Wilf's formula and a generalization of the Choi – Lee – Srivastava identities

Ulrich Abel

Technische Hochschule Mittelhessen, Fachbereich MND

Wilhelm-Leuschner-Straße 13, 61169 Friedberg, Germany.

e-mail: Ulrich.Abel@mnd.thm.de

Abstract

The identities of Choi, Lee, and Srivastava imply a formula proposed by Wilf. We show that these identities are immediate consequences of the well-known product formulas for the sine function and the cosine function. Moreover, we prove a generalization.

Keywords: Euler–Mascheroni constant, infinite product formulas.

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1 Introduction

Herbert S. Wilf [1] proposed in the problem section of *The American Mathematical Monthly* to prove the identity

$$\cosh\left(\frac{\pi}{2}\right) = \frac{\pi}{2} e^{\gamma} \prod_{k=1}^{\infty} e^{-1/k} \left(1 + \frac{1}{k} + \frac{1}{2k^2}\right), \tag{1}$$

where γ denotes the Euler–Mascheroni constant. In the following there appeared several proofs ([2], cf. [3, 4]). Chen and Paris [5, Theorem 1] gave explicit expressions for infinite products of the form

$$\prod_{k=1}^{\infty} e^{-p_1/k} \left(1 + \frac{p_1}{k} + \frac{p_2}{k^2} + \cdots + \frac{p_m}{k^m}\right),$$

where $p_1, \cdots, p_m \in \mathbb{C}$ and m is any positive integer (see also [6]). Choi, Lee, and Srivastava [7] derived the following generalization

$$\begin{aligned} \sinh(\pi z) &= \pi z \left(1 + z^2\right) e^{2\gamma} \prod_{k=1}^{\infty} \left(1 + \frac{2}{k} + \frac{1 + z^2}{k^2}\right), \\ \cosh(\pi z) &= \pi \left(\frac{1}{4} + z^2\right) e^{\gamma} \prod_{k=1}^{\infty} e^{-1/k} \left(1 + \frac{1}{k} + \frac{1/4 + z^2}{k^2}\right). \end{aligned} \tag{2}$$

Recently, C. Hernández-Aguilar, J. López-Bonilla, and R. López-Vázquez [8], proved the latter identities [8, Eqs. (3) and (2)] using a certain relation involving an infinite product and the gamma function [8, Eq. (4)]. In this note we show that these identities are immediate consequences of the well-known product formulas

$$\sin(\pi z) = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right), \qquad \cos(\pi z) = \prod_{k=1}^{\infty} \left(1 - \frac{4z^2}{(2k-1)^2}\right) \tag{3}$$

for the sine function and the cosine function, respectively. Moreover, we derive the following generalization.

Theorem 1 *For $r \in \mathbb{N}$ and $z \in \mathbb{C}$, the hyperbolic functions possess the representations*

$$\begin{aligned} \sinh(\pi z) &= \pi z \left(\prod_{j=1}^{r} \left(j^2 + z^2\right)\right) e^{2r\gamma} \prod_{k=1}^{\infty} e^{-2r/k} \left(1 + \frac{2r}{k} + \frac{r^2 + z^2}{k^2}\right), \\ \cosh(\pi z) &= \pi \left(\prod_{j=1}^{r} \left(\left(j - \frac{1}{2}\right)^2 + z^2\right)\right) e^{(2r-1)\gamma} \prod_{k=1}^{\infty} e^{-(2r-1)/k} \left(1 + \frac{2r-1}{k} + \frac{\left(r - \frac{1}{2}\right)^2 + z^2}{k^2}\right). \end{aligned}$$

In the special case $r = 1$ the formulas reduce to the identities (2), which are valid also in the cases $z = \pm i$ and $z = \pm i/2$, respectively. For $z = 1/2$, we obtain

$$\cosh\left(\frac{\pi}{2}\right) = \pi\left(\prod_{j=1}^{r}\left(j^2 - j + \frac{1}{2}\right)\right) e^{(2r-1)\gamma} \prod_{k=1}^{\infty} e^{-(2r-1)/k}\left(1 + \frac{2r-1}{k} + \frac{r^2 - r + \frac{1}{2}}{k^2}\right).$$

Wilf's formula (1) is the special case $r = 1$.

2 Proof of Theorem 1

The product representation (3) of sine implies

$$\sinh(\pi z) = -i\sin(i\pi z) = \pi z \lim_{n\to\infty} f_n(z),$$

where

$$f_n(z) = \prod_{k=1}^{n}\frac{k^2+z^2}{k^2} = \left(\prod_{j=1}^{r}\frac{j^2+z^2}{(n+j)^2+z^2}\right)\prod_{k=1}^{n}\frac{(k+r)^2+z^2}{k^2}.$$

Furthermore,

$$\prod_{k=1}^{n} e^{-1/k} = e^{-(\ln n + \gamma_n)} = \frac{1}{n}e^{-\gamma_n},$$

with positive reals γ_n tending to $\lim_{n\to\infty}\gamma_n = \gamma$. Hence,

$$f_n(z) = \left(\prod_{j=1}^{r}\frac{j^2+z^2}{(n+j)^2+z^2}\right)\cdot n^{2r}e^{2r\gamma_n}\prod_{k=1}^{n}\left(e^{-2r/k}\frac{(k+r)^2+z^2}{k^2}\right).$$

The limit letting $n \to \infty$ leads to the first formula of the theorem.
Analogously, the product representation (3) of cosine implies

$$\cosh(\pi z) = \cos(i\pi z) = \lim_{n\to\infty} g_n(z),$$

where

$$g_n(z) = \prod_{k=1}^{n}\frac{(2k-1)^2+4z^2}{(2k-1)^2} = \left(\prod_{j=1}^{r}\frac{(2j-1)^2+4z^2}{(2n+2j-1)^2+4z^2}\right)\prod_{k=1}^{n}\frac{(2k+2r-1)^2+4z^2}{(2k-1)^2}.$$

As above we conclude that

$$\begin{aligned} g_n(z) &= \left(\prod_{j=1}^{r}\frac{(j-1/2)^2+z^2}{(n+j-1/2)^2+z^2}\right) n^{2r-1}e^{(2r-1)\gamma_n}\prod_{k=1}^{n}\left(e^{-(2r-1)/k}\frac{(k+r-1/2)^2+z^2}{k^2}\cdot\frac{k^2}{(k-1/2)^2}\right) \\ &\to \pi\left(\prod_{j=1}^{r}\left(\left(j-\frac{1}{2}\right)^2+z^2\right)\right) e^{(2r-1)\gamma}\prod_{k=1}^{\infty}\left(e^{-(2r-1)/k}\frac{k^2+(2r-1)k+\left(r-\frac{1}{2}\right)^2+z^2}{k^2}\right) \end{aligned}$$

as $n \to \infty$, since it is well-known (see, e.g., [9, (6.1.46)]), that

$$\prod_{k=1}^{n}\frac{k}{k-1/2} = \Gamma(1/2)\frac{\Gamma(n+1)}{\Gamma(n+1/2)} \sim \sqrt{\pi n} \qquad (n\to\infty).$$

This completes the proof.

Conclusion

The above note presents a generalization of the identities by Choi, Lee, and Srivastava. We show that these identities are immediate consequences of the well-known product formulas for the sine function and the cosine function. They imply a formula proposed by Herbert S. Wilf.

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Conflicts of Interest

The author declares that there is no conflict of interests.

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