The relation among Lie-B acklund Transformations andSymmetries For Soliton Equation

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Abstract

In this paper, we establish how symmetries of a system of differential equations R^{k+1} may be combined with a Bäcklund map for R^{k+1} to produce another Bäcklund map for R^{k+1}.And use prolongations of Bäcklund map to generate symmetries for the system of differential equations we show that one can obtain Symmetries of Soliton equations from their Bäcklund Transformations. Finally We explain briefly the concepts of LB transformations group,for Kdv equation.

Keywords: Bäcklund transformations ; Soliton Equation; Symmetries;Lie-Bäcklund symmetries; Kdv equations.

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Introduction

One of the main properties of equations of admitting soliton solutions , is that can be represented as the integrability conditions of Bäcklund maps, the prolongations of auto-Bäcklund transformation are again auto-Bäcklund transformation

Also symmetries of partial differential equations[4] , the symmetries group for partial differential equations is the largest loacal group of transformations acting on independent and dependent variables of a system with property that it transform soluions of the system to another solution. It is known that extend point transformation of jet bundles[1] ,[6] can be combined with Baäcklund map of Systems of differential equations
to give families of Bäcklund map for the system [1], [3]. This may be
Seen a follows:
Let (Φ, ϕ) be a point transformation of a system of differential equations $R^{(k+1)}$ living on the jet bundle
$J^k(M, N)$, where M is a space of independent variables of the system, N is a space of dependent variables
of the system and k is the degree of the system . This means that:
The symmetry of Bäcklund map is always defined by the maps

$$\phi : M \to M$$

And

$$\Phi : J^0(M, N) \to J^0(M, N)$$

LetΨ be a Bäcklund map for R^{k+1} ie:

$$\Psi : J^k(M, N) \times \acute{N} \to J^1(M, N)$$

Where
M is a space of independent variables with coordinates X ,
N is a space of dependent variables with coordinates Z,
and
$\acute{N}$ is a space of new dependent variables with coordinates Y
And
let (Φ, ϕ) be diffeomorphism of $J^0(M, \acute{N})$,

and
$P^{k,0}(\acute{\Phi},\Phi)$ is diffeomorphism of $J^k(M,N)) \times J^0(M,\acute{N})$
This means that

$$P^{k,0}(\acute{\Phi},\Phi) : J^k(M,N) \times J^0(M,\acute{N}) \to J^k(M,N) \times J^0(M,\acute{N})$$

Then we obtain the one parameter of Bäcklund map $\check{\Psi}$,[]given by
$P^{k,0}(\acute{\Phi},\Phi)$ is diffeomorphism of $J^k(M,N)) \times J^0(M,\acute{N})$,
This means that

$$P^{k,0}(\acute{\Phi},\Phi) : J^k(M,N) \times J^0(M,\acute{N}) \to J^k(M,N) \times J^0(M,\acute{N})$$

Then we obtain the one parameter family of Bäcklund maps $\check{\Psi}$;[1],[2]
where

$$\check{\Psi} : J^K(M,N) \times J^0(M,\acute{N}) \to J^1(M,\acute{N})$$

Example 1:

Let Ψ be a Bäcklund map for Kdv potential equation [11] defined by

$$\acute{Y_1} = {}_1 = -z_1 + (y-z)^2$$

$$\acute{Y_2} = \Psi_2 = -z + 4[z_1^2 + z_1(y_1 - z)^2 + z_{11}(y-z)]$$

And let (Φ_t, ϕ_t) be one- parameter group of symmetries of K-dv equation by:

$$(x^1, x^2, z) \to (x^1 - 12tx^2, x^2, z + tx^1 - 6tx^2)(1)$$

And let $(\acute{\Phi_t}, \acute{\phi_t})$ be one- parameter group of diffeomorphism of $J^0(M,N)$ given by

$$(\acute{x^1}, \acute{x^2}, \acute{y}) \to (\acute{x^1} - 12t\acute{x^2}, \acute{x^2}, y + t\acute{x^1} - 6t\acute{x^2})(2)$$

And let $P^{2,0}(\Phi_t \times \acute{\Phi_t})$ is given by

$$P^{2,0}(\Phi_t \times \acute{\Phi_t}) : (x^1, x^2, z, z_1, z_2, z_{11}, z_{12}, z_{22}, y) \to$$

$$(x^1 - 12tx^2, x^2, z + tx^1 - 6t^2x^2, z_1 + t, z_2 + 12tz_1 + 6t^2, z_{11}, z_{12} + tx^1 - 6t^2x^2)(3)$$

And let $P^2\acute{\Phi}^{-1}$ are given by

$$P^2\acute{\Phi}^{-1} : (\acute{x}^1, \acute{x}^2, \acute{y}, \acute{y}_1, \acute{y}_2, \acute{y}_{11}, \acute{y}_{12}, \acute{y}_{22})$$

$$\to (\acute{x^1} - 12t\acute{x}^2, \acute{x}^2, y + t\acute{x}^1 - 6t^2\acute{x}^2, y_1 + t, y_2 + 12ty_1 + 6t^2, y_{12}, ...)(4)$$

From (3)and (4) and (1) we get:

$$\check{\Psi}_1 = P^2\acute{\Phi}^{-1} \circ \Psi \circ P^{2,0}(\Phi \times \acute{\Phi})$$

$$= -z_2 + 4[4t^2 + 2tz_1 - 2t(y-z)^2 + z_1^2 + z_1(y-z)^2 + z_{11}(y-z)]$$

This is Bäcklund map for K.dv equation gotten from the one- parameter group of symmetries .As one can show that its integrability conditions yield the $K.dv$ equation [11].

1 Symmetries

Consider R^k to be the system of differential equation.A point transformation [14],[9],[5] (Φ,ϕ) is said to be symmetry of R^k if

$$P^k\Phi(R^k) = R^k(I)$$

Is satisfied for some (r). However that a symmetry [5] (Φ,ϕ) may be shown as follows: Let $f \in C^\infty(M,N)$ be a solution of R^k and suppose that the condition (I) holds. Then since $im(\Psi) \subset R^k$, it follows that:

$$im(P^k\Phi \circ \Psi \circ \phi^{-1}) \subset P^k\Phi(R^k) \subset R^k.$$

Whence:

$$im(\check{\Psi}) \subset R^k.$$

So $\check{\Psi}$ is another soliton of R^k.

If $\{(\Phi_t, \phi_t)\}$ is a one - parameter group of point transformations , it is said to be a one - parameter group of symmetries of R^k whenever

$$P^k \Phi_t R^k = R^k$$

The fact that for each $[t, (\Phi_t, \phi_t)]$ is a point transforation implies that if f is a solution of R^k , then $\check{\Psi}$ is a one parameter family solutions. We shall use prolongations of Bäcklund maps to generate symmetries for systems of differential equations via these examples.

2 Special Cases and Examples

Example 1:

The K.dv equation

$$z_{111} + z_2 - 6z_1^2 = 0$$

Has a Bäcklund map [2], witch is defined by :

$$\acute{Y_1} = \Psi_1 = -z_1 + (y - z)^2$$

$$\acute{Y_2} = \Psi_2 = -z_2 + 4[z_1^2 + z_1(y - z)^2 + z_{11}(y - z)]$$

The third prolongations of the group of point transformation [6]given by :

$$P^3\Phi_1 : (x^1, x^2, z_1, z_2, z_{11}, z_{12}, z_{22}, ..., z_{111}, ...) \rightarrow$$

$$(x^1 - 12tx^2, x^2, z + tx^1 - 6tx^2, z_1 + t, z_2 + 12tz_1 + 6t^2, z_{111}, z_{12}, ..., z_{111}, ...)$$

Then Every point under $q\epsilon\ J^3(M, N)$ for witch $z_{111} + z_2 - 6z_1^2 = 0$ Is mapped under $P^3\Phi_t$ to point $\acute{q}$ with

$$\acute{z_{111}} + \acute{z_2} - 6\acute{z_1^2} = 0$$

thus

$[\Phi_t, \phi_t]$ is one - parameter group of K.dv equation.

Example 2:

The Sin -Gordon equation:

$$z_{12} = sinz$$

HasBäcklund map Ψ given by [8]

$$Y_1 = z_1 + 2sin[\frac{y + z}{2}]$$

$$Y_2 = z_2 + 2sin[\frac{y - z}{2}]$$

Let $M = R^2$,$N = R$, and let (Φ_t, ϕ_t) be one - parameter group of point transformations given by

$$\Phi_t : (x^1, x^2, z) \rightarrow (e^t x^1, e^{-t} x^1, z)$$

$$\phi_t : (x^1, x^2) \rightarrow (e^t x^1, e^{-t} x^2)$$

The first prolongation of $(\Phi_t, \phi_t$ is determined by

$$P^1\Phi_t : (x^1, x^2, z, z_1, z_2) \rightarrow (e^t x^1, e^{-t} x^2, z, e^{-t} z_1, e^t z_2)$$

The second prolongation of the group of point of transformations given by

$$P^2\Phi_t : (x^1, x^2, z, z_1, z_2, z_{11}, z_{12}, z_{22}) \to (e^t x^1, e^{-t}x^2, z, e^t z^1, e^t z_2, e^{-2t}z_{11}, z_{12}, e^{2t}z_{22})$$

Is mapped via $P^2\Phi_t$ to a point $q^{'}$ with $z^{'}_{12} = sinz^{'}$ thus (Φ_t, ϕ_t) is one - parameter group of symmetries of sin -Gordon equation.

3 The K.dv and modified K.dv equations

In the preceding two examples , the Bäcklund transformation [10] [8],[10] determined by Bäcklund map is an "auto-Bäcklund transformation, we now exhibit
a one -parameter family of Bäcklund maps , obtained from invariance of the $K.dv$ equation, which form of the modified $K.dv$ equation
Thus when we combine Bäcklund map of $K.dv(z_{111} + z_2 + 12zz_1)$ with symmetries we have Bäcklund map of $mK.dv$ $(z_{111} + z_2 - 6z^2 z_1 + 12tz_1)$

Example 3:

The $K.dv$ equation is

$$z_{111} + z_2 + 12zz_1$$

Has a Bäcklund map given by :

$$Y\acute{}_1 = \Psi_1 = -z_1 + (y - z)^2$$

$$Y\acute{}_2 + \Psi_2 = 8z^2 + 4zy^2 + 2z_{11} - 4z_1 y$$

The $p^2\Phi^{\acute{}-1}$ of point of transformations given by

$$(x^{\acute{1}}, x^{\acute{2}}, y\acute{}, y\acute{}_1, \acute{}y_2, y\acute{}_{11}, y\acute{}_{12}, y\acute{}_{22}) \to$$

$$(x^{\acute{1}} - 12tx^{\acute{2}}, x^{\acute{2}}, y\acute{} - t, y\acute{}_1, y\acute{}_2, ...)$$

Then

$$\tilde{\Psi}_t := P^2\Phi^{'-1} \circ \Psi \circ P^{2,0}(\Phi \times \Phi^{'})$$

Then

$$Y_1 = -2z - y^2 + 2t$$

$$Y_2 = 4(z + 2t)(2z + y^2 - 2t) + 2z_{11} - 4z_1 y$$

The image under $\tilde{\Psi}_t$ is

$$z_{111} + z_2 - z^2 z_1 + 12tz_1 = 0$$

Which is $m.K.dv$ equation
When

$$P^3\Phi_t : (x^1, x^2, z, z_1, z_2, z_{11}, z_{12}, z_{22}, z_{111}, z_{112}, ...) \to$$

$$(x^1 - 12t, x^2, z - t, z_1, z_2 + 12tz_1, ..., z_{11}, ..., z_{111}, ...)$$

Then every point $q \in J^3(M, N)$ for which $z_{111} + z_2 + 12zz_1 = 0$ is mapped under $P^3\Phi_t$
to a point $q\acute{}$ with $\acute{z}_{111} + \acute{z}_2 + 12\acute{z}\acute{z}_1 = 0$
Thus (Φ_t, ϕ_t) is one-parameter group of symmetries of [2] , [5] $K.dv$ equations.

4 Lie-Bäcklund Symmetries Group And Infinitesimal Symmetries.

Consider Partial differential equation on the form

$$H = H(x, t, u_t, u_1, ..., u_n)(1)$$

The most general infintesimal operator of one parameter LB group of transformations leaving (1)invariant may be taken as

$$X\eta = \eta\frac{\partial}{\partial u} + D_t\eta\frac{\partial}{\partial u_t} + D_tD_x\frac{\partial}{\partial u_{xt}} + D_x\eta\frac{\partial}{\partial u_1} + D_x^2\eta\frac{\partial}{\partial u_2} + D_x^3\eta\frac{\partial}{\partial u_3} + ... + D_x^n\frac{\partial}{\partial u_n}$$

where η is a function of x, t, u and partial derivatives of u with respect to x and t of any arbitrary, but finite order D_x and D_t are the total derivative operator with respect to x and t respectively defined by

$$D_x = \frac{\partial}{\partial x} + u_1\frac{\partial}{\partial u} + u_{1,t}\frac{\partial}{\partial u_t} + u_{2,t}\frac{\partial}{\partial u_{1,t}} + u_2\frac{\partial}{\partial u_1} + u_3\frac{\partial}{\partial u_2} + ...$$

With an analogous for D_t .Then the invariance condition of (1) with respect to the group whose infintesimal generators are tangent vectors (2) is given by

$$X(\eta)H|_{H=0} = 0(3)$$

The condition (3) provides an algorithm for finding η . Each independent η satisfying (3) is called LB symmetry of equation (1)[15] .We can obtain an infinite sequence of LB symmetries(if they exist) which are ploynomials in u and its derivatives with respect the variable x by finding a recursion operator, which generates infinitley many symmetries from a given symmetry.

5 LB Symmetries for the K-dv equations

The equation which we consider is:

$$w_t + w_3 + \frac{1}{2}w_1^3 = 0(4)$$

Now Eq.(4) admits a LB symmetry $\eta(w, w_1, w_2, ..., w_n)$ for a finite n if and only if for every solution $w = w(x, t)$ of Eq.(4)

$$X(\eta)[w_t + w_3 + \frac{1}{2}w_1^3 = 0(5)$$

or

$$D_t(\eta) + D_x^3\eta + \frac{3}{2}D_x\eta = 0(6)$$

By using (2)in (5),the condition (5) reduces to

$$D_x^3\eta + \frac{3}{2}w_1^2D_x\eta = \eta_u(w_3 + \frac{1}{2}w_1^3) + \sum_{j=1}^{n} D_x^j(w_3 + \frac{1}{2}w_1^3)(7)$$

one can easily find that the following are the solutions of equations (7) for η

$$\eta^1 = w_1(8)$$

$$\eta^{11} = w_3 + \frac{1}{2}w_1^3(9)$$

η^1 and η^{11} correspond,respectively,to $x$and$t$ translation symmetries[14],[15] of Eq(4). Now we assume that there exists a recursion operator Δwhich generater (9) from (8). Using this Δ we may take the next LB symmetry in the form

$$\eta^{111} = w_5 + B(w, w_1, w_2, w_3, w_4).(10)$$

Substitution of (10)in (7) results in polynpmial form in w_5, w_6, whose coefficients should vanish since for every solution of (4)the condition(7)must hold.This leads to the following determining equations for B:

$$\frac{\partial B}{\partial u_4} = 0, D_x\frac{\partial B}{\partial u_3} = \frac{5}{2}w_1^2$$

solving these equations we obtain

$$B = \frac{5}{2}w_1^2 w_3 + F(w, w_1, w_2)$$

where $F$is arbitrary. Following the same lines of argument as before we obtain η^{111} in a recursive manner. In this way we get

$$\eta^{111} = w_5 + \frac{5}{2}w_1^2 w_3 + \frac{5}{2}w_1 w_2^2 + \frac{3}{8}w_1^5.(11)$$

The form of (8),(9),(11) suggests that the recursion operator Δ is of the form

$$\Delta = D_x^2 + p(w_1) + q(w_1)D_x^{-1}(w_2).(12)$$

To find p and q we note that Δ generates (11) from (9) and (9) from (8) , and therefore we have.

$$p + \frac{1}{2}qw_1 = \frac{1}{2}w_1^2(13)$$

$$3w_1 w_2^2 + \frac{3}{2}w_1^2 w_3 + p(w_3 + \frac{1}{2}w_1^3 + q(\frac{1}{2}w_2^2 + \frac{1}{8}w_1^4 = \frac{5}{2}w_1^2 w_3 + \frac{5}{2}w_1 w_2^2 + \frac{3}{8}w_1^5.(14)$$

Solving (13) and (14) for p and q and inserting these in (12)we find

$$\Delta = D_x^2 + w_1^2 - w_1 D_x^{-1} w_2.$$

The proof that this Δ is indeed the recursion operator[14].

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