Weak and Strong Convergence of Implicit and Explicit Algorithms for Total Asymptotically Nonexpansive Mappings

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Abstract

In this paper, we prove weak and strong convergence of implicit and explicit iterative algorithms for approximation of common fixed point of finite family of total asymptotically nonexpansive mappings. Our recursion formulas seem more efficient than those recently announced by several authors for the same problem. Our theorems improve, generalize and extend several recently announced results.

Keywords: Asymptotically Nonexpansive Mappings, Modulus of Convexity, Total Asymptotically Quasi-Nonexpansive Mappings, Uniformly Convex Real Banach Spaces.

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1 Introduction

Let K be a nonempty subset of a real normed space E. A mapping $T : K \to K$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. The mapping T is called *asymptotically nonexpansive* if there exists a sequence $\{\mu_n\}_{n\geq 1} \subset [0, \infty)$ with $\lim\limits_{n\to\infty} \mu_n = 0$ such that for all $x, y \in K$,

$$\|T^n x - T^n y\| \leq (1 + \mu_n)\|x - y\| \;\forall\; n \geq 1.$$

The operator T is called *uniformly $L-$Lipschitzian* if there exists a constant $L \geq 0$ such that for all $x, y \in K$,

$$\|T^n x - T^n y\| \leq L\|x - y\| \;\forall\; n \geq 1.$$

It is easy to see that every asymptotically nonexpansive mapping is uniformly $L-$Lipschitzian.

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [19] as a generalization of the class of nonexpansive mappings. They proved that if K is a bounded closed convex nonempty subset of a uniformly convex real Banach space and T is an asymptotically nonexpansive self-mapping of K, then T has a fixed point.

A mapping $T : K \to K$ is said to be *total asymptotically nonexpansive* (see e.g. [2, 3, 12, 21]) if there exist nonnegative real sequences $\{\mu_n\}$ and $\{l_n\}$, $n \geq 1$ with $\mu_n \to 0$, $l_n \to 0$ as $n \to \infty$ and nondecreasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for all $x, y \in K$,

$$\|T^n x - T^n y\| \leq \|x - y\| + \mu_n \phi(\|x - y\|) + l_n,\; n \geq 1. \quad (1)$$

Remark 1 *If $\phi(t) = 0 \;\forall\; t \in [0, +\infty)$, then (1) reduces to*

$$\|T^n x - T^n y\| \leq \|x - y\| + l_n,\; n \geq 1,$$

so that if K is bounded and T^N is continuous for some integer $N \geq 1$, then the mapping T is of asymptotically nonexpansive type. (the class of mappings which are of asymptotically nonexpansive type includes the class of mappings which are asymptotically nonexpansive in the intermediate sense and the class of nearly asymptotically nonexpansive mappings which had been studied by several authors, see e.g. [11, 12, 32]). If $\phi(t) = t$, then (1) becomes

$$\|T^n x - T^n y\| \leq (1+\mu_n)\|x-y\| + l_n,\ n \geq 1.$$

In addition, if $l_n = 0$ for all $n \geq 1$, then total asymptotically nonexpansive mappings coincide with asymptotically nonexpansive mappings. If $\mu_n = 0$ and $l_n = 0$ for all $n \geq 1$, we obtain from (1) the class of mappings that includes the class of nonexpansive mappings.

Alber et al. [2] *introduced the class of total asymptotically nonexpansive mappings as a more general class of asymptotically nonexpansive mappings. The idea behind the introduction of the class of total asymptotically nonexpansive mappings is to unify various definitions of classes of mappings associated with the class of asymptotically nonexpansive mappings and to prove a general convergence theorems applicable to all these classes of nonlinear mappings.*

At this juncture, we shall pause and provide an example to show that class of total asymptotically nonexpansive mappings properly contains the class of asymptotically nonexpansive mappings.

Example 2 *(See Ofoedu and Madu [29]) Let $E = \mathbb{R} \times \ell_1$ be endowed with the norm $\|.\|_E = |.| + \|.\|_{\ell_1}$. Let K be a subset of E defined by $K := [0,1] \times B$, where B is the closed unit ball of ℓ_1. For all $u \in [0,1]$ and $\overline{x} \in B$ define $T : K \to K$ by*

$$T(u,\overline{x}) = \begin{cases} \Big(\frac{1}{3}, (0, \frac{|x_1|^2}{3}, \frac{x_2}{3}, \frac{x_3}{3}, \frac{x_4}{3}, \ldots)\Big), \text{ if } u \in \Big[0, \frac{1}{3}\Big] \\ \Big(0, (0, \frac{|x_1|^2}{3}, \frac{x_2}{3}, \frac{x_3}{3}, \frac{x_4}{3}, \ldots)\Big), \text{ if } u \in \Big(\frac{1}{3}, 1\Big]. \end{cases} \tag{2}$$

We can easily check that T given by (2) is not continuous and thus cannot be asymptotically nonexpansive (since every asymptotically nonexpansive mapping is uniformly L-Lipschitzian, so Lipschitz and every Lipschitz mapping is continuous). Next, let $\{l_n\}_{n\geq 1}$ be a sequence of real numbers such that $l_1 = \frac{1}{3}$ and $\lim_{n\to\infty} l_n = 0$. Observe that for all $(u,\overline{x}), (v,\overline{y}) \in K$,

$$\Big\|T(u,\overline{x}) - T(v,\overline{y})\Big\|_E \leq |u-v| + l_1 + \frac{1}{3}\max\Big\{|x_1|+|y_1|, 1\Big\}\|\overline{x}-\overline{y}\|_{\ell_1}.$$

Moreover, we can equally check easily that for all $n \geq 2$ and for all $(u,\overline{x}), (u,\overline{y}) \in K$,

$$T^n(u,\overline{x}) = \Big(\frac{1}{3}, (\underbrace{0,0,\ldots,0,0}_{n-\text{times}}, \frac{|x_1|^2}{3^n}, \frac{x_2}{3^n}, \frac{x_3}{3^n}, \frac{x_4}{3^n}, \ldots)\Big)$$

and

$$\Big\|T^n(u,\overline{x}) - T^n(v,\overline{y})\Big\|_E \leq \frac{1}{3^n}\max\Big\{|x_1|+|y_1|, 1\Big\}\|\overline{x}-\overline{y}\|_{\ell_1}.$$

So, for all $n \geq 1$,

$$\begin{aligned} \Big\|T^n(u,\overline{x}) - T^n(v,\overline{y})\Big\|_E \quad &\leq \quad |u-v| + \|\overline{x}-\overline{y}\|_{\ell_1} \\ &\quad + \frac{2}{3^n}\Big[|u-v| + \|\overline{x}-\overline{y}\|_{\ell_1}\Big] + l_n. \end{aligned} \tag{3}$$

Thus, with $\phi : [0,+\infty) \to [0,+\infty)$ defined by $\phi(t) = 2t$, $\mu_n = \frac{1}{3^n}$ for all $n \geq 1$ and $\{l_n\}_{n\geq 1}$ any null sequence with $l_1 = \frac{1}{3}$, we obtain from (3) that

$$\Big\|T^n(u,\overline{x}) - T^n(v,\overline{y})\Big\|_E \leq \Big\|(u,\overline{x}) - (v,\overline{y})\Big\|_E + \mu_n\phi\Big(\Big\|(u,\overline{x}) - (v,\overline{y})\Big\|_E\Big) + l_n.$$

So, the mapping T given by (2) is total asymptotically nonexpansive but not asymptotically nonexpansive.

Considerable research efforts have been devoted to developing iterative methods for approximation of common fixed points (when they exist) of finite families of nonlinear mappings. (see e.g., [4, 8, 11, 15, 27, 22, 23, 35, 39, 40, 44]).

Zhou and Chang [45] introduced the following implicit iteration process for approximation of common fixed point of finite family of asymptotically nonexpansive mappings:

$$x_0 \in K, x_n = \alpha_n x_{n-1} + \beta_n T^n_{n(modN)} x_n + \gamma_n u_n \; \forall \; n \geq 1. \tag{4}$$

In fact, Zhou and Chang [45] proved the following theorem:

Theorem ZC. *Let E be a real uniformly convex Banach space satisfying Opial's condition, K be a nonempty closed convex subset of E, $T_1, T_2, ..., T_N : K \to K$ be N asymptotically nonexpansive mappings with sequences $\{k_n^{(i)}\}_{n\geq 1}$, $i = 1, 2, .., N$ in $[1, +\infty)$ such that $\lim_{n\to\infty} k_n = 1$ and $F := \bigcap_{n=1}^{N} F(T_i) \neq \emptyset$. Let u_n be a bounded sequence in K, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three sequences in $[0, 1]$ and $k_n = \max\{k_n^{(1)}, k_n^{(2)}, ..., k_n^{(N)}\}$ such that*
(i) $\alpha_n + \beta_n + \gamma_n = 1$, $\forall \; n \geq 1$; (ii) $\sum_{n=1}^{\infty}(k_n - 1) < +\infty$;
(iii) there exist constants $\tau_1, \tau_2 \in (0, \frac{1}{\sigma})$ such that $\tau_1 \leq \beta_n \leq \tau_2$, $\forall \; n \geq 1$, where $\sigma = \sup_{n\geq 1} k_n \geq 1$; (iv) $\sum_{n=1}^{\infty} \gamma_n < +\infty$; (v) there exists a constant $L > 0$ such that for any $i, j \in \{1, 2, ..., N\}$, $i \neq j$, $\|T_i^n x - T_j^n y\| \leq L\|x - y\|$, $\forall \; n \geq 1$, $\forall \; x, y \in K$. Then, the implicit iterative sequence $\{x_n\}$ defined by (4) converges weakly to a common fixed point of $\{T_1, T_2, ..., T_N\}$.

Recently, Y. Hao [21] proved the following theorem:

Theorem H. *Let H be a real Hilbert space and K be a nonempty closed convex and bounded subset of H such that $K + K \subset K$. Let $T_i : K \to K$ be a uniformly L_i-Lipschitz total asymptotically nonexpansive mapping with the function ψ_i and sequences $\{\mu_n^{(i)}\}$, $\{\ell_n^{(i)}\}$ for each $i \in \{1, 2, ..., N\}$. Assume that $\sum_{n=1}^{\infty} \mu_n^{(i)} < \infty$ and $\sum_{n=1}^{\infty} \ell_n^{(i)} < \infty$ for each $i \in \{1, 2, ..., N\}$. Let $\{u_n\}_{n\geq 1}$ be a bounded sequence in K such that $\sum_{n=1}^{\infty} \|u_n\| < \infty$ and $\{\alpha_n\}$ a sequence in $\left[\frac{1-L}{L}, a\right]$, where $L = \max_{1\leq i\leq N}\{L_i\} > 1$ and a is some constant in $(0, 1)$. Assume that $F := \bigcap_{n=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T^{k(n)}_{i(n)} x_n + u_n, \; \forall \; n \geq 1, \tag{5}$$

then the sequence $\{x_n\}$ converges weakly to some point $x^* \in F$.

Remark 3 *We observe that Theorem ZC holds for finite family of asymptotically nonexpansive mappings and that condition (v) in the Theorem ZC is strong. In Theorem H, the further assumption that the sequence $\{u_n\}$ in K is bounded is superflous since it is already assumed that K is bounded. Besides, the conditions $K + K \subset K$, $\sum_{n=1}^{\infty} \|u_n\| < \infty$ and the boundedness condition on K are rather too strong. The one that is most worrisome is the condition $\sum_{n=1}^{\infty} \|u_n\| < \infty$, where $\{u_n\}_{n\geq 1}$ is sequence of error terms. It has been severally objected (see e.g. [28]) that the condition $\sum \|u_n\| < \infty$ (as imposed by Hao) is not compatible with the randomness of the occurence of errors (since it implies in particular, that the sequence of errors tend to zero as n tends to infinity). This is almost impossible to verify in application.*

It is our purpose in this paper to prove weak and strong convergence of implicit and explicit iteration processes for approximation of common fixed point of finite family of total asymptotically nonexpansive mappings. Our altimate aim is to take care of the anomalies pointed out in Remark 3. Our theorems unify, extend and generalize the corresponding results of Alber *et. al.* [2], Hao [21], Sahu [32], Shahzad and Udomene [33], Zhou and Chang [45] and a host of other results recently announced for the approximation of common fixed points of finite families of several classes of nonlinear mappings. Our method of proof is of independent interest.

2 Preliminary

Let E be a real normed linear space with dual E^*. We denote by J_q *the generalized duality mapping* from E to 2^{E^*} defined by

$$J_q x := \{f^* \in E^* : \langle x, f^*\rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\},$$

where $\langle .,.\rangle$ denotes the generalized duality pairing between members of E and members of E^*. For $q = 2$, the mapping $J = J_2$ from E to 2^{E^*} is called *the normalized duality mapping*. It is well known that if E is uniformly smooth or E^* is strictly convex, then duality mapping is single-valued. If $E = H$ is a Hilbert space then the duality mapping becomes the identity map of H.

Let E be a real normed space. The modulus of convexity of E is the function $\delta_E : [0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = inf\{1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1,\ \epsilon = \|x - y\|\}. \tag{6}$$

The space E is uniformly convex if and only if $\delta_E(\epsilon) > 0\ \forall\ \epsilon \in (0, 2]$. It is well known (see e.g., [1]) that in a uniformly convex space, δ_E is continuous, strictly increasing, and $\delta_E(0) = 0$; and that $\frac{\delta_E(\epsilon)}{\epsilon}$ is non-decreasing for all $\epsilon \in (0, 2]$.

Let K be a closed convex nonempty subset of a Banach space E. A mapping $T : K \to E$ is said to be *demiclosed at* x_0 if and only if whenever a sequence $\{x_n\}_{n\geq1}$ in K converges weakly to $x^* \in K$ and the sequence $\{Tx_n\}_{n\geq1}$ converges strongly to $x_0 \in E$ we have that $Tx^* = x_0$. The operator T is said to be completely continuous if and only if for any bounded sequence $\{y_n\}_{n\geq1}$ in K, the sequence $\{Ty_n\}_{n\geq1}$ has a subsequence (say $\{Ty_{n_k}\}_{k\geq1}$) which converges strongly to some $y^* \in E$.

In the sequel, we shall need the following Lemmas:

Lemma 4 *(see e.g., [1]). Let E be a uniformly convex real Banach space, $\lambda \in [0, 1]$, $x,\ y \in E$. Then*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - 2\lambda(1 - \lambda)C^2\delta_E\Big(\frac{\|x - y\|}{2C}\Big), \tag{7}$$

where $C = \sqrt{\frac{\|x\|^2+\|y\|^2}{2}}$.

Remark 5 *(see [1]) If $\|x\| \leq R$ and $\|y\| \leq R$, where R is some positive number, then $C \leq R$ and $2C^2\delta_E\Big(\frac{\|x-y\|}{2C}\Big) \geq \frac{R^2\delta_E\left(\frac{\|x-y\|}{2R}\right)}{2L^*}$, where L^* is a constant (the Figiel constant, see e.g., [18]) such that $1 < L^* < 1.7$.*

Lemma 6 *Let $\{a_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences of nonnegative real numbers such that*

$$\lambda_{n+1} \leq (1 + \beta_n)\lambda_n + \gamma_n.$$

Suppose that $\sum_{n=1}^{\infty}\beta_n < \infty$ and $\sum_{n=1}^{\infty}\gamma_n < \infty$. Then $\{\lambda_n\}$ is bounded and $\lim_{n\to\infty}\lambda_n$ exists.

Lemma 7 *(see Corollary 2.6 of [3]) Let E be a reflexive Banach space with weakly sequentially continuous normalized duality mapping J. Let K be a closed convex subset of E and $T : K \to K$ a uniformly continuous total asymptotically nonexpansive mapping with bounded orbits. Then $I - T$ is demiclosed at zero.*

3 Main results

Let K be a nonempty closed convex subset of a real normed space E. Let $T_1, T_2, ..., T_m : K \to K$ be m total asymptotically nonexpansive mappings and $\{\alpha_n\}_{n\geq 1} \subset (0,1)$. We define the implicit iteration process $\{x_n\}$ by $x_0 \in K$,

$$\begin{aligned}
&x_1 = \alpha_1 x_0 + (1-\alpha_1)T_1 x_1\\
&x_2 = \alpha_2 x_1 + (1-\alpha_2)T_2 x_2\\
&\vdots\\
&x_m = \alpha_m x_{m-1} + (1-\alpha_m)T_m x_m\\
&x_{m+1} = \alpha_{m+1} x_m + (1-\alpha_{m+1})T_1^2 x_{m+1}\\
&\vdots\\
&x_{2m} = \alpha_{2m} x_{2m-1} + (1-\alpha_{2m})T_m^2 x_{2m}\\
&x_{2m+1} = \alpha_{2m+1} x_{2m} + (1-\alpha_{2m+1})T_1^3 x_{2m+1}\\
&\vdots
\end{aligned} \tag{8}$$

Since for all $z \in \mathbb{Z}$ (where $\mathbb{Z}$ is the set of integers), there exists $j(z) \in \{1, 2, ..., m\}$ such that $z - j(z)$ is divisible by m (that is $j(z) = z \, mod \, m$), then there exists $q(z) \in \mathbb{Z}$ with $\lim_{z\to+\infty} q(z) = +\infty$ such that

$$z = \big(q(z) - 1\big)m + j(z). \tag{9}$$

Thus, we may write (8) in a more compact form as

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_{j(n)}^{q(n)} x_n, \ \forall \, n \geq 1. \tag{10}$$

By similar procedure as in (8), the following explicit iteration process is generated:

$$z_1 \in K, \ z_{n+1} = \alpha_n z_n + (1-\alpha_n)T_{j(n)}^{q(n)} z_n, \ \forall \, n \geq 1. \tag{11}$$

Remark 8 *Since $n - m \in \mathbb{Z}$ for all $n \in \mathbb{N}$ (where $\mathbb{N}$ denotes the set of positive integers), we obtain from (9) (for the particular case $n - m \in \mathbb{Z}$) that*

$$n - m = \big(q(n-m) - 1\big)m + j(n-m). \tag{12}$$

Also substituting $n \in \mathbb{N}$ for z in (9) and subtracting m from both sides of the resulting equation gives

$$n - m = \Big(\big(q(n) - 1\big) - 1\Big)m + j(n). \tag{13}$$

Comparing (12) and (13) we obtain (by unique representation theorem) that

$$q(n-m) = q(n) - 1 \quad \text{and } j(n-m) = j(n) \ \forall \, n \in \mathbb{N}. \tag{14}$$

We shall now proceed to prove weak and strong convergence of the schemes (10) and (11) in real Banach spaces. We start as follows.

Proposition 9 *Let K be a nonempty subset of a real normed space E and $T_1, T_2, ..., T_m : K \to K$ be m total asymptotically nonexpansive mappings, then there exist sequences $\{\mu_n\}$, $\{\ell_n\} \subset [0, +\infty)$ with $\lim_{n\to\infty} \mu_n = 0 = \lim_{n\to\infty} \ell_n$ and a nondecreasing continuous function $\phi : [0, +\infty) \to [0, +\infty)$, with $\phi(0) = 0$ such that for all $x, y \in K$,*

$$\|T_i^n x - T_i^n y\| \leq \|x - y\| + \mu_n \phi(\|x - y\|) + \ell_n \ \forall \, n \geq 1, \ i = 1, 2, ..., m.$$

Proof. Let $I := \{1, 2, ..., m\}$. Since $T_1, T_2, ..., T_m : K \to K$ are m total asymptotically nonexpansive mappings, then there exist sequences $\{\mu_{in}\}$, $\{\ell_{in}\} \subset [0, +\infty)$ with $\lim_{n\to\infty} \mu_{in} = 0 = \lim_{n\to\infty} \ell_{in}$ and nondecreasing continuous function $\phi_i : [0, +\infty) \to [0, +\infty)$, with $\phi_i(0) = 0$ such that for all $x, y \in K$,

$$\|T_i^n x - T_i^n y\| \leq \|x - y\| + \mu_{in}\phi_i(\|x - y\|) + \ell_{in}\ \forall\, n \geq 1,\ \forall\, i \in I.$$

Setting $\mu_n := \max_{i\in I}\{\mu_{in}\}$, $\ell_n := \max_{i\in I}\{\ell_{in}\}$ and defining $\phi : [0, +\infty) \to [0, +\infty)$ by $\phi(t) = \max_{i\in I}\{\phi_i(t)\}$, $\forall\, t \in [0, +\infty)$, then ϕ is nondecreasing continuous with $\phi(0) = 0$; the sequences $\{\mu_n\}$, $\{\ell_n\}$ belong to $[0, +\infty)$ and are such that $\lim_{n\to\infty} \mu_n = 0 = \lim_{n\to\infty} \ell_n$ and for all $x, y \in K$,

$$\|T_i^n x - T_i^n y\| \leq \|x - y\| + \mu_n\phi(\|x - y\|) + \ell_n\ \forall\, n \geq 1,\ i \in I.$$

This completes the proof. □

Remark 10 *In what follows,* $\mu_n := \max_{i\in I}\{\mu_{in}\}$, $\ell_n := \max_{i\in I}\{\ell_{in}\}$ *and* $\phi(t) = \max_{i\in I}\{\phi_i(t)\}$, $\forall\, t \in [0, +\infty)$. *We shall assume that* $\sum_{n=1}^{\infty} \mu_n < +\infty$ *and* $\sum_{n=1}^{\infty} \ell_n < +\infty \Leftrightarrow \sum_{n=1}^{\infty} \ell_{in} < +\infty$, $\forall\, i \in I$ *and that there exists constants* $M_0 > 0, M_1 > 0$ *such that* $\phi(t) \leq M_0 t$ *for all* $t > M_1$. *We shall also assume that the real sequence* $\{\alpha_n\}_{n\geq 1}$ *is such that* $\eta_1 \leq 1 - \alpha_n \leq \eta_2\ \forall\, n \in \mathbb{N}$ *and for some* $\eta_1, \eta_2 \in \left(0, \frac{1}{\xi}\right)$, *where* $\xi := \sup_{n\geq 1}(1 + M_0\mu_n) \geq 1$.

3.1 Convergence of implicit iteration scheme.

We now state and prove the following Theorem.

Theorem 11 *Let* E *be a real normed space,* K *be a nonempty closed convex subset of* E *and* $T_i : K \to K,\ i = 1, 2, ..., m$ *be* m *total asymptotically nonexpansive mappings such that* $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. *Let* $\{x_n\}$ *be given by* (10), *then* $\{x_n\}$ *is bounded and for all* $p \in F$, $\lim_{n\to\infty} \|x_n - p\|$ *exists.*

Proof. Let $p \in F$. Then we obtain using (10) that

$$\begin{aligned} \|x_n - p\| &= \left\|\alpha_n(x_{n-1} - p) + (1 - \alpha_n)(T_{j(n)}^{q(n)} x_n - p)\right\| \\ &\leq \alpha_n\|x_{n-1} - p\| + (1 - \alpha_n)\left\|T_{j(n)}^{q(n)} x_n - p\right\| \\ &\leq \alpha_n\|x_{n-1} - p\| + (1 - \alpha_n)\Big[\|x_n - p\| + \mu_{q(n)}\phi\Big(\|x_n - p\|\Big) + \ell_{q(n)}\Big]. \end{aligned} \tag{15}$$

Since ϕ is a continuous function, it follows that ϕ attains a maximum (say M) in the interval $[0, M_1]$ and (by our assumption, see Remark 10) $\phi(t) \leq M_0 t$ whenever $t > M_1$. In either case, we have

$$\phi(t) \leq M + M_0 t\ \forall\, t \in [0, +\infty). \tag{16}$$

Thus, using (15) and (16), we get

$$\|x_n - p\| \leq \alpha_n\|x_{n-1} - p\| + (1 - \alpha_n)\Big[\|x_n - p\| + \mu_{q(n)}\big(M + M_0\|x_n - p\|\big) + \ell_{q(n)}\Big],$$

so that

$$\Big(1 - (1 - \alpha_n)(1 + \mu_{q(n)}M_0)\Big)\|x_n - p\| \leq \alpha_n\|x_{n-1} - p\| + \eta_2\Big[\mu_{q(n)}M + \ell_{q(n)}\Big]. \tag{17}$$

But $0 < (1-\alpha_n)(1+\mu_{q(n)}M_0) \leq \eta_2\xi < 1 \ \forall \ n \in \mathbb{N}$. So, $1-(1-\alpha_n)(1+\mu_{q(n)}M_0) \geq 1-\eta_2\xi > 0 \ \forall \ n \in \mathbb{N}$. We thus obtain from (17) that

$$\begin{aligned} \|x_n - p\| &\leq \frac{\alpha_n}{1-(1-\alpha_n)(1+\mu_{q(n)}M_0)}\|x_{n-1}-p\| + \frac{\eta_2}{1-\eta_2\xi}\Big[\mu_{q(n)}M + \ell_{q(n)}\Big] \\ &= \frac{1}{\Big(1-\frac{(1-\alpha_n)\mu_{q(n)}M_0}{\alpha_n}\Big)}\|x_n - p\| + \frac{\eta_2}{1-\eta_2\xi}\Big[\mu_{q(n)}M + \ell_{q(n)}\Big]. \end{aligned} \tag{18}$$

Observe that

$$0 < \frac{(1-\alpha_n)\mu_{q(n)}M_0}{\alpha_n} < \frac{\xi^{-1}(\xi-1)}{1-\xi^{-1}} = 1 \tag{19}$$

and that

$$0 < \frac{(1-\alpha_n)\mu_{q(n)}M_0}{\alpha_n} \leq \frac{\eta_2\mu_{q(n)}M_0}{1-\eta_2} \tag{20}$$

So, defining $\omega_n = \frac{(1-\alpha_n)\mu_{q(n)}M_0}{\alpha_n}$, we obtain from (19) that $0 < \omega_n < 1$ and from (20) that $\sum_{n=1}^{\infty}\omega_n < +\infty$, which implies that $\lim_{n\to\infty}\omega_n = 0$. Thus, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$1-\omega_n > k_0 \Leftrightarrow \frac{1}{1-\omega_n} < \frac{1}{k_0} \text{ for some } \mathrm{k}_0 \in (0,1).$$

Therefore, for all $n \geq n_0$,

$$\begin{aligned} \frac{1}{1-\omega_n} &= \sum_{p=0}^{\infty}\omega_n^p = 1 + \omega_n + \omega_n^2\sum_{p=0}^{\infty}\omega_n^p \\ &\leq 1 + \omega_n\Big(1 + \frac{1}{1-\omega_n}\Big) \\ &< 1 + \omega_n(1 + \frac{1}{k_0}) = 1 + \Big[\frac{k_0+1}{k_0}\Big]\omega_n. \end{aligned} \tag{21}$$

Using (21), we obtain from (18) that

$$\begin{aligned} \|x_n - p\| &\leq \frac{1}{1-\omega_n}\|x_{n-1}-p\| + \frac{\eta_2}{1-\eta_2\xi}\Big[\mu_{q(n)}M + \ell_{q(n)}\Big] \\ &\leq \Big(1 + \Big[\frac{k_0+1}{k_0}\Big]\omega_n\Big)\|x_{n-1}-p\| + \frac{\eta_2}{1-\eta_2\xi}\Big[\mu_{q(n)}M + \ell_{q(n)}\Big]. \end{aligned} \tag{22}$$

Now, if we define

$$\lambda_n := \|x_n - p\|, \ \ \beta_n := \Big[\frac{k_0+1}{k_0}\Big]\omega_n \ \text{ and } \ \gamma_n := \frac{1}{1-\eta_2\xi}\Big[\mu_{q(n)}M + \ell_{q(n)}\Big],$$

then (22) becomes

$$\lambda_n \leq (1+\beta_n)\lambda_{n-1} + \gamma_n. \tag{23}$$

It is easy to see that $\sum_{n=1}^{\infty}\beta_n < +\infty$ and $\sum_{n=1}^{\infty}\gamma_n < +\infty$. So, using (23), we obtain from Lemma 6 that the sequence $\{x_n\}$ is bounded and that $\lim_{n\to\infty}\|x_n - p\|$ exists. This completes the proof. □

Theorem 12 *Let E be a uniformly convex real Banach space, K be a nonempty closed convex subset of E and $T_i : K \to K$, $i = 1, 2, ..., m$ be m uniformly continuous total asymptotically nonexpansive mappings such that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (10), then $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0, i = 1, 2, ..., m$.*

Proof. We first show that $\lim_{n\to\infty}\|x_n - T_{j(n)}^{q(n)}x_n\| = 0, i = 1, 2, ..., m$. Let $p \in F$; and observe that from recursion formula (10), Lemma 4 and Remark 5, the following inequality holds:

$$
\begin{aligned}
\|x_n - p\|^2 &= \Big\|\alpha_n(x_{n-1} - p) + (1-\alpha_n)\Big(T_{j(n)}^{q(n)}x_n - p\Big)\Big\|^2 \\
&\leq \alpha_n\|x_{n-1} - p\|^2 + (1-\alpha_n)\Big\|T_{j(n)}^{q(n)}x_n - p\Big\|^2 \\
&\quad -2\alpha_n(1-\alpha_n)C^2\delta_E\Big(\frac{\|x_{n-1} - T_{j(n)}^{q(n)}x_n\|}{2C}\Big) \\
&\leq \alpha_n\|x_{n-1} - p\|^2 + (1-\alpha_n)\Big[\|x_n - p\| + \mu_{j(n)}\phi(\|x_n - p\|) + \ell_{j(n)}\Big]^2 \\
&\quad -2\eta_1(1-\eta_2)\frac{R_0^2\delta_E(\|x_{n-1} - T_{j(n)}^{q(n)}x_n\|/2R_0)}{2L^*} \\
&\leq \alpha_n\|x_{n-1} - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 + Q_0(\mu_{j(n)} + \ell_{j(n)}) \\
&\quad -2\eta_1(1-\eta_2)\frac{R_0^2\delta_E(\|x_{n-1} - T_{j(n)}^{q(n)}x_n\|/2R_0)}{2L^*},
\end{aligned}
$$

for some $Q_0 > 0,\ R_0 > 0$. This implies that

$$
\begin{aligned}
\frac{R_0^2\delta_E(\|x_{n-1} - T_{j(n)}^{q(n)}x_n\|/2R_0)}{2L^*} &\leq \frac{1}{2\eta_1(1-\eta_2)}\Big[\alpha_n\|x_{n-1} - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\
&\quad -\|x_n - p\|^2 + Q_0(\mu_{j(n)} + \ell_{j(n)})\Big] \\
&= \frac{1}{2\eta_1(1-\eta_2)}\Big[\alpha_n\big(\|x_{n-1} - p\|^2 - \|x_n - p\|^2\big) \\
&\quad +Q_0\big(\mu_{j(n)} + \ell_{j(n)}\big)\Big] \\
&\leq \frac{1}{2\eta_1(1-\eta_2)}\Big[(1-\eta_1)\big(\|x_{n-1} - p\|^2 - \|x_n - p\|^2\big) \\
&\quad +Q_0\big(\mu_{j(n)} + \ell_{j(n)}\big)\Big]
\end{aligned} \tag{24}
$$

Since by Theorem 11, $\lim_{n\to\infty}\|x_n - p\|$ exists, we obtain from (24) that

$$
\lim_{n\to\infty}\frac{R_0^2\delta_E(\|x_{n-1} - T_{j(n)}^{q(n)}x_n\|/2R_0)}{2L^*} = 0 \tag{25}
$$

Thus, by continuity of δ_E and the fact that $\delta_E(0) = 0$, (25) gives

$$
\lim_{n\to\infty}\Big\|x_{n-1} - T_{j(n)}^{q(n)}x_n\Big\| = 0. \tag{26}
$$

Next, from (10),

$$
\|x_n - x_{n-1}\| = (1-\alpha_n)\Big\|x_{n-1} - T_{j(n)}^{q(n)}x_n\Big\|.
$$

Thus,

$$
\lim_{n\to\infty}\|x_n - x_{n-1}\| = 0. \tag{27}
$$

Furthermore,

$$
\Big\|T_{j(n)}^{q(n)}x_n - x_n\Big\| \leq \Big\|T_{j(n)}^{q(n)}x_n - x_{n-1}\Big\| + \|x_{n-1} - x_n\|. \tag{28}
$$

So, we obtain from (26), (27) and (28) that

$$
\lim_{n\to\infty}\Big\|T_{j(n)}^{q(n)}x_n - x_n\Big\| = 0. \tag{29}
$$

It is easy to see using (27) that

$$\lim_{n\to\infty}\|x_n - x_{n-i}\| = 0 = \lim_{n\to\infty}\|x_n - x_{n+i}\|, i = 1,2,...,m. \quad (30)$$

By uniform continuity of $T_i,\ i = 1,2,...,m$, there exists a continuous increasing function $\pi : \mathbb{R} \to \mathbb{R}$ with $\pi(0) = 0$ such that

$$\begin{aligned}\left\|x_{n-1} - T_{j(n)}x_n\right\| &\le \left\|x_{n-1} - T_{j(n)}^{q(n)}x_n\right\| + \left\|T_{j(n)}^{q(n)}x_n - T_{j(n)}x_n\right\| \\ &\le \left\|x_{n-1} - T_{j(n)}^{q(n)}x_n\right\| + \pi\Big(\left\|T_{j(n)}^{q(n)-1}x_n - x_n\right\|\Big). \quad (31)\end{aligned}$$

Observe that from the second summand on the right hand side (second line) of (31), we get

$$\begin{aligned}\left\|T_{j(n)}^{q(n)-1}x_n - x_n\right\| &\le \left\|T_{j(n)}^{q(n)-1}x_n - T_{j(n-m)}^{q(n)-1}x_{n-m}\right\| \\ &\quad + \left\|T_{j(n-m)}^{q(n)-1}x_{n-m} - x_{n-m}\right\| + \|x_{n-m} - x_n\|. \quad (32)\end{aligned}$$

But, by (14),

$$q(n-m) = q(n) - 1 \ \text{ and } \ j(n-m) = j(n).$$

Considering the first two summands on the right hand side of (32), it therefore follows that the first summand

$$\begin{aligned}\left\|T_{j(n)}^{q(n)-1}x_n - T_{j(n-m)}^{q(n)-1}x_{n-m}\right\| &= \left\|T_{j(n)}^{q(n)-1}x_n - T_{j(n)}^{q(n)-1}x_{n-m}\right\| \\ &\le \|x_n - x_{n-m}\| + \mu_{q(n)-1}\phi(\|x_n - x_{n-m}\|) \\ &\quad + \ell_{q(n)-1}. \quad (33)\end{aligned}$$

Thus, (33) implies that

$$\lim_{n\to\infty}\left\|T_{j(n)}^{q(n)-1}x_n - T_{j(n-m)}^{q(n)-1}x_{n-m}\right\| = 0. \quad (34)$$

Moreover, the second summand

$$\left\|T_{j(n-m)}^{q(n)-1}x_{n-m} - x_{n-m}\right\| = \left\|T_{j(n-m)}^{q(n-m)}x_{n-m} - x_{n-m}\right\| \to 0 \text{ as } n \to +\infty \quad (35)$$

So, using (34) and (35) in (32), we obtain that

$$\lim_{n\to\infty}\|T_{j(n)}^{q(n)-1}x_n - x_n\| = 0.$$

As a result, we obtain from (31) (using the property of π) that

$$\lim_{n\to\infty}\|x_{n-1} - T_{j(n)}x_n\| = 0. \quad (36)$$

Furthermore,

$$\|x_n - T_{j(n)}x_n\| \le \|x_n - x_{n-1}\| + \|x_{n-1} - T_{j(n)}x_n\|. \quad (37)$$

Thus, using (27) and (36), we obtain from (37) that

$$\lim_{n\to\infty}\|x_n - T_{j(n)}x_n\| = 0. \quad (38)$$

Again, using the fact that $T_i, i = 1,2,...,m$ are uniformly continuous, we have that there exists continuous increasing functions $\pi_i : \mathbb{R} \to \mathbb{R}$ with $\pi_i(0) = 0, i = 1,2,...,m$ such that $\|T_ix - T_iy\| \le \pi_i(\|x - y\|)\ \forall\ x, y \in K,\ i = 1,2,...,m$. Thus, defining $\pi_0 : \mathbb{R} \to \mathbb{R}$ by $\pi_0(t) = \max_{i\in I}\pi_i(t)$ (where $I = \{1,2,...,m\}$), we have that π_0 is continuous increasing function, $\pi_0(0) = 0$ and

$$\begin{aligned}\|x_n - T_{j(n)+i}x_n\| &\le \|x_n - x_{n+i}\| + \|x_{n+i} - T_{j(n)+i}x_{n+i}\| \\ &\quad + \|T_{j(n)+i}x_{n+i} - T_{j(n)+i}x_n\| \\ &\le \|x_n - x_{n+i}\| + \|x_{n+i} - T_{j(n)+i}x_{n+i}\| \\ &\quad + \pi_0(\|x_{n+i} - x_n\|). \quad (39)\end{aligned}$$

So, using (30), (38) and (39), we have that

$$\lim_{n\to\infty} \|x_n - T_{j(n)+i}x_n\| = 0, \ i = 1, 2, ..., m. \tag{40}$$

But for all $i \in \{1, 2, ..., m\}$, there exists $\theta_i \in \{1, 2, ..., m\}$ such that

$$j(n) + \theta_i = i \ (mod \ m).$$

It therefore follows from (40) that

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = \lim_{n\to\infty} \|x_n - T_{j(n)+\theta_i}x_n\| = 0.$$

Hence, $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for all $i \in \{1, 2, ..., m\}$. This completes the proof. □

3.2 Weak convergence of implicit iteration process.

Theorem 13 *Let K be a closed convex nonempty subset of a uniformly convex real Banach space with weakly continuous normalized duality mapping. Let $T_1, T_2, ..., T_m : K \to K$ be m uniformly continuous total asymptotically nonexpansive mappings such that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{x_n\}_{n\geq 1}$ be the sequence defined by (10), then $\{x_n\}_{n\geq 1}$ converges weakly to a common fixed point of $T_1, T_2, ..., T_m$.*

Proof. Since E is a reflexive Bananch space and by Theorem 11, $\{x_n\}_{n\geq 1}$ is bounded, there exists a subsequence $\{x_{n_k}\}_{k\geq 1}$ of $\{x_n\}_{n\geq 1}$ such that $\{x_{n_k}\}_{k\geq 1}$ converges weakly to some $x^* \in K$; and since by Theorem 12 $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for all $i \in \{1, 2, ..., m\}$, we obtain by Lemma 7 that $x^* \in F$. We now show that $\{x_n\}_{n\geq 1}$ converges weakly to x^*. Suppose for contradiction that there exists another subsequence $\{x_{n_s}\}_{s\geq 1}$ of $\{x_n\}_{n\geq 1}$ such that $\{x_{n_s}\}_{s\geq 1}$ converges weakly to $q^* \neq x^*$. Lemma 7 again shows that $q^* \in F$. By Theorem 11, $r_1 = \lim_{n\to\infty} \|x_n - x^*\|$ and $r_2 = \lim_{n\to\infty} \|x_n - q^*\|$ exist, where $r_1 \geq 0$ and $r_2 \geq 0$. Then since E has weakly continuous normalized duality mapping, it follows (see e. g. Gossez and Lami Dozo [20]) that E satisfies Opial's condition (see Opial [30]). Thus,

$$\begin{aligned} r_1 &= \limsup_{k\to\infty} \|x_{n_k} - x^*\| < \limsup_{k\to\infty} \|x_{n_k} - q^*\| \\ &= r_2 = \limsup_{s\to\infty} \|x_{n_s} - q^*\| < \limsup_{s\to\infty} \|x_{n_s} - x^*\| = r_1, \end{aligned}$$

a contradiction. Hence, $x^* = q^*$. This implies that $\{x_n\}_{n\geq 1}$ converges weakly to $x^* \in F$. This completes the proof. □

The following corollaries easily follow from our presentation so far.

Corollary 14 *Let K be a closed convex nonempty subset of a uniformly convex real Banach space with weakly continuous normalized duality mapping. Let $T_1, T_2, ..., T_m : K \to K$ be m uniformly L-Lipschitzian total asymptotically nonexpansive mappings such that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{x_n\}_{n\geq 1}$ be the sequence defined by (10), then $\{x_n\}_{n\geq 1}$ converges weakly to a common fixed point of $T_1, T_2, ..., T_m$.*

Corollary 15 *Let K be a closed convex nonempty subset of a uniformly convex real Banach space with weakly continuous normalized duality mapping. Let $T_1, T_2, ..., T_m : K \to K$ be m asymptotically nonexpansive mappings such that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{x_n\}_{n\geq 1}$ be the sequence defined by (10), then $\{x_n\}_{n\geq 1}$ converges weakly to a common fixed point of $T_1, T_2, ..., T_m$.*

Corollary 16 *Let K be a closed convex nonempty subset of a uniformly convex real Banach space with weakly continuous normalized duality mapping. Let $T_1, T_2, ..., T_m : K \to K$ be m nonexpansive mappings such that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{x_n\}_{n\geq 1}$ be the sequence defined by (10), then $\{x_n\}_{n\geq 1}$ converges weakly to a common fixed point of $T_1, T_2, ..., T_m$.*

Corollary 17 *Let K be a closed convex nonempty subset of a real Hilbert space H. Let $T_1, T_2, ..., T_m : K \to K$ be m uniformly continuous total asymptotically nonexpansive mappings such that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{x_n\}_{n\geq 1}$ be the sequnce defined by (10), then $\{x_n\}_{n\geq 1}$ converges weakly to a common fixed point of $T_1, T_2, ..., T_m$.*

3.3 Strong convergence of implicit iteration process.

Theorem 18 *Let E be a uniformly convex real Banach space, K be a nonempty closed convex subset of E and $T_i : K \to K$, $i = 1, 2, ..., m$ be m uniformly continuous total asymptotically nonexpansive mappings such that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (10). Suppose that one of $T_1, T_2, ..., T_m$ is compact, then $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, ..., T_m$.*

Proof. We obtain from Theorem 12 that

$$\lim_{n\to\infty} \|T_i x_n - x_n\| = 0, \ i = 1, 2, ..., m. \tag{41}$$

Let T_{i_0} be compact for some $i_0 \in \{1, 2, ..., m\}$. Since T_{i_0} is continuous and compact, it is completely continuous. Thus, there exists a subsequence $\{T_{i_0} x_{n_k}\}$ of $\{T_{i_0} x_n\}$ such that $T_{i_0} x_{n_k} \to x^*$ as $k \to \infty$ for some $x^* \in K$. Since by (41) $\lim_{k\to\infty} \|x_{n_k} - T_{i_0} x_{n_k}\| = 0$, we have that $\lim_{k\to\infty} x_{n_k} = x^*$. Observe that for all $i \in I$,

$$\|x^* - T_i x^*\| \ \ \leq \ \ \|x^* - x_{n_k}\| + \|x_{n_k} - T_i x_{n_k}\| + \|T_i x_{n_k} - T_i x^*\|. \tag{42}$$

So, we obtain from (42) that Taking limit as $k \to \infty$ in (42) using the fact that $T_i \ \forall \ i \in I$ is continuous, we have that $x^* = T_i x^* \ \forall \ i \in I$ and so $x^* \in F(T_i) \ \forall \ i \in I$. But by Theorem 11, $\lim_{n\to\infty} \|x_n - p\|$ exists for $p \in F$. Thus, $\lim_{n\to\infty} \|x_n - x^*\| = 0$. Hence, $\{x_n\}$ converges strongly to $x^* \in F$. This completes the proof. □

Corollary 19 *Let E be a uniformly convex real Banach space, K be a nonempty closed convex subset of E and $T_i : K \to K$, $i = 1, 2, ..., m$ be m uniformly L-Lipschitzian total asymptotically nonexpansive mappings such that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (10). Suppose that one of $T_1, T_2, ..., T_m$ is compact, then $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, ..., T_m$.*

Corollary 20 *Let E be a uniformly convex real Banach space, K be a nonempty closed convex subset of E and $T_i : K \to K$, $i = 1, 2, ..., m$ be m asymptotically nonexpansive mappings such that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (10). Suppose that one of $T_1, T_2, ..., T_m$ is compact, then $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, ..., T_m$.*

Corollary 21 *Let E be a uniformly convex real Banach space, K be a nonempty closed convex subset of E and $T_i : K \to K$, $i = 1, 2, ..., m$ be m nonexpansive mappings such that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (10). Suppose that one of $T_1, T_2, ..., T_m$ is compact, then $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, ..., T_m$.*

Remark 22 *Though Theorem 18 holds in both real ℓ_p and L_p spaces $1 < p < +\infty$, Theorem 13 does not hold in L_p spaces $1 < p < +\infty$, $p \neq 2$ since it is well known that for $p \neq 2$, L_p spaces do not possess weakly continuous duality mapping (see e.g. [5]-[7]). It is thus easy to see that both Theorems 13 and 18 hold in real Hilbert spaces.*

3.4 Convergence of explicit iteration process.

Theorem 23 *Let E be a real normed space, K be a nonempty closed convex subset of E and $T_i : K \to K,\ i = 1, 2, ..., m$ be m total asymptotically nonexpansive mappings such that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. From arbitrary $z_1 \in E$, define the sequence $\{z_n\}$ by (11), then, $\{z_n\}_{n\geq 1}$ is bounded and for all $p \in F$, $\lim_{n\to\infty} \|z_n - p\|$ exists.*

Proof. Let $p \in F$. Then using (11), we obtain that

$$\begin{aligned} \|z_{n+1} - p\| &= \Big\|\alpha_n(z_n - p) + (1 - \alpha_n)(T^{q(n)}_{j(n)} z_n - p)\Big\| \\ &\leq \alpha_n\|z_n - p\| + (1 - \alpha_n)\Big[\|z_n - p\| + \mu_{q(n)}\phi(\|z_n - p\|) + \ell_{q(n)}\Big] \\ &\leq \alpha_n\|z_n - p\| + (1 - \alpha_n)\Big[\|z_n - p\| + \mu_{q(n)}(M + M_0\|z_n - p\|) + \ell_{q(n)}\Big], \end{aligned} \tag{43}$$

where M is the maximum of the continuous function ϕ on the interval $[0, M_1]$ and $M_0,\ M_1$ as in Remark 10. Thus, we obtain from (43) that

$$\|z_{n+1} - p\| \leq \Big(1 + M_0\mu_{q(n)}\Big)\|z_n - p\| + \mu_{q(n)}M + \ell_{q(n)}. \tag{44}$$

So,

$$\|z_{n+1} - p\| \leq \Big(1 + \delta_n\Big)\|z_n - p\| + \sigma_n, \tag{45}$$

where $\delta_n = M_0\mu_{q(n)}$ and $\sigma_n = \mu_{q(n)}M + \ell_{q(n)}$. Observe that $\sum_{n=1}^{\infty} \delta_n < +\infty$ and $\sum_{n=1}^{\infty} \sigma_n < +\infty$. Hence, by Lemma 6, we have that $\{z_n\}_{n\geq 1}$ is bounded and $\lim_{n\to\infty} \|z_n - p\|$ exists for all $p \in F$. This completes the proof. □

Theorem 24 *Let E be a uniformly convex real Banach space, K be a nonempty closed convex subset of E and $T_i : K \to K,\ i = 1, 2, ..., m$ be m uniformly continuous total asymptotically nonexpansive mappings such that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. From arbitrary $z_1 \in E$, define the sequence $\{z_n\}$ by (11), then, $\lim_{n\to\infty} \|z_n - T_i z_n\| = 0,\ i = 1, 2, ..., m$.*

Proof. Let $p \in F(T)$, then from recursion formula (11), Lemma 4 and Remark 5, we get that

$$\begin{aligned} \|z_{n+1} - p\|^2 &= \Big\|\alpha_n(z_n - p) + (1 - \alpha_n)\Big(T^{q(n)}_{j(n)} z_n - p\Big)\Big\|^2 \\ &\leq \alpha_n\|z_n - p\|^2 + (1 - \alpha_n)\Big\|T^{q(n)}_{j(n)} z_n - p\Big\|^2 \\ &\quad -2\alpha_n(1 - \alpha_n)C^2\delta_E\Big(\frac{\|z_n - T^{q(n)}_{j(n)} z_n\|}{2C}\Big) \\ &\leq \alpha_n\|z_n - p\|^2 + (1 - \alpha_n)\Big[\|z_n - p\| + \mu_{j(n)}\phi(\|z_n - p\|) + \ell_{j(n)}\Big]^2 \\ &\quad -2\eta_1(1 - \eta_2)\frac{R_1^2\delta_E(\|z_n - T^{q(n)}_{j(n)} z_n\|/2R_1)}{2L^*} \\ &\leq \alpha_n\|z_n - p\|^2 + (1 - \alpha_n)\|z_n - p\|^2 + Q_1(\mu_{j(n)} + \ell_{j(n)}) \\ &\quad -2\eta_1(1 - \eta_2)\frac{R_1^2\delta_E(\|z_n - T^{q(n)}_{j(n)} z_n\|/2R_1)}{2L^*} \end{aligned}$$

for some $Q_1 > 0,\ R_1 > 0$. This implies that

$$\begin{aligned} \frac{R_1^2\delta_E(\|z_n - T^{q(n)}_{j(n)} z_n\|/2R_1)}{2L^*} &\leq \frac{1}{2\eta_1(1 - \eta_2)}\Big[\|z_n - p\|^2 - \|z_{n+1} - p\|^2 \\ &\quad +Q_0(\mu_{j(n)} + \ell_{j(n)})\Big] \end{aligned} \tag{46}$$

Since by Theorem 23, $\lim_{n\to\infty}\|z_n-p\|$ exists, we obtain from (46) that

$$\lim_{n\to\infty}\frac{M^2\delta_E(\|z_n-T^{q(n)}_{j(n)}z_n\|/2M)}{2L^*}=0 \tag{47}$$

Thus, since δ_E is strictly increasing, continuous and $\delta_E(0)=0$, (47) gives

$$\lim_{n\to\infty}||z_n-T^{q(n)}_{j(n)}z_n||=0. \tag{48}$$

Also, from (11),

$$\|z_{n+1}-z_n\|\le||z_n-T^{q(n)}_{j(n)}z_n||.$$

Thus,

$$\lim_{n\to\infty}\|z_{n+1}-z_n\|=0\Leftrightarrow\lim_{n\to\infty}\|z_n-z_{n-1}\|=0.$$

The rest follows as in the proof of Theorem 12. This completes the Proof. □

Follwing the method of proof of Theorem 13 and Theorem 18, we (respectively) obtain the following theorems:

3.5 Weak convergence of explicit iteration process.

Theorem 25 *Let K be a closed convex nonempty subset of a uniformly convex real Banach space with weakly continuous normalized duality mapping. Let $T_1,T_2,...,T_m:K\to K$ be m uniformly continuous total asymptotically nonexpansive mappings such that $F:=\bigcap_{i=1}^{m}F(T_i)\neq\emptyset$. Let $\{z_n\}_{n\ge1}$ be the sequence defined by (11), then $\{z_n\}_{n\ge1}$ converges weakly to a common fixed point of $T_1,T_2,...,T_m$.*

3.6 Strong convergence of explicit iteration process.

Theorem 26 *Let E be a uniformly convex real Banach space, K be a nonempty closed convex subset of E and $T_i:K\to K$, $i=1,2,...,m$ be m uniformly continuous total asymptotically nonexpansive mappings such that $F:=\bigcap_{i=1}^{m}F(T_i)\neq\emptyset$. Let $\{x_n\}$ be the sequence defined by (10). Suppose that one of $T_1,T_2,...,T_m$ is compact, then $\{x_n\}$ converges strongly to a common fixed point of $T_1,T_2,...,T_m$.*

Remark 27 *Corollaries similar to those obtained above are also obtainable in subsections 3.5 and 3.6.*

3.7 Necessary and sufficient conditions for convergence in real Banach spaces.

Using (23), (45) and following the method of proof of Theorem 3.2 of [12], we (respectively) obtain the following theorems.

Theorem 28 *Let E be a real Banach space, K be a nonempty closed convex subset of E and $T_i:K\to K$, $i=1,2,...,m$ be m continuous total asymptotically nonexpansive mappings such that $F:=\bigcap_{i=1}^{m}F(T_i)\neq\emptyset$. Suppose that $\{x_n\}$ is given by (10), then the sequence $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^m$ if and only if $\liminf_{n\to\infty}d\Big(x_n,F\Big)=0$, where $d\Big(x_n,F\Big)=\inf_{y\in F}\|x_n-y\|$, $n\in\mathbb{N}$.*

Theorem 29 *Let E be a real Banach space, K be a nonempty closed convex subset of E and $T_i : K \to K,\ i = 1, 2, ..., m$ be m continuous total asymptotically nonexpansive mappings such that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Suppose that $\{z_n\}$ is given by (11), then the sequence $\{z_n\}$ converges strongly to a common fixed point of $T_i,\ i = 1, 2, ..., m$ if and only if $\liminf_{n\to\infty} d\Big(z_n, F\Big) = 0$.*

Definition 30 *A mapping $T : K \to K$ is said to be* total asymptotically quasi-nonexpansive *if $F(T) \neq \emptyset$ and there exist nonnegative real sequences $\{\mu_n\}$ and $\{l_n\}$, $n \geq 1$ with $\mu_n,\ l_n \to 0$ as $n \to \infty$ and strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for all $x \in E,\ x^* \in F(T)$,*

$$\|T^n x - x^*\| \leq \|x - x^*\| + \mu_n \phi(\|x - x^*\|) + l_n,\ n \geq 1. \tag{49}$$

Remark 31 *If $\phi(t) \equiv 0$, then (1) reduces to $\|T^n x - x^*\| \leq \|x - x^*\| + l_n,\ n \geq 1$, so that if K is bounded and T^N is continuous for some integer $N \geq 1$, the mapping T is of quasi asymptotically nonexpansive type which includes the class of mappings which are quasi asymptotically nonexpansive in the intermediate sense. If $\phi(\lambda) = \lambda$, then (49) reduces to*

$$\|T^n x - x^*\| \leq (1 + \mu_n)\|x - x^*\| + l_n,\ n \geq 1. \tag{50}$$

In addition, if $l_n = 0$ for all $n \geq 1$, then total asymptotically quasi-nonexpansive mappings coincide with asymptotically quasi-nonexpansive mappings studied by various authors. If $\mu_n = 0$ and $l_n = 0$ for all $n \geq 1$, we obtain from (50) the class of quasi-nonexpansive mappings. Observe that the class of total asymptotically nonexpansive mappings with nonempty fixed point sets belongs to the class of total asymptotically quasi-nonexpansive mappings.

It is trivial to observe that all the Theorems of this paper carry over to the class of total asymptotically quasi-nonexpansive mappings with little or no modifications.

A subset K of a real normed linear space E is said to be a retract of E if there exists a continuous map $P : E \to K$ such that $Px = x$ for all $x \in K$. A map $P : E \to E$ is said to be a retraction if $P^2 = P$. It follows that if a map P ia a retraction, then $Py = y$ for all y in the range of P. The mapping P is called a sunny nonexpansive retraction if for all $x \in E$ and $t \in (0, 1)$ $P((1 - t)x + tP(x)) = P(x)$. If K is a nonempty closed convex subset of a Hilbert space H, then the nearest point projection P_K from H to K is the sunny nonexpansive retraction. This however is not true for Banach spaces since nonexpansivity of projections P_K characterizes Hilbert spaces. On the other hand, a sunny nonexpansive retraction can play a similar role in a Banach space as a projection does in Hilbert spaces. For existence of nonexpansive retracts outside Hilbert spaces, one may see [31].

Definition 32 *Let K be a nonempty closed and convex subset of E. Let $P : E \to K$ be the nonexpansive retraction of E onto K. A* non-self *map $T : K \to E$ is said to be* total asymptotically nonexpansive *if there exist sequences $\{\mu_n\}_{n\geq1},\ \{l_n\}_{n\geq1}$ in $[0, +\infty$ with $\mu_n,\ l_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : [0, +\infty) \to [0, +\infty)$ with $\phi(0) = 0$ such that for all $x, y \in K$,*

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq \|x - y\| + \mu_n \phi(\|x - y\|) + l_n,\ n \geq 1.$$

Let $T_1, T_2, ..., T_m : K \to E$ be m total asymptotically nonexpansive non-self *maps; assuming existence of common fixed points of these operators, our theorems and method of proof easily carry over to this class of mappings using the iterative sequences $\{x_n\}$ and $\{z_n\}$ defined by*

$$x_0 \in K,\ x_n = P\Big(\alpha_n x_{n-1} + (1 - \alpha_n) T_{j(n)} (PT_{j(n)})^{q(n)-1} x_n\Big),\ n \geq 1$$

and

$$z_1 \in K,\ z_{n+1} = P\Big(\alpha_n z_n + (1 - \alpha_n) T_{j(n)} (PT_{j(n)})^{q(n)-1} z_n\Big),\ n \geq 1$$

(respectively) instead of (10) and (11) provided the well definedness of P as a sunny nonexpansive retraction is guaranteed.

Remark 33 *We note that for the class of asymptotically nonexpansive mappings, the condition - there exist* $M_0 > 0$ *and* $M_1 > 0$ *such that* $\phi(t) \leq M_0 t$ *for all* $t > M_1$*- is not needed. A prototype for* $\phi : [0, \infty) \to [0, \infty)$ *satisfying the conditions of our theorems is* $\phi(\lambda) = \lambda^s,\ 0 < s \leq 1$.

Remark 34 *Addition of* bounded *(or the so called mean) error terms to the iteration process studied in this paper leads to no further generalization. In fact, if we consider the sequences* $\{x_n\}_{n\geq 1}$, $\{z_n\}_{n\geq 1}$ *generated by*

$$x_0 \in K, x_n = \alpha_n x_{n-1} + \beta_n T_{j(n)}^{q(n)} x_n + \gamma_n u_n,\ n \geq 1, \tag{51}$$

$$z_1 \in K, z_{n+1} = \alpha_n z_{n-1} + \beta_n T_{j(n)}^{q(n)} z_n + \gamma_n u_n,\ n \geq 1 \tag{52}$$

(for self total asymptotically nonexpansive mappings) and the iterative sequences $\{x_n\}$ *and* $\{z_n\}$ *defined by*

$$x_0 \in K,\ x_n = P\Big(\alpha_n x_{n-1} + \beta_n T_{j(n)}(PT_{j(n)})^{q(n)-1} x_n + \gamma_n u_n\Big),\ n \geq 1 \tag{53}$$

and

$$z_1 \in K,\ z_{n+1} = P\Big(\alpha_n z_n + \beta_n T_{j(n)}(PT_{j(n)})^{q(n)-1} z_n + \gamma_n u_n\Big),\ n \geq 1 \tag{54}$$

(for nonself total asymptotically nonexpansive mappings), where $\{\alpha_n\}_{n\geq 1}$, $\{\beta_n\}_{n\geq 1}$ *and* $\{\gamma_n\}_{n\geq 1}$ *are real sequnces in* $(0, 1)$ *such that* $\alpha_n + \beta_n + \gamma_n = 1\ \forall\ n \geq 1$, $\sum_{n=1}^{\infty} \gamma_n < +\infty$, $\eta_1 \leq \beta_n \leq \eta_2$ *for some* $\eta_1, \eta_2 \in (0, \frac{1}{\xi})$ *(where* ξ *is as defined in Remark 10) and* $\{u_n\}_{n\geq 1}$ *is a bounded sequence in* K*, then repeating the argument of this paper, we obtain weak and strong convergence of* (51), (52), (53) *and* (54) *to a common fixed point of corresponding* $T_1, T_2, ..., T_m$*. The iteration scheme* (54) *improves and generalizes the so called three step iteration scheme with error recently introduced by L. Yang and X. Xie [25] for three nonself asymptotically nonexpansive mappings.*

Remark 35 *Our theorems unify, extend and generalize the corresponding results of Alber* et al. *[2], L. Yang and X. Xie [25], Sahu [32], Shahzad and Udomene [33] and a host of other results recently announced (see e.g. [8]-[17], [24, 26], [33]-[37], [38, 41, 42, 43],)for the approximation of common fixed points of finite families of several classes of nonlinear mappings.*

Remark 36 *Finally, observe that the condition (5) as imposed in Theorem ZC of Zhou and Chang [45], the conditions* $K + K \subset K$, $\sum_{n=1}^{\infty} \|u_n\| < +\infty$ *and the boundedness condition imposed on* K *by Hao [21] are all dispenced with in this paper. Our iteration schemes* (10) *and* (51) *are, therefore, far better and more efficient than the schemes* (4) *and* (5) *respectively introduced and studied by Zhou and Chang [45] and Hao [21]. Our method of proof Theorem 24 closed the gap observed in the proof of Theorem 3.5 of [12]. Theorem 13 extends the corresponding result of Zhou and Chang [45] from the class of asymptotically nonexpansive mappings to the class of uniformly continuous total asymptotically nonexpansive mappings; while the corresponding result of Hao [21] is improved, generalized and extended from real Hilbert space to uniformly convex real Banach spaces and from the class of uniformly L-Lipschitzian total asymptotically nonexpansive mappings to the class of uniformly continuous asymptotically nonexpansive mappings. Our explicit iteration schemes and strong convergence theorems are also of independent interest.*

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