Weak Insertion of a Contra-Baire-1 (Baire-.5) Function

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Abstract

A sufficient condition in terms of lower cut sets are given for the weak insertion of a Baire-.5 function between two comparable real-valued functions on the topological spaces that $\mathcal{F}_{0}$-kernel of sets are $\mathcal{F}_{0}$-sets.

Indexing terms/Keywords: Weak insertion, Strong binary relation, Baire-.5 function, kernel-sets, Lower cut set.


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J. Dontchev in [5] introduced a new class of mappings called contra-continuity. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 3, 7, 8, 9, 11, 12, 19].

Results of Kat`etov [13, 14] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a necessary and sufficient condition...
1 Introduction

A generalized class of closed sets was considered by Maki in 1986 [16]. He investigated the sets that can be represented as union of closed sets and called them \( V \)-sets. Complements of \( V \)-sets, i.e., sets that are intersection of open sets are called \( \Lambda \)-sets [16].

Recall that a real-valued function \( f \) defined on a topological space \( X \) is called \( A \)-continuous [20] if the preimage of every open subset of \( \mathbb{R} \) belongs to \( A \), where \( A \) is a collection of subsets of \( X \). Most of the definitions of function used throughout this paper are consequences of the definition of \( A \)-continuity. However, for unknown concepts the reader may refer to [4, 10]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity. For the insertion of a Baire-.5 function between two comparable real-valued functions on the topological spaces that \( \mathcal{F} \)-kernel of sets are \( \mathcal{F} \)-sets.

A real-valued function \( f \) defined on a topological space \( X \) is called contraBaire-.5 (Baire-.5) if the preimage of every open subset of \( \mathbb{R} \) is a G\( \delta \)-set [21].

If \( g \) and \( f \) are real-valued functions defined on a space \( X \), we write \( g \leq f \) in case \( g(x) \leq f(x) \) for all \( x \) in \( X \).

The following definitions are modifications of conditions considered in [15].

A property \( P \) defined relative to a real-valued function on a topological space is a \( B \)--.5-property provided that any constant function has property \( P \) and provided that the sum of a function with property \( P \) and any Baire-.5 function also has property \( P \). If \( P1 \) and \( P2 \) are \( B \)--.5-properties, the following terminology is used: A space \( X \) has the weak \( B \)--.5-insertion property for \( (P1,P2) \) iff for any functions \( g \) and \( f \) on \( X \) such that \( g \leq f \), \( g \) has property \( P1 \) and \( f \) has property \( P2 \), then there exists a Baire-.5 function \( h \) such that \( g \leq h \leq f \).

In this paper, for a topological space that \( \mathcal{F} \)-kernel of sets are \( \mathcal{F} \)-sets, is given a sufficient condition for the weak \( B \)--.5-insertion property. Also several insertion theorems are obtained as corollaries of these results.

2 The Main Result

Before giving a sufficient condition for insertability of a Baire-.5 function, the necessary definitions and terminology are stated.

Definition 2.1. Let \( A \) be a subset of a topological space \( (X, \tau) \). We define the subsets \( A^\Lambda \) and \( A^V \) as follows:

\[
A^\Lambda = \bigcap \{O : O \supseteq A, O \in (X, \tau)\} \quad \text{and} \quad A^V = \bigcup \{F : F \subseteq A, F^c \in (X, \tau)\}.
\]

In [6, 17, 18], \( A^\Lambda \) is called the kernel of \( A \).

We define the subsets \( G\delta(A) \) and \( \mathcal{F}\sigma(A) \) as follows:

\[
G\delta(A) = \bigcup \{O : O \subseteq A, O \text{ is } G\delta \text{ set}\} \quad \text{and} \quad \mathcal{F}\sigma(A) = \bigcap \{F : F \supseteq A, F \text{ is } \mathcal{F}\sigma \text{ set}\}.
\]

\( \mathcal{F}\sigma(A) \) is called the \( \mathcal{F}\sigma \)-kernel of \( A \).

The following first two definitions are modifications of conditions considered in [13, 14].

Definition 2.2. If \( p \) is a binary relation in a set \( S \) then \( p^- \) is defined as follows: \( x \) \( p^- \) \( y \) if and only if \( y \) \( p^- \) \( v \) implies \( x \) \( p^- \) \( y \) for any \( u \) and \( v \) in \( S \).

Definition 2.3. A binary relation \( p \) in the power set \( P(X) \) of a topological space \( X \) is called a strong binary relation in \( P(X) \) in case \( p \) satisfies each of the following conditions:

1) If \( A_i \) \( p \) \( B_j \) for any \( i \in \{1, \ldots, m\} \) and for any \( j \in \{1, \ldots, n\} \), then there exists a set \( C \) in \( P(X) \) such that \( A_i \) \( p \) \( C \) and \( C \) \( p \) \( B_j \) for any \( i \in \{1, \ldots, m\} \) and any \( j \in \{1, \ldots, n\} \).

2) If \( A \subseteq B \), then \( A \) \( p^- \) \( B \).

3) If \( A \) \( p \) \( B \), then \( \mathcal{F}\sigma(A) \subseteq B \) and \( A \subseteq G\delta(B) \).
The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

Definition 2.4. If f is a real-valued function defined on a space X and if \( \{ x \in X : f(x) < l \} \subseteq A(f, l) \subseteq \{ x \in X : f(x) \leq l \} \) for a real number \( l \), then \( A(f, l) \) is a lower indefinite cut set in the domain of \( f \) at the level \( l \).

We now give the following main results:

Theorem 2.1. Let \( g \) and \( f \) be real-valued functions on the topological space \( X \), that \( F_\sigma \)-kernel of sets in \( X \) are \( F_\sigma \)-sets, with \( g \leq f \). If there exists a \( F_\sigma \)-kernel \( \rho \) on the power set of \( X \) and there exist lower indefinite cut sets \( A(f, t) \) and \( A(g, t) \) in the domain of \( f \) and \( g \) at the level \( t \) for each rational number \( t \) such that if \( t_1 < t_2 \) then \( A(f, t_1) \rho A(g, t_2) \), then there exists a Baire-.5 function \( h \) defined on \( X \) such that \( g \leq h \leq f \).

Proof. Let \( g \) and \( f \) be real-valued functions defined on the \( X \) such that \( g \leq f \). By hypothesis there exists a strong binary relation \( \rho \) on the power set of \( X \) and there exist lower indefinite cut sets \( A(f, t) \) and \( A(g, t) \) in the domain of \( f \) and \( g \) at the level \( t \) for each rational number \( t \) such that if \( t_1 < t_2 \) then \( A(f, t_1) \rho A(g, t_2) \).

Define functions \( F \) and \( G \) mapping the rational numbers \( Q \) into the power set of \( X \) by \( F(t) = A(f, t) \) and \( G(t) = A(g, t) \). If \( t_1 \) and \( t_2 \) are any elements of \( Q \) with \( t_1 < t_2 \), then \( F(t_1) \rho \bar{G}(t_2), \overline{G(t_1)} \rho G(t_2) \), and \( F(t_1) \rho G(t_2) \). By Lemmas 1 and 2 of [14] it follows that there exists a function \( H \) mapping \( Q \) into the power set of \( X \) such that if \( t_1 \) and \( t_2 \) are any rational numbers with \( t_1 < t_2 \), then \( F(t_1) \rho H(t_2), H(t_1) \rho H(t_2) \) and \( H(t_1) \rho G(t_2) \). For any \( x \) in \( X \), let \( h(x) = \inf \{ t \in Q : x \in H(t) \} \).

We first verify that \( g \leq h \leq f \). If \( x \) is in \( H(t) \) then \( x \) is in \( G(t') \) for any \( t' > t \); since \( x \) in \( G(t') = A(g, t') \) implies that \( g(x) \leq t' \), it follows that \( g(x) \leq t \). Hence \( g \leq h \). If \( x \) is not in \( H(t) \), then \( x \) is not in \( F(t'/t) \) for any \( t'/t < t \); since \( x \) is not in \( F(t'/t) = A(f, t'/t) \) implies that \( f(x) > t'/t \), it follows that \( f(x) \geq t \). Hence \( h \leq f \).

Also, for any rational numbers \( t_1 \) and \( t_2 \) with \( t_1 < t_2 \), we have \( h^{-1}(t_1,t_2) = G_\delta(H(t_2)) \setminus F_\sigma(H(t_1)) \). Hence \( h^{-1}(t_1,t_2) \) is a \( G_\delta \)-set in \( X \), i.e., \( h \) is a Baire-.5 function on \( X \).

The above proof used the technique of theorem 1 of [13].

3 Applications

Definition 3.1. A real-valued function \( f \) defined on a space \( X \) is called contra-upper semi-Baire-.5 (resp. contra-lower semi-Baire-.5) if \( f^{-1}(-\infty,t) \) (resp. \( f^{-1}(t, +\infty) \)) is a \( G_\delta \)-set for any real number \( t \).

The abbreviations usc, lsc, cusB.5 and clsB.5 are used for upper semicontinuous, lower semicontinuous, contra-upper semi-Baire-.5, and contra-lower semi-Baire-.5, respectively.

Remark 1. [13, 14]. A space \( X \) has the weak c−insertion property for (usc, lsc) if and only if \( X \) is normal.

Before stating the consequences of theorem 2.1, we suppose that \( X \) is a topological space that \( F_\sigma \)-kernel of sets are \( F_\sigma \)-sets.

Corollary 3.1. For each pair of disjoint \( F_\sigma \)-sets \( F_1,F_2 \), there are two \( G_\delta \)-sets \( G_1 \) and \( G_2 \) such that \( F_1 \subseteq G_1, F_2 \subseteq G_2 \) and \( G_1 \cap G_2 = \emptyset \) if and only if \( X \) has the weak B-.5−insertion property for (cusB-.5, clsB-.5).

Proof. Let \( g \) and \( f \) be real-valued functions defined on the \( X \), such that \( f \) is lsB1, \( g \) is usB1, and \( g \leq f \). If a binary relation \( \rho \) is defined by \( A \rho B \) in case \( F_\sigma(A) \subseteq G_\delta(B) \), then by hypothesis \( \rho \) is a strong binary relation in the power set of \( X \). If \( t_1 \) and \( t_2 \) are any elements of \( Q \) with \( t_1 < t_2 \), then

\[
A(f, t_1) \subseteq \{ x \in X : f(x) \leq t_1 \} \subseteq \{ x \in X : g(x) < t_2 \} \subseteq A(g, t_2);
\]

since \( \{ x \in X : f(x) \leq t_1 \} \) is a \( F_\sigma \)-set and since \( \{ x \in X : g(x) < t_2 \} \) is a \( G_\delta \)-set, it follows that \( F_\sigma(A(f, t_1)) \subseteq G_\delta(A(g, t_2)) \). Hence \( t_1 < t_2 \) implies that \( A(f, t_1) \rho A(g, t_2) \). The proof follows from Theorem 2.1.

On the other hand, let \( F_1 \) and \( F_2 \) are disjoint \( F_\sigma \)-sets. Set \( f = \chi F_1 \) and \( g = \chi F_2 \), then \( f \) is clsB-.5, \( g \) is cusB-.5, and \( g \leq f \). Thus there exists Baire-.5 function \( h \) such that \( g \leq h \leq f \). Set
G1 = \{x \in X : h(x) < 1/2\} and

G2 = \{x \in X : h(x) > 1/2\}, then G1 and G2 are disjoint Gδ–sets such that F1 \subseteq G1 and F2 \subseteq G2.

Remark 2. [22]. A space X has the weak c−insertion property for (lsc, usc) if and only if X is extremally disconnected.

Corollary 3.2. For every G of Gδ–set, Fσ(G) is a Gδ–set if and only if X has the weak B−.5−insertion property for (clsB−.5, cusB−.5).

Before giving the proof of this corollary, the necessary lemma is stated.

Lemma 3.1. The following conditions on the space X are equivalent:

(i) For every G of Gδ−set we have Fσ(G) is a Gδ−set.

(ii) For each pair of disjoint Gδ−sets as G1 and G2 we have Fσ(G1) \cap Fσ(G2)= \emptyset.

The proof of lemma 3.1 is a direct consequence of the definition Fσ−kernel sets. We now give the proof of corollary 3.2.

Proof. Let g and f be real-valued functions defined on the X, such that f is clB−.5,g is cusB−.5, and f \leq g. If a binary relation ρ is defined by AρB in case Fσ(A) \subseteq G \subseteq Fσ(G) \subseteq Gδ(B) for some Gδ−set g in X, then by hypothesis and lemma 3.1 ρ is a strong binary relation in the power set of X. If t1 and t2 are any elements of Q with t1 < t2, then

A(g, t1)= \{x \in X : g(x) < t1\} \subseteq \{x \in X : f(x) \leq t2\};

= A(f, t2);

since (x \in X : g(x) < t1) is a Gδ−set and since (x \in X : f(x) \leq t2) is a Fσ−set, by hypothesis it follows that A(g, t1) \rho A(f, t2). The proof follows from Theorem 2.1.

On the other hand, Let G1 and G2 are disjoint Gδ−sets. Set f = \chi_{G2}

and g = \chi_{G1} c , then f is clB−.5,g is cusB−.5, and f \leq g.

Thus there exists Baire-.5 function h such that f \leq h \leq g. Set F1 = \{x \in X : h(x) \leq 1/3\} and F2 = \{x \in X : h(x) \geq 2/3\} then F1 and F2 are disjoint Fσ−sets such that G1 \subseteq F1 and G2 \subseteq F2. Hence Fσ(F1) \cap Fσ(F2)= \emptyset.

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References


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