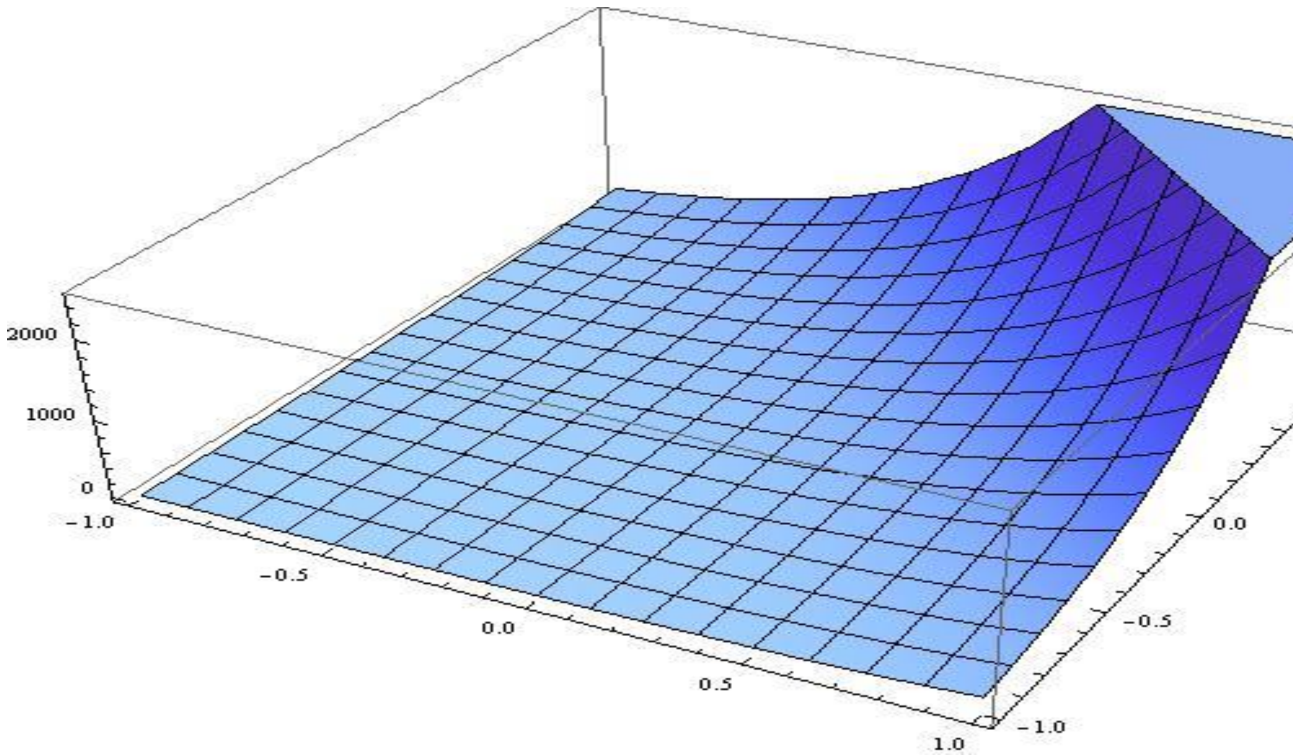

Fig.14 3D-plot of Eq. (3.1.51) when ($\alpha = 1, \beta = 2, \sigma = 0, a_0 = 5, z = 1, c = -1$).

5. Conclusion:

In this paper, we introduce a new modification of the Khater method. Khater method is considered as one of the most powerful generalized methods in nonlinear partial differential equation field. Especially, it concludes all solutions that can be obtained by using many different methods. We implement the modified Khater method on two significant modules in mathematical physics. We find a new form of solitary traveling solutions for Schwarzian Korteweg de Vries is so closed to (2+1) Ablowitz-Kaup-Newell-Segur (AKNS) equation. We plot some of our obtained solutions Fig. [1] - Fig. [14] to show the solitary and contour plot of these solutions. The earned solitary solutions show the physical features of each model. This renders examination the capabilities of these models and how they are applied in normal life. This helps in the progress and well-being of mankind.

6. Acknowledgements

(Mostafa Khater) I would like to dedicate this paper to my mother and the soul of my father; he was there for the beginning of this degree and did not make it to the end. His love, support, and constant care will never be forgotten. He is very much missed.

References

[1] Yang, Xiao-Jun, Feng Gao, and Hari M. Srivastava. "Exact travelling wave solutions for the local fractional two-dimensional Burgers-type equations." *Computers & Mathematics with Applications* 73, no. 2 (2017): 203-210.

[2] Seadawy, Aly R., Dianchen Lu, and Mostafa MA Khater. "Bifurcations of solitary wave solutions for the three dimensional Zakharov–Kuznetsov–Burgers equation and Boussinesq equation with dual dispersion." *Optik-International Journal for Light and Electron Optics* 143 (2017): 104-114.

[3] Seadawy, Aly R. "Stability analysis solutions for nonlinear three-dimensional modified Korteweg–de Vries–Zakharov–Kuznetsov equation in a magnetized electron–positron plasma." *Physica A: Statistical Mechanics and its Applications* 455 (2016): 44-51.

[4] Khater, Mostafa MA, Dianchen Lu, and Emad HM Zahran. "Solitary wave solutions of the Benjamin–Bona–Mahoney–Burgers equation with dual power-law nonlinearity." *Appl. Math. Inf. Sci* 11, no. 5 (2017): 1-5.

[5] Cheemaa, Nadia, and Muhammad Younis. "New and more exact traveling wave solutions to integrable (2+ 1)-dimensional Maccari system." *Nonlinear Dynamics* 83, no. 3 (2016): 1395-1401.

[6] Lu, Dianchen, Aly R. Seadawy, and Mostafa MA Khater. "Bifurcations of new multi soliton solutions of the van der Waals normal form for fluidized granular matter via six different methods." *Results in Physics* 7 (2017): 2028-2035.

[7] Wang, Haiyan, and Xiang-Sheng Wang. "Traveling wave phenomena in a Kermack–McKendrick SIR model." *Journal of Dynamics and Differential Equations* 28, no. 1 (2016): 143-166.

[8] Seadawy, Aly R., Dianchen Lu, and Mostafa MA Khater. "Bifurcations of traveling wave solutions for Dodd–Bullough–Mikhailov equation and coupled Higgs equation and their applications." *Chinese Journal of Physics* 55, no. 4 (2017): 1310-1318.

[8] Khater, Mostafa MA. "Exact traveling wave solutions for the generalized Hirota-Satsuma couple KdV system using the exp (− φ (ξ))-expansion method." *Cogent Mathematics* 3, no. 1 (2016): 1172397.

[9] Khater, Mostafa MA, Aly R. Seadawy, and Dianchen Lu. "Elliptic and solitary wave solutions for Bogoyavlenskii equations system, couple Boiti-Leon-Pempinelli equations system and Time-fractional Cahn-Allen equation." *Results in Physics* 7 (2017): 2325-2333.

[10] Darvishi, MTa, Sb Arbabi, Mb Najafi, and AMc Wazwaz. "Traveling wave solutions of a (2+ 1)-dimensional Zakharov-like equation by the first integral method and the tanh method." *Optik-International Journal for Light and Electron Optics* 127, no. 16 (2016): 6312-6321.

[11] Seadawy, Aly R., Dianchen Lu, and Mostafa MA Khater. "New wave solutions for the fractional-order biological population model, time fractional burgers, Drinfel'd–Sokolov–Wilson and system of shallow water wave equations and their applications." *European Journal of Computational Mechanics* (2017): 1-17.

[12] Zahran, Emad HM, and Mostafa MA Khater. "Modified extended tanh-function method and its applications to the Bogoyavlenskii equation." *Applied Mathematical Modelling* 40, no. 3 (2016): 1769-1775.

[13] Khater, Mostafa MA, Aly R. Seadawy, and Dianchen Lu. "Dispersive optical soliton solutions for higher order nonlinear Sasa-Satsuma equation in mono mode fibers via new auxiliary equation method." *Superlattices and Microstructures* (2017).

[14] Cheemaa, Nadia, and Muhammad Younis. "New and more general traveling wave solutions for nonlinear Schrödinger equation." *Waves in Random and Complex Media* 26, no. 1 (2016): 30-41.

[15] Khater, Mostafa MA, Aly R. Seadawy, and Dianchen Lu. "Dispersive solitary wave solutions of new coupled Konno-Oono, Higgs field and Maccari equations and their applications." *Journal of King Saud University-Science* (2017).

[16] Khater, Mostafa MA, Emad HM Zahran, and Maha SM Shehata. "Solitary wave solution of the generalized Hirota–Satsuma coupled KdV system." *Journal of the Egyptian Mathematical Society* 25, no. 1 (2017): 8-12.

[17] Khater, Mostafa MA, Aly R. Seadawy, and Dianchen Lu. "Optical soliton and rogue wave solutions of the ultra-short femto-second pulses in an optical fiber via two different methods and its applications." *Optik-International Journal for Light and Electron Optics* (2017).

[18] Gao, Feng, Xiao-Jun Yang, and Hari Mohan Srivastava. "Exact-Traveling-Wave Solutions for Linear and Nonlinear Heat-Transfer Equations." *Thermal Science* (2017).

[19] Khater, Mostafa MA, and Dipankar Kumar. "New exact solutions for the time fractional coupled Boussinesq–Burger equation and approximate long water wave equation in shallow water." *Journal of Ocean Engineering and Science* 2, no. 3 (2017): 223-228.

[20] Seadawy, Aly. "Stability analysis of traveling wave solutions for generalized coupled nonlinear KdV equations." *Applied Mathematics & Information Sciences* 10, no. 1 (2016): 209.

[21] Khater, Mostafa MA, and Dipankar Kumar. "Implementation of three reliable methods for finding the exact solutions of (2+ 1) dimensional generalized fractional evolution equations." *Optical and Quantum Electronics* 49, no. 12 (2017): 427.

[22] Yang, Xiao-Jun, J. A. Tenreiro Machado, Dumitru Baleanu, and Carlo Cattani. "On exact traveling-wave solutions for local fractional Korteweg-de Vries equation." *Chaos: An Interdisciplinary Journal of Nonlinear Science* 26, no. 8 (2016): 084312.

[23] Baskonus, Haci Mehmet, and Hasan Bulut. "Exponential prototype structures for (2+ 1)-dimensional Boiti-Leon-Pempinelli systems in mathematical physics." *Waves in Random and Complex Media* 26, no. 2 (2016): 189-196.

[24] Yang, Xiao-Jun, JA Tenreiro Machado, and Dumitru Baleanu. "Exact traveling-wave solution for local fractional Boussinesq equation in fractal domain." *Fractals* 25, no. 04 (2017): 1740006.

[25] Krichever, Igor Moiseevich, and Sergei Petrovich Novikov. "Holomorphic bundles over algebraic curves and non-linear equations." *Russian Mathematical Surveys* 35, no. 6 (1980): 53.

[26] Bruzón, M. S., M. L. Gandarias, Concepción Muriel, J. Ramirez, and F. R. Romero. "Traveling-Wave Solutions of the Schwarz-Korteweg-de Vries Equation in 2+ 1 Dimensions and the Ablowitz-Kaup-Newell-Segur Equation Through Symmetry Reductions." *Theoretical and mathematical physics* 137, no. 1 (2003): 1378-1389.

[27] Toda, Kouichi, and Song-Ju Yu. "The investigation into the Schwarz–Korteweg–de Vries equation and the Schwarz derivative in (2+ 1) dimensions." Journal of Mathematical Physics 41, no. 7 (2000): 4747-4751.

[28] Aslan, İsmail. "The Exp-function approach to the Schwarzian Korteweg–de Vries equation." Computers & Mathematics with Applications 59, no. 8 (2010): 2896-2900.

[29] Xiao-Yan, Tang, and Hu Heng-Chun. "Characteristic manifold and Painlevé integrability: Fifth-order Schwarzian Korteweg-de Vries type equation." Chinese physics letters 19, no. 9 (2002): 1225.

[30] Weiss, John. "The Painlevé property for partial differential equations. II: Bäcklund transformation, Lax pairs, and the Schwarzian derivative." Journal of Mathematical Physics 24, no. 6 (1983): 1405-1413.

[31] Chun-Long, Zheng, and Zhang Jie-Fang. "General solution and fractal localized structures for the (2+ 1)-dimensional generalized Ablowitz-Kaup-Newell-Segur system." Chinese physics letters 19, no. 10 (2002): 1399.

[32] Ishimori, Yuji. "A relationship between the Ablowitz-Kaup-Newell-Segur and Wadati-Konno-Ichikawa schemes of the inverse scattering method." Journal of the Physical Society of Japan 51, no. 9 (1982): 3036-3041.

[33] Najafi, Mohammad, Maliheh Najafi, and M. T. Darvishi. "New Exact Solutions to the (2+ 1)-Dimensional Ablowitz-Kaup-Newell-Segur Equation: Modification of the Extended Homoclinic Test Approach." *Chinese Physics Letters* 29, no. 4 (2012): 040202.

[34] Chun-Long, Zheng, Zhang Jie-Fang, Wu Feng-Min, Sheng Zheng-Mao, and Chen Li-Qun. "Solitons in a generalized (2+ 1)-dimensional Ablowitz-Kaup-Newell-Segur system." *Chinese Physics* 12, no. 5 (2003): 472.