The p-k-Wright Function

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Abstract

In this article we will presented a new definition of the function ${}_pW^{\gamma}_{k,\alpha,\beta}(z)$.

Some elementary properties of the new ${}_pW^{\gamma}_{k,\alpha,\beta}(z)$ are presented and his Laplace transform is obtained. Also it has been shown that the fractional Riemann-Liouville integral transform such functions with powers multipliers into functions of the same form with a very interesting relation between indices

Key Words: p-k-Wrigth function, Laplace Transform, Fractional calculus.

Mathematics Subject Classification: 26A33, 33E12.

I Preliminares

The importance of the role played by the Wright function $W_{\alpha,\beta}(z)$ in partial differential equation of fractional order is well known and was widely treated in papers by several authors including Gorenflo, Luchko, Mainardi (cf. [5]), Mainardi (cf. [9]), Mainardi, Pagnini (cf. [10]).

Romero - Cerutti (cf. [14]) introduce a generalization of the Wright function denoted by $W^{\gamma}_{k,\alpha,\beta}(z)$, that we will named the k-Wright function, in whose definition is used the k-Gamma function $\Gamma_k(z)$ and the k-Pochhammer symbol $(\gamma)_{n,k}$ defined Diaz and Pariguan (cf. [2]) .

Definition 1. Let $k \in \mathbb{R}^+$ $z, \alpha, \beta, \gamma \in \mathbb{C}$, $Re(\alpha) > 0$,$Re(\beta) > 0$.The k-Wrigth function is defined as

$$W^{\gamma}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n + \beta)} \frac{(z)^n}{(n!)^2} \tag{I.1}$$

where $(\gamma)_{n,k}$ is the k-Pochhammer symbol given by

$$(\gamma)_{n,k} = \gamma(\gamma + k)(\gamma + 2k) \ldots (\gamma + (n-1)k)$$

and $\Gamma_k(z)$ is the k-Gamma function $\Gamma_k(z)$

$$\Gamma_k(z) = \int_0^{\infty} e^{-\frac{t^k}{k}} t^{z-1}, \quad z \in \mathbb{C}\, ; k \in \mathbb{R}^+, \Re(z) > 0 \tag{I.2}$$

Recently, K.S. Gehlot (cf.[4]) has introduced a modification of the k-Gamma function by means of the following integral

$$ {}_p\Gamma_k(z) = \int_0^\infty e^{-\frac{t^k}{p}} t^{z-1} dt, \quad z \in \mathbb{C}\,; k, p \in \mathbb{R}^+, \Re(z) > 0 \tag{I.3} $$

Also, he has define a new Pochhammer symbol

$$ {}_p(z)_{n,k} = \left(\frac{zp}{k}\right) \cdot \left(\frac{zp}{k} + p\right) \cdot \left(\frac{zp}{k} + 2p\right) \dots \left(\frac{zp}{k} + (n-1)p\right) \tag{I.4} $$

$$ = \frac{{}_p\Gamma(z+nk)}{{}_p\Gamma(z)} \quad ; z \in \mathbb{C}, \Re(z) > 0 \tag{I.5} $$

where $k, p \in \mathbb{R}^+$,$n \in \mathbb{N}$ and also establishes the relation between the function introduced by him and k-Gamma and the clasical Gamma function . In fact, we have following.

Lemma 1. *For the p-k-Gamma function, the k-Gamma function and the classical Gamma function it is verified:*

$$ {}_p\Gamma_k(z) = \left(\frac{p}{k}\right)^{\frac{z}{k}} \Gamma_k(z) = \frac{p^{\frac{z}{k}}}{k} \Gamma\left(\frac{z}{k}\right) \quad ; z \in \mathbb{C}\,; k, p \in \mathbb{R}^+, \Re(z) > 0 \tag{I.6} $$

And, for the ${}_p(z)_{n,k}$ Pochhammer symbol, we have the following relation

Lemma 2. *For the p-k-Pochhammer symbol, the k-Pochhammer symbol and the classical Pochhammer symbol it has*

$$ {}_p(z)_{n,k} = \left(\frac{p}{k}\right)^n (z)_{n,k} = p^n (z)_n \quad ; z \in \mathbb{C}\,; k, p \in \mathbb{R}^+, \Re(z) > 0 \tag{I.7} $$

The proof Lemma 1 and Lemma 2 could be seen in (cf.[4]).□

In a recent paper (cf. [1]) have introduced a p-k-generalization of the classical Mittag-Leffler function

$$ {}_pE^{\gamma}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n + \beta)} \frac{(z)^n}{n!} \tag{I.8} $$

where $k, p \in \mathbb{R}^+$,$z, \alpha, \beta, \gamma \in \mathbb{C}$, $Re(\alpha) > 0$,$Re(\beta) > 0$. As particular cases of ${}_pE^{\gamma}_{k,\alpha,\beta}(z)$ we have the classical two parameters Mittag- Leffler function $E_{\alpha,\beta}(z)$ for $p = k = \gamma = 1$ and for $p = k$ one gets the k-Mittag- Leffler function $E^{\gamma}_{k,\alpha,\beta}(z)$

In the development of this paper we use fractional integrals and fractional derivatives, and also Laplace transform, so we introduce the definitions and notations.

Definition 2. Let f be a sufficiently well-behaved function with support in $\mathbb{R}^+$, and let ν be a real number, $\nu > 0$. The Riemann-Liouville fractional integral of order ν, $I^{\nu}_{+} f$ is given by

$$ I^{\nu}_{+} f(t) \doteq \frac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} f(\tau) d\tau \tag{I.9} $$

here $\Gamma(z)$ denotes the Gamma function of Euler

$$ \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \qquad , \Re(z) > 0 \tag{I.10} $$

It is known that the semigroup property is verified

$$I_+^{\nu} I_+^{\mu} = I_+^{\nu+\mu} \tag{I.11}$$

where by I_+^0 we denote the Identity operator.

The Riemann-Liouville fractional derivative of order $\nu > 0$, D_+^{ν} is defined as the left inverse of the Riemann-Liouville integral of order ν; i. e,

$$D_+^{\nu} I_+^{\nu} = I \qquad ,\nu > 0$$

Another way to defined this fractional derivative is as follows.

Definition 3. Let $\nu > 0$ be a real number, and let be m the integer sucht that $m-1 < \nu < m$. Then the Riemann-Liouville fractional derivative of order ν is given by

$$D_+^{\nu} f(t) = D^m I_+^{m-\nu} f(t) \tag{I.12}$$

Definition 4. Let $\mathfrak{L}(f)(s)$ be the Laplace transform of an exponential order function and piecewise continuous where

$$\mathfrak{L}\left[f(t)\right](s) = \int_0^{\infty} e^{-st} f(t)dt \tag{I.13}$$

$t \in \mathbb{R}^+$, and $s \in \mathbb{C}$.

Exist the relationship between the k-Wrigth function and the k-Mittag-Leffler function obtained throung the Laplace transform (cf[14]). In fact, we have the following

$$\mathfrak{L}[W^{\gamma}_{k,\alpha,\beta}(z)](s) = \frac{1}{s} E^{\gamma}_{k,\alpha,\beta}(s^{-1}) \tag{I.14}$$

where $k \in \mathbb{R}^+$,$\alpha, \beta, \gamma \in \mathbb{C}$, $\Re(\alpha) > 0$,$\Re(\beta) > 0$,$\Re(\gamma) > 0$ and $\Re(s) > 0$.

II Main Results

II.1 Definition and convergence

In view of the expressions (I.1),(I.3),(I.8) and (I.14) we introduce a new function of Wright type that we will call the p-k-Wright function by means of the following

Definition 5. *Let $k, p \in \mathbb{R}^+$,$\alpha, \beta, \gamma \in \mathbb{C}$, $Re(\alpha) > 0$,$Re(\beta) > 0$.The p-k-Wrigth function is defined as*

$${}_pW^{\gamma}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n + \beta)} \frac{(z)^n}{(n!)^2} \tag{II.1}$$

where ${}_p\Gamma_k(z)$ is given by (I.3), *and ${}_p(\gamma)_{n,k}$ is the Pochhammer symbol given by* (I.4).

Easily we can prove that ${}_pW^{\gamma}_{k,\alpha,\beta}(z) \to W^{\gamma}_{k,\alpha,\beta}(z)$ as $p \to k$, because ${}_p(\gamma)_{n,k} \to (\gamma)_{n,k}$ and ${}_p\Gamma_k \to \Gamma_k$.

As particular cases of ${}_pW^{\gamma}_{k,\alpha,\beta}(z)$ we have the classical two parameters Wright function $W_{\alpha,\beta}(z)$ for $p = k = \gamma = 1$.

Theorem 1. *The p-k-Wright function, defined in* (II.1),*is an entire function.*

Proof. Rewriting the series

$$ {}_pW^{\gamma}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta)} \frac{(z)^n}{(n!)^2} =: \sum_{n=0}^{\infty} c_n z^n \tag{II.2} $$

then radius of convergence of the p-k-Wright function will be called R such that

$$ R = \lim_{n\to\infty} \sup \left| \frac{c_n}{c_{n+1}} \right| \tag{II.3} $$

Taking into account the proof of Theorem 1. that the p-k-Mittag-Leffler function is an entire function,the definition of function (II.1) differs in that the factorials is squared ($(n!)^2$), therefore in an analogous way we can obtein apliying Lemma 1 and (I.5)

$$ \left| \frac{c_n}{c_{n+1}} \right| = (n+1)^2 |p^{\frac{\alpha}{k}-1}| \left| \frac{\Gamma\left(\frac{\alpha}{k}n + \frac{\alpha}{k} + \frac{\beta}{k}\right)}{\Gamma\left(\frac{\alpha}{k}n + \frac{\beta}{k}\right)} \right| \left| \frac{\Gamma\left(n+\frac{\gamma}{k}\right)}{\Gamma\left(n+\frac{\gamma}{k}+1\right)} \right| \tag{II.4} $$

$$ \approx (n+1)|p^{\frac{\alpha}{k}-1}| \left| \left(\frac{\alpha}{k} n\right)^{\frac{\alpha}{k}} \right| \to \infty \tag{II.5} $$

Thus, the p-k-Wright function is an entire function. □

II.2 Elementary Properties of the p-k-Wright function

In this section we obtain several elementary properties of our p-k-Wright defined by (II.1) and some others associated with the p-k-function Mittag-Leffler function obtained by means of the Laplace Transform.

Lemma 3 *Let* $k, p \in \mathbb{R}^+$,$\alpha, \beta, \gamma \in \mathbb{C}$, $Re(\alpha) > 0$,$Re(\beta) > 0$. *Then there holds the formula*

$$ \frac{d}{dz}\left[{}_pW^{\gamma}_{k,\alpha,\beta}(z)\right] = \left(\frac{\gamma p}{k}\right) \sum_{n=0}^{\infty} \frac{{}_p(\gamma+k)_{n,k}}{{}_p\Gamma_k(\alpha n+\alpha+\beta)} \frac{z^n}{(n+1)(n!)^2} \tag{II.6} $$

Proof Let $\alpha, \beta, \gamma \in \mathbb{C}$, $Re(\alpha) > 0$,$Re(\beta) > 0$.From definition (II.1), and Lemma 2 deduced the well known relations for the p-k-Pochhammer symbol

$$ {}_p(\gamma)_{n+1,k} = \left(\frac{\gamma p}{k}\right) {}_p(\gamma+k)_{n,k} $$

then

$$ \begin{aligned} \frac{d}{dz}\left[{}_pW^{\gamma}_{k,\alpha,\beta}(z)\right] &= \frac{d}{dz} \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta)} \frac{(z)^n}{(n!)^2} \\ &= \sum_{n=1}^{\infty} \frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta)} n \frac{z^{n-1}}{(n!)^2} \\ &= \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{n+1,k}}{{}_p\Gamma_k(\alpha(n+1)+\beta)} (n+1) \frac{z^n}{((n+1)!)^2} \\ &= \left(\frac{\gamma p}{k}\right) \sum_{n=0}^{\infty} \frac{{}_p(\gamma+k)_{n,k}}{{}_p\Gamma_k(\alpha n+\alpha+\beta)} \frac{z^n}{(n+1)(n!)^2} \qquad \square. \end{aligned} $$

Theorem 2 *Let $k, p \in \mathbb{R}^+$,$\alpha, \beta, \gamma \in C$, $Re(\alpha) > 0$,$Re(\beta) > 0$. Then there holds the formula*

$$ {}_pW^{\gamma+k}_{k,\alpha,\beta}(z) - {}_pW^{\gamma}_{k,\alpha,\beta}(z) = \left(\frac{k}{\gamma}\right) z \frac{d}{dz}\left[{}_pW^{\gamma}_{k,\alpha,\beta}(z)\right] \tag{II.7} $$

Proof Let $\alpha, \beta, \gamma \in C$, $Re(\alpha) > 0$,$Re(\beta) > 0$.From definition (II.1)

$$ {}_pW^{\gamma+k}_{k,\alpha,\beta}(z) - {}_pW^{\gamma}_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma+k)_{n,k} - ({}_p\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta)} \frac{(z)^n}{(n!)^2} \tag{II.8} $$

Taking into account that the Pochhammer p-k-symbol verified ${}_p(\gamma+k)_{n,k} = \left[1 + n\left(\frac{k}{\gamma}\right)\right] {}_p(\gamma)_{n,k}$ it result

$$ {}_p(\gamma+k)_{n,k} - {}_p(\gamma)_{n,k} = n\left(\frac{k}{\gamma}\right) {}_p(\gamma)_{n,k} \tag{II.9} $$

Replacing (II.9) in (II.8) we have

$$ \begin{aligned} {}_pW^{\gamma+k}_{k,\alpha,\beta}(z) - {}_pW^{\gamma}_{k,\alpha,\beta}(z) &= \sum_{n=0}^{\infty} \frac{n\left(\frac{k}{\gamma}\right) {}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta)} \frac{(z)^n}{(n!)^2} \\ &= \sum_{n=1}^{\infty} \frac{n\left(\frac{k}{\gamma}\right) {}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta)} \frac{(z)^n}{(n!)^2} \\ &= \sum_{n=0}^{\infty} \frac{(n+1)\left(\frac{k}{\gamma}\right) {}_p(\gamma)_{n+1,k}}{{}_p\Gamma_k(\alpha(n+1)+\beta)} \frac{(z)^{(n+1)}}{((n+1)!)^2} \\ &= \left(\frac{k}{\gamma}\right) z \left(\frac{\gamma p}{k}\right) \sum_{n=0}^{\infty} \frac{{}_p(\gamma+k)_{n,k}}{{}_p\Gamma_k(\alpha n+\alpha+\beta)} \frac{z^n}{(n+1)(n!)^2} \end{aligned} $$

Then, by Lemma 3. it results

$$ {}_pW^{\gamma+k}_{k,\alpha,\beta}(z) - {}_pW^{\gamma}_{k,\alpha,\beta}(z) = \left(\frac{k}{\gamma}\right) z \frac{d}{dt}\left[{}_pW^{\gamma}_{k,\alpha,\beta}(z)\right] \qquad \square. $$

Remark Furthermore, it can be proved that verified the following differential equation

$$ {}_pW^{\gamma+k}_{k,\alpha,\beta}(z) = \left(\frac{k}{\gamma}\right) z \frac{d}{dt}\left[{}_pW^{\gamma}_{k,\alpha,\beta}(z)\right] + {}_pW^{\gamma}_{k,\alpha,\beta}(z) \tag{II.10} $$

Theorem 3 *Let $k, p \in \mathbb{R}^+$,$\alpha, \beta, \gamma \in C$, $Re(\alpha) > 0$,$Re(\beta) > 0$. Then there holds the formula*

$$ {}_pW^{\gamma}_{k,\alpha,\beta}(z) = \left(\frac{\alpha p}{k}\right) z \frac{d}{dz}\left[{}_pW^{\gamma}_{k,\alpha,\beta+k}(z)\right] + \left(\frac{\beta p}{k}\right) {}_pW^{\gamma}_{k,\alpha,\beta+k}(z) \tag{II.11} $$

Proof. By definition (II.1) we have

$$\left(\frac{\alpha p}{k}\right) z\frac{d}{dz}\left[{}_pW^{\gamma}_{k,\alpha,\beta+k}(z)\right] + \left(\frac{\beta p}{k}\right) {}_pW^{\gamma}_{k,\alpha,\beta+k}(z) =$$

$$\left(\frac{\alpha p}{k}\right) z\frac{d}{dz}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta+k)}\frac{z^n}{(n!)^2} + \left(\frac{\beta p}{k}\right)\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta+k)}\frac{z^n}{(n!)^2} =$$

$$\sum_{n=0}^{\infty}\frac{\left(\frac{\alpha p}{k}\right){}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta+k)}\frac{znz^{n-1}}{(n!)^2} + \sum_{n=0}^{\infty}\frac{\left(\frac{\beta p}{k}\right){}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta+k)}\frac{z^n}{(n!)^2} =$$

$$\sum_{n=0}^{\infty}\frac{\frac{(\alpha n+\beta)p}{k}{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta+k)}\frac{z^n}{(n!)^2}$$

Taking into account ${}_p\Gamma(z+k) = \frac{zp}{k}{}_p\Gamma(z)$

$$\left(\frac{\alpha p}{k}\right) z\frac{d}{dz}\left[{}_pW^{\gamma}_{k,\alpha,\beta+k}(z)\right] + \left(\frac{\beta p}{k}\right) {}_pW^{\gamma}_{k,\alpha,\beta+k}(z) = \sum_{n=0}^{\infty}\frac{\frac{(\alpha n+\beta)p}{k}{}_p(\gamma)_{n,k}}{\frac{(\alpha n+\beta)p}{k}{}_p\Gamma_k(\alpha n+\beta)}\frac{z^n}{(n!)^2}$$

$$\left(\frac{\alpha p}{k}\right) z\frac{d}{dz}\left[{}_pW^{\gamma}_{k,\alpha,\beta+k}(z)\right] + \left(\frac{\beta p}{k}\right) {}_pW^{\gamma}_{k,\alpha,\beta+k}(z) = \sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta)}\frac{z^n}{(n!)^2} = {}_pW^{\gamma}_{k,\alpha,\beta}(z)$$

Then , we have

$$ {}_pW^{\gamma}_{k,\alpha,\beta}(z) = \left(\frac{\alpha p}{k}\right) z\frac{d}{dz}\left[{}_pW^{\gamma}_{k,\alpha,\beta+k}(z)\right] + \left(\frac{\beta p}{k}\right) {}_pW^{\gamma}_{k,\alpha,\beta+k}(z)\Box. \tag{II.12}$$

In what follows shows the relationship between the p-k-Wrigth function and the p-k-Mittag-Leffler function obtained throung the Laplace transform. In fact, we have the following

Theorem 1. Let $k,p\in\mathbb{R}^+$,$\alpha,\beta,\gamma\in\mathbb{C}$, $Re(\alpha)>0$,$Re(\beta)>0$ $Re(\gamma)>0$,$Re(s)>0$

$$\mathfrak{L}[{}_pW^{\gamma}_{k,\alpha,\beta}(z)](s) = \frac{1}{s}{}_pE^{\gamma}_{k,\alpha,\beta}(s^{-1}) \tag{II.13}$$

where ${}_pE^{\gamma}_{k,\alpha,\beta}(z)$ is the p-k-Mittag-Leffler function given by Cerutti; Luque and Dorrego. (cf. [1])

Proof. From definition of Laplace Transform and from (II.1) we have

$$\mathfrak{L}[{}_pW^{\gamma}_{k,\alpha,\beta}(z)](s) = \int_0^{\infty} e^{-sz}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta)}\frac{z^n}{(n!)^2}dz =$$

$$\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta)}\frac{1}{(n!)^2}\int_0^{\infty} e^{-sz}z^n dz \tag{II.14}$$

Taking into account that the integral in (II.14) is

$$\int_0^{\infty} e^{-sz}z^n dz = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}} \tag{II.15}$$

From (II.14) and (II.15) we have

$$\mathfrak{L}[{}_pW^{\gamma}_{k,\alpha,\beta}(z)](s) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta)} \frac{(s^{-1})^{n+1}}{n!} =$$

$$\frac{1}{s}\sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta)} \frac{(s^{-1})^{n}}{n!} = \frac{1}{s}{}_pE^{\gamma}_{k,\alpha,\beta}(s^{-1})$$

Then

$$\mathfrak{L}[{}_pW^{\gamma}_{k,\alpha,\beta}(z)](s) = \frac{1}{s}{}_pE^{\gamma}_{k,\alpha,\beta}(s^{-1}) \tag{II.16}$$

III Fractional Integral and Fractional Derivative of the $\mathcal{E}(t,p,k,\alpha,\beta)$ function

In this subsection we will defined the function $\mathcal{E}(t,p,k,\alpha,\beta)$ in term of the p-k- Wrigth function and then we evaluate its Riemann-Liouville fractional integral and derivative.

Definition 6. Let α,β,γ be complex numbers that $\Re(\alpha)>0$, $\Re(\beta)>0$ and $\Re(\gamma)>0$, $k>0$ and $t\in\mathbb{R}$. We define the auxiliary function $\mathcal{E}(t,p,k,\alpha,\beta)$ by the following relation

$$\mathcal{E}(t,p,k,\alpha,\beta) = t^{\frac{\beta}{k}-1}{}_pW^{\gamma}_{k,\alpha,\beta}(t^{\frac{\alpha}{k}}) \tag{III.1}$$

Easily we can prove that $\mathcal{E}(t,p,k,\alpha,\beta)\to\mathcal{E}(t,k,\alpha,\beta)$ as $p\to k$, where $\mathcal{E}(t,k,\alpha,\beta)$ is the function in term of the k- Wrigth function defined (c.f[14])

Proposition 1. Let α,β,γ be complex numbers that $\Re(\alpha)>0$, $\Re(\beta)>0$ and $\Re(\gamma)>0$, $k>0$ and $t\in\mathbb{R}$ then

$$I^{\nu}_{+}(\mathcal{E}(t,p,k,\alpha,\beta))(x) = p^{\nu}x^{\frac{\beta}{k}+\nu-1}{}_pW^{\gamma}_{k,\alpha,\beta+k\nu}(x^{\frac{\alpha}{k}}) \tag{III.2}$$

Proof. By definition (I.9) and (III.1) we have

$$I^{\nu}_{+}(\mathcal{E}(t,p,k,\alpha,\beta))(x) = \frac{1}{\Gamma(\nu)}\int_0^x t^{\frac{\beta}{k}-1}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta)}\frac{t^{\frac{\alpha n}{k}}}{(n!)^2}(x-t)^{\nu-1}dt$$

$$\frac{1}{\Gamma(\nu)}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta)(n!)^2}\int_0^x t^{\frac{\alpha n+\beta}{k}-1}(x-t)^{\nu-1}dt \tag{III.3}$$

making the change of variable

$$t=\xi x, \quad x-t=x(1-\xi), \quad dt=xd\xi$$

and replacing in (III.3) it result

$$I^{\nu}_{+}(\mathcal{E}(t,p,k,\alpha,\beta))(x) = \frac{1}{\Gamma(\nu)}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta)(n!)^2}\int_0^1 (\xi x)^{\frac{\alpha n+\beta}{k}-1}[x(1-\xi)]^{\nu-1}x^1 d\xi =$$

$$\frac{1}{\Gamma(\nu)}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta)(n!)^2}x^{\frac{\alpha n+\beta}{k}+\nu-1}\int_0^1 \xi^{\frac{\alpha n+\beta}{k}-1}(1-\xi)^{\nu-1}d\xi = \tag{III.4}$$

The integral in (III.4) result

$$\int_0^1 \xi^{\frac{\alpha n+\beta}{k}-1}(1-\xi)^{\nu-1}d\xi = B\left(\frac{\alpha n+\beta}{k},\nu\right)$$

where $B(z,w)$ is the Beta function. Then

$$I_+^\nu(\mathcal{E}(t,p,k,\alpha,\beta))(x) = \frac{1}{\Gamma(\nu)}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta)(n!)^2}x^{\frac{\alpha n+\beta}{k}+\nu-1}B\left(\frac{\alpha n+\beta}{k},\nu\right) =$$

$$\frac{1}{\Gamma(\nu)}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta)(n!)^2}x^{\frac{\alpha n+\beta}{k}+\nu-1}\frac{\Gamma\left(\frac{\alpha n+\beta}{k}\right)\Gamma(\nu)}{\Gamma\left(\frac{\alpha n+\beta}{k}+\nu\right)} =$$

$$\frac{1}{\Gamma(\nu)}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta)(n!)^2}x^{\frac{\alpha n+\beta}{k}+\nu-1}\frac{\frac{p^{\frac{\alpha n+\beta}{k}}}{k}p^\nu\Gamma\left(\frac{\alpha n+\beta}{k}\right)\Gamma(\nu)}{\frac{p^{\frac{\alpha n+\beta+k\nu}{k}}}{k}\Gamma\left(\frac{\alpha n+\beta+k\nu}{k}\right)} =$$

Appliying Lemma 1. , ${}_p\Gamma_k(z) = \frac{p^{\frac{z}{k}}}{k}\Gamma(\frac{z}{k})$

$$I_+^\nu(\mathcal{E}(t,p,k,\alpha,\beta))(x) = \frac{1}{\Gamma(\nu)}x^{\frac{\beta}{k}+\nu-1}p^\nu\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{n,k}(x^{\frac{\alpha}{k}})^n}{{}_p\Gamma_k(\alpha n+\beta)(n!)^2}\frac{{}_p\Gamma_k(\alpha n+\beta)\Gamma(\nu)}{{}_p\Gamma_k(\alpha n+\beta+k\nu)}$$

Then,

$$I_+^\nu(\mathcal{E}(t,p,k,\alpha,\beta))(x) = p^\nu x^{\frac{\beta}{k}+\nu-1}{}_pW^\gamma_{k,\alpha,\beta+k\nu}(x^{\frac{\alpha}{k}})$$

Proposition 2. Let α,β,γ be complex numbers that $\Re(\alpha)>0$, $\Re(\beta)>0$ and $\Re(\gamma)>0$, $k>0$ and $t\in\mathbb{R}$ then

$$D_+^\nu(\mathcal{E}(t,p,k,\alpha,\beta))(x) = p^{-\nu}x^{\frac{\beta}{k}-\nu-1}{}_pW^\gamma_{k,\alpha,\beta+k(-\nu-1)}(x^{\frac{\alpha}{k}}) \quad \text{(III.5)}$$

Proof. The Riemann-Liouville fractional integral of order $r-\nu$ of $\mathcal{E}(t,p,k,\alpha,\beta)$ result

$$I_+^{r-\nu}(\mathcal{E}(t,p,k,\alpha,\beta))(x) = p^{r-\nu}x^{\frac{\beta}{k}+r-\nu-1}{}_pW^\gamma_{k,\alpha,\beta+k(r-\nu)}(x^{\frac{\alpha}{k}}) \quad \text{(III.6)}$$

Now we evaluate the derivative of order r with respect to x of (III.6)

$$\frac{d^r}{dx^r}p^{r-\nu}x^{\frac{\beta}{k}+r-\nu-1}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta+k(r-\nu))}\frac{x^{\frac{\alpha n}{k}}}{(n!)^2} =$$

$$\frac{d^r}{dx^r}p^{r-\nu}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k(\alpha n+\beta+k(r-\nu))}\frac{x^{\frac{\alpha n+\beta}{k}+r-\nu-1}}{(n!)^2} =$$

$$p^{r-\nu}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{n,k}\left(\frac{\alpha n+\beta}{k}+r-\nu-1\right)\ldots\left(\frac{\alpha n+\beta}{k}-\nu-1\right)}{{}_p\Gamma_k(\alpha n+\beta+k(r-\nu))}\frac{x^{\frac{\alpha n+\beta}{k}-\nu-1}}{(n!)^2} =$$

$$p^{r-\nu}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{n,k}\Gamma\left(\frac{\alpha n+\beta+k(r-\nu)}{k}-1\right)}{p^{\frac{\alpha n+\beta+k(r-\nu)}{k}-1}\Gamma\left(\frac{\alpha n+\beta+k(r-\nu)}{k}-1\right)}\frac{x^{\frac{\alpha n}{k}}x^{\frac{\beta}{k}-\nu-1}}{(n!)^2\Gamma\left(\frac{\alpha n+\beta-k\nu}{k}-1\right)} =$$

$$p^{r-\nu}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{n,k}}{p^r p^{\frac{\alpha n+\beta-k\nu}{k}-1}\Gamma\left(\frac{\alpha n+\beta-k\nu}{k}-1\right)}\frac{x^{\frac{\alpha n}{k}}x^{\frac{\beta}{k}-\nu-1}}{(n!)^2}=$$

$$p^{-\nu}x^{\frac{\beta}{k}-\nu-1}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{n,k}}{{}_p\Gamma_k\left(\alpha n+\beta-k\nu-k\right)}\frac{x^{\frac{\alpha n}{k}}}{(n!)^2}=$$

then

$$D_+^{\nu}(\mathcal{E}(t,p,k,\alpha,\beta))(x)=p^{-\nu}x^{\frac{\beta}{k}-\nu-1}{}_pW^{\gamma}_{k,\alpha,\beta+k(-\nu-1)}(x^{\frac{\alpha}{k}})\qquad .\square$$

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