An Introduction To Complex Arithmetic Calculus And An Original Reformulation Of The Goldbach Conjecture.

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Abstract
Prime numbers are well known (for simple characterizations of primes via divisibility, see [11] and [12] and [13] and [14] and [15]), and the Goldbach conjecture (see [1] or [2] or [3] or [4] or [5] or [6] or [7] or [8] or [9] or [10] or [16]) states that every even integer \( e \geq 4 \) is of the form \( e = p + q \), where \( q \) and \( p \) are prime. In this paper, we give an original reformulation of the Goldbach conjecture via complex arithmetic calculus. This reformulation shows that the Goldbach conjecture can be attacked without using strong investigations that have been on this conjecture in the past.

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Introduction

Prologue. This paper is divided into three sections. In Section.1, we introduce definitions that are not standard and we present some elementary properties deduced from these definitions. In Section.2, using definitions of Section.1, we give the trivial reformulation of the Goldbach conjecture that we will use in Section.3. In Section.3, we prove a proposition linked to complex arithmetic calculus and we use it to give an original reformulation of the Goldbach conjecture via the trivial reformulation of Section.2. This original reformulation shows that the Goldbach conjecture can be attacked without using strong investigations that have been on this conjecture in the past, and by using only complex arithmetic calculus.

1. Non-standard definitions and simple properties.

Definition 1.0. We say that \( e \) is goldbach, if \( e \) is an even integer \( \geq 4 \) and is of the form \( e = p + q \), where \( p \) and \( q \) are prime. Note that the Goldbach conjecture (see Abstract) states that every even integer \( e \geq 4 \) is goldbach.

Example 1.0. 4 is goldbach, since 4 is an even integer \( \geq 4 \) and 4 = 2 + 2, where 2 is prime; 6 is goldbach, since 6 is an even integer \( \geq 4 \) and 6 = 3 + 3, where 3 is prime; 8 is goldbach, since 8 is an even integer \( \geq 4 \) and 8 = 3 + 5, where 3 and 5 are prime; 10 is goldbach, since 10 is an even integer \( \geq 4 \) and 10 = 3 + 7, where 3 and 7 are prime; 12 is goldbach, since 12 is an even integer \( \geq 4 \) and 12 = 7 + 5, where 5 and 7 are prime; and 1764 is also goldbach, because 1764 is an even integer \( \geq 4 \) and is of the form 1764 = 883 + 881, where 883 and 881 are prime.

That being so, let us define:

Definition 1.1. We say that \( e \) is goldbachian, if \( e \) is an even integer \( \geq 4 \) and if every even integer \( v \) such that \( 4 \leq v \leq e \) is of the form \( v = p_v + q_v \), where \( p_v \) and \( q_v \) are prime; in other worlds, we say that \( e \) is goldbachian, if \( e \) is an even integer \( \geq 4 \) and if every even integer \( v \) with \( 4 \leq v \leq e \) is goldbach (see Definition 1.0 for the meaning of goldbach); in other terms again, we say that \( e \) is goldbachian, if \( e \) is an even integer \( \geq 4 \) and \( v \) is an even integer of the form \( 4 \leq v \leq e \), implies that \( v \) is goldbach. Using the previous definition, then we have the following trivial remarks.
Remark 1.1.0 12 is goldbachian.

Proof. Indeed, observe (by using Example 1.0.0 of Definition 1.0) that 12 is an even integer \( \geq 4 \), and every even integer \( v \) of the form \( 4 \leq v \leq 12 \) is goldbach; consequently 12 is goldbachian.

Remark 1.1.1. If \( d \) is goldbachian and if \( d' \) is an even integer of the form \( 4 \leq d' \leq d \), then \( d' \) is also goldbachian.

Proof. Immediate and is a trivial consequence of the definition of goldbachian introduced above.

Remark 1.1.2. 12 and 10 and 8 and 6 and 4 are simultaneously goldbachian.

Proof. Immediate and is a trivial consequence of Remark 1.1.0 and Remark 1.1.1.

Remark 1.1.3. For every integer \( n \in \{1, 2, 3, 4, 5\} \), \( 2n + 2 \) is goldbachian.

Proof. Immediate and is a trivial consequence of Remark 1.1.2.

Note that goldbachian implies goldbach; so there is no confusion between goldbachian and goldbach. Having defined goldbach and goldbachian, then it comes:

Definitions 1.2. For every integer \( n \geq 2 \), we define \( G(n) \), \( g_n \), \( G(n+1) \) and \( g_{n+1} \) as follows:

\[
G(n) = \{ g; 1 \leq g \leq 2n \text{ and } g \text{ is goldbachian} \}, \quad g_n = \max_{g \in G(n)} g.
\]

Using the definitions of \( G(n) \) and \( g_n \), then it becomes trivial to deduce that for every integer \( n \geq 1 \), we clearly have

\[
G(n+1) = \{ g; 1 \leq g \leq 2n+2 \text{ and } g \text{ is goldbachian} \}, \quad g_{n+1} = \max_{g \in G(n+1)} g.
\]

It is immediate that \( G(n) \subseteq G(n+1) \) for every integer \( n \geq 2 \), and therefore \( g_n \leq g_{n+1} \) for every integer \( n \geq 2 \). Using the previous definitions, then we have the following trivial remarks.

Remark 1.2.0 If \( n \geq 2 \), then \( G(n) \subseteq G(n+1) \) and \( g_n \leq g_{n+1} \). Proof. Immediate (it suffices to use the definitions of \( G(n) \), \( g_n \), \( G(n+1) \), \( g_n \) and \( g_{n+1} \).

Remark 1.2.1. If \( g_{n+1} \neq 2n+2 \), then \( G(n+1) = G(n) \) and \( g_{n+1} = g_n \). Proof. Immediate and is a trivial consequence of the definition of \( G(n) \), \( g_n \), \( G(n+1) \), \( g_{n+1} \) introduced above.

Remark 1.2.2. If \( g_{n+1} \leq 2n \), then \( G(n+1) = G(n) \) and \( g_{n+1} = g_n \). Proof. Observe that \( g_{n+1} \neq 2n+2 \) and use Remark 1.2.1.

Remark 1.2.3. For every integer \( n \in \{1, 2, 3, 4, 5\} \), we have \( g_{n+1} = 2n + 2 \). Proof. Immediate and is a trivial consequence of Remark 1.1.3 and the definition of \( g_{n+1} \).

Now using the definitions of \( G(n+1) \) and \( g_{n+1} \), then the following Proposition becomes trivial.

Proposition 1.3. Let \( n \) be an integer \( \geq 2 \). We have the following seven trivial properties.

1. \( g_{n+1} \) is even and \( g_{n+1} \leq 2n+2 \).
2. \( g_{n+1} = 2n+2 \), if and only if, \( 2n+2 \) is goldbachian (in other words, \( g_{n+1} \neq 2n+2 \), if and only if, \( 2n+2 \) is not goldbachian).
3. \( g_n \leq g_{n+1} \).
4. If \( g_{n+1} < 2n+2 \), then \( 2n+2 \) is not goldbachian.
5. If \( 2n+2 \leq e \) and if \( e \) is goldbachian, then \( 2n+2 \) is goldbachian.
6. (An implicit use of the Goldbach formula). If \( g_{n+1} < 2n+2 \), then \( 2n+2 \) is not goldbachian and there exists an integer \( e \) such that \( 1 \leq e \leq n \) and \( 2e+2 \) can not be of the form \( 2e+2 = p_e + q_e \), where \( p_e \) and \( q_e \) are prime.
7. (An explicit use of the Goldbach formula). \( g_{n+1} = 2n+2 \), if and only if, for every integer \( n' \) such that \( 1 \leq n' \leq n \), we have \( 2n' + 2 = p_{n'} + q_{n'} \), where \( p_{n'} \) and \( q_{n'} \) are prime.

Proof. Properties (1.3.0) and (1.3.1) are immediate (it suffices to use the definition of \( g_{n+1} \)). Property (1.3.2) is trivial (it suffices to use the definition of \( g_{n+1} \) via the definition of \( g_n \)); and property (1.3.3) is a trivial consequence of property (1.3.1). Property (1.3.4) is an immediate consequence of Remark 1.1.1 of Definition 1.1. Property (1.3.5) is trivial (it suffices to use property (1.3.3) and the definition of goldbachian (see Definition 1.1)), and property (1.3.6) is an immediate consequence of the definition of goldbachian and the definition of \( g_{n+1} \) (see Definition 1.1 for goldbachian and Definitions 1.2 for \( g_{n+1} \)).
We will use \( g_{n+1} \) to give the trivial reformulation of the Goldbach conjecture.

2. The trivial reformulation of the Goldbach conjecture.

Theorem 2.1. The following are equivalent.
(1). For every integer \( n' \geq 1 \), we have \( 2n' + 2 = p_{n'} + q_{n'} \), where \( p_{n'} \) and \( q_{n'} \) are prime.
(2) The Goldbach conjecture is true [i.e. every even integer \( e \geq 4 \) is of the form \( e = p_e + q_e \), where \( p_e \) and \( q_e \) are prime].
(3) For every integer \( n \geq 1 \), \( 2n + 2 \) is goldbachian.
(4) For every integer \( n \geq 1 \), we have \( g_{n+1} = 2n + 2 \).

Proof. (1) \( \Rightarrow \) (2) Immediate [since property (2) is only the obvious reformulation of property (1)]; (2) \( \Rightarrow \) (3) Immediate [it suffices to use the meaning of the Goldbach conjecture and the definition of goldbachian]; (3) \( \Rightarrow \) (4) Immediate [it suffices to use the definition of goldbachian and the definition of \( g_{n+1} \)]; (4) \( \Rightarrow \) (1) Immediate, by using property (1.3.6) of Proposition 1.3.

Theorem 2.1 is the trivial reformulation of the Goldbach conjecture. Theorem 2.1 will help us in Section 3 to give an original reformulation of the Goldbach conjecture via complex arithmetic calculus. Before, we need the following elementary combinatoric remark.

Remark 2.2. Let \( n \) be an integer \( \geq 1 \); consider \( \mathcal{G}(n + 1) \) and \( g_{n+1} \) (see Definitions 1.2). We have the following four properties.

(2.2.0.) \( g_{n+1} \) is even and \( 4 \leq g_{n+1} \leq 2n + 2 \).
(2.2.1.) If \( g_{n+1} \neq 2n + 2 \), then: \( n > 5 \) and \( g_{n+1} = g_n \).
(2.2.2.) \( \{ \) An implicit using of the Goldbach formula \( \} \). If \( g_{n+1} \neq 2n + 2 \), then: \( n > 5 \) and \( g_{n+1} = g_n \) and there exists an integer \( e \) such that \( 1 \leq e \leq n \) and \( 2e + 2 \) can not be of the form \( 2e + 2 = p + q \), where \( p \) and \( q \) are prime.
(2.2.3.) \( \{ \) Another implicit using of the Goldbach formula \( \} \). If \( g_{n+1} \leq 2n \), then: \( n > 5 \) and \( g_{n+1} = g_n \) and there exists an integer \( e \) such that \( 1 \leq e \leq n \) and \( 2e + 2 \) can not be of the form \( 2e + 2 = p + q \), where \( p \) and \( q \) are prime.

Proof. Property (2.2.0) is immediate. Indeed, it is immediate (by using the definition of \( g_{n+1} \)) that \( g_{n+1} \) is even. It is trivial that 4 is goldbachian (use Remark 1.1.2 of Definition 1.1) and so 4 \( \in \mathcal{G}(n + 1) \) and therefore \( g_{n+1} \leq 4 \). It is immediate that \( g_{n+1} \leq 2n + 2 \) (use the definition of \( g_{n+1} \)). Now using the previous two inequalities, then it becomes trivial to deduce that \( 4 \leq g_{n+1} \leq 2n + 2 \). Property (2.2.0) follows. Property (2.2.1) is also immediate. Indeed, if \( g_{n+1} \neq 2n + 2 \), clearly \( n > 5 \) (since \( g_{n+1} = 2n + 2 \) for \( n \in \{1, 2, 3, 4, 5\} \), by using Remark 1.2.3 of Definitions 1.2), and clearly
\[
\mathcal{G}(n + 1) = \mathcal{G}(n) \text{ and } g_{n+1} = g_n
\]

(observe that \( g_{n+1} \neq 2n + 2 \) and use Remark 1.2.2 of Definitions 1.2). Property (2.2.1) follows. Property (2.2.2) is only the trivial reformulation of property (2.2.1), by using the definition of \( g_{n+1} \) and \( g_n \) (see Definitions 1.2). Property (2.2.3) is an immediate consequence of property (2.2.2) \{ Indeed observe that \( g_{n+1} \neq 2n + 2 \) and use property (2.2.2) \}. Remark 2.2 immediately follows.

We will use Remark 2.2 in Section 3 to give an original reformulation of the Goldbach conjecture via complex arithmetic calculus.

3. Properties linked to complex arithmetic calculus and an original reformulation of the Goldbach conjecture.

In this section, we prove a proposition linked to complex arithmetic calculus and we use it to give an original reformulation of the Goldbach conjecture. This original reformulation via complex arithmetic calculus shows that the Goldbach conjecture can be attacked without using strong investigations that have been done on this conjecture in the past. Before, we need the following last definition.

Definition 3.0 (Fundamental). Let \( n \) be an integer \( \geq 1 \) and let \( g_{n+1} \) (see Definitions 1.2); then \( \phi_n \) is defined as follows.
\[
\phi_n = \left( i g_{n+1}^2 - 2 i g_n g_{n+1} + 2n \right)^2 , \quad i^2 = -1.
\]

It is immediate that for every integer \( n \geq 1 \), \( \phi_n \) is well defined and gets sense. Now using Definition 3.0, then we have the following elementary Proposition linked to complex arithmetic calculus.

Proposition 3.1 Let \( n \) be an integer \( \geq 2 \) and let \( g_{n+1} \) (see Definitions 1.2); now look at \( \phi_n \) introduced in Definition 3.0, and via \( \phi_n \), consider \( \phi_{n-1} \) (this consideration gets sense, since \( n \geq 2 \), and therefore \( n - 1 \geq 1 \)). If \( g_{n+1} \neq 2n + 2 \), then we have the following two simple properties.

(3.1.0.) \( \{ \) Implicit using of the Goldbach formula \( \} \). \( g_{n+1} = g_n \) and there exists an integer \( e \) such that \( 1 \leq e \leq n \) and
$2e + 2$ cannot be of the form $2e + 2 = p + q$, where $p$ and $q$ are prime.

(3.1.1) \[ \phi_n - \phi_n = -4g^5_{n+1} - 4g^4_{n+1} - 4ig^3_{n+1} - 8ig^2_{n+1} - 8n + 4 + 8ng^4_{n+1} + 16ing^2_{n+1}. \]

**Proof.** (3.1.0) Indeed observing (by the hypothesis) that $g_{n+1} \neq 2n + 2$, clearly $g_{n+1} = g_n$ and there exists an integer $e$ such that $1 \leq e \leq n$ and $2e + 2$ cannot be of the form $2e + 2 = p + q$, where $p$ and $q$ are prime. (use property (2.2.2) of Remark 2.2). Property (3.1.0) follows.

(3.1.1) Indeed, look at $\phi_n$, and observe (by Definition 3.0) that

\[ \phi_n = (ig^3_{n+1} - 2ing^2_{n+1} + 2n)^2 \]  

(3.1).

Now let $n - 1$ and look at $\phi_{n-1}$; then using equality (3.1), it becomes trivial to deduce that

\[ \phi_{n-1} = (ig^3_{n+1} - 2(n-1)g^2_{n+1} + 2(n-1))^2 \]  

(3.2).

Noticing (by property (3.1.0)) that $g_{n+1} = g_n$ and using the preceding equality, then it becomes trivial to deduce that equality (3.2) clearly says that

\[ \phi_{n-1} = (ig^3_{n+1} - 2(n-1)g^2_{n+1} + 2(n-1))^2 \]  

(3.3).

It is elementary to see that equality (3.3) clearly says that

\[ \phi_{n-1} = (ig^3_{n+1} - 2n^2g^2_{n+1} + 2n + 2ig^2_{n+1} - 2)^2 \]  

(3.4).

Look at equality (3.4); observing (by elementary computation) that

\[ (ig^3_{n+1} - 2n^2g^2_{n+1} + 2n + 2ig^2_{n+1} - 2)^2 = \lambda_n \]  

(3.5), where

\[ \lambda_n = (ig^3_{n+1} - 2n^2g^2_{n+1} + 2n)^2 + 2(2ig^2_{n+1} - 2)(ig^3_{n+1} - 2n^2g^2_{n+1} + 2n) + (2ig^2_{n+1} - 2)^2 \]  

(3.5').

then, using equalities (3.5) and (3.5') it becomes trivial to deduce that equality (3.4) says that

\[ \phi_{n-1} = (ig^3_{n+1} - 2n^2g^2_{n+1} + 2n)^2 + 2(2ig^2_{n+1} - 2)(ig^3_{n+1} - 2n^2g^2_{n+1} + 2n) + (2ig^2_{n+1} - 2)^2 \]  

(3.6).

Using equality (3.1), then it becomes elementary do deduce that equality (3.6) says that

\[ \phi_{n-1} = \phi_n + 2(2ig^2_{n+1} - 2)(ig^3_{n+1} - 2n^2g^2_{n+1} + 2n) + (2ig^2_{n+1} - 2)^2 \]  

(3.7).

Observing (by elementary computation and the fact that $i^2 = -1$) that

\[ 2(2ig^2_{n+1} - 2)(ig^3_{n+1} - 2n^2g^2_{n+1} + 2n) + (2ig^2_{n+1} - 2)^2 = \lambda'_n \]  

(3.8), where

\[ \lambda'_n = -4g^5_{n+1} - 4g^4_{n+1} - 4ig^3_{n+1} - 8ig^2_{n+1} - 8n + 4 + 8ng^4_{n+1} + 16ng^2_{n+1} \]  

(3.8').

then, using equalities (3.8) and (3.8'), it becomes trivial to deduce that equality (3.7) says that

\[ \phi_{n-1} = \phi_n + 4g^5_{n+1} + 4g^4_{n+1} + 4ig^3_{n+1} + 8ig^2_{n+1} - 8n + 4 + 8ng^4_{n+1} + 16ng^2_{n+1} \]  

(3.9).

Using equality (3.9), then we immediately deduce that

\[ \phi_{n-1} - \phi_n = -4g^5_{n+1} - 4g^4_{n+1} - 4ig^3_{n+1} + 8ig^2_{n+1} - 8n + 4 + 8ng^4_{n+1} + 16ng^2_{n+1}. \]

Property (3.1.1) follows and Proposition 3.1 immediately follows.

Having proved the previous simple Proposition linked to complex arithmetic calculus, we are now ready to give an original reformulation of the Goldbach conjecture.

**Theorem 3.2 (An original reformulation of the Goldbach conjecture).** The following are equivalent.

1. The Goldbach conjecture is true [i.e. every even integer $e \geq 4$ is of the form $e = p + q$, where $p$ and $q$ are prime].
2. For every integer $n \geq 2$

\[ \phi_n - \phi_n \neq -4g^5_{n+1} - 4g^4_{n+1} - 4ig^3_{n+1} - 8ig^2_{n+1} - 8n + 4 + 8ng^4_{n+1} + 16ng^2_{n+1}. \]

**Proof.** (1) $\Rightarrow$ (2). Observe (by remarking that the Goldbach conjecture is true and by using Theorem 2.1) that

For every integer $n \geq 1$, we have $g_{n+1} = 2n + 2$ \hspace{1cm} (3.10).

Using the definition of $g_{n+1}$, then it becomes trivial to deduce that (3.10) clearly implies that

For every integer $n \geq 2$, we have $g_n = 2n$ \hspace{1cm} (3.11).
Now look at $\phi_n$ introduced in Definition 3.0; then using equality of (3.10), it becomes trivial to deduce that

$$\phi_n = (2ig_{n+1}^2 + 2n)^2 \quad (3.12).$$

That being so, consider $\phi_{n-1}$ (this consideration gets sense, since $n \geq 2$, and therefore $n - 1 \geq 1$), then using equality of (3.11), it becomes trivial to deduce that

$$\phi_{n-1} = (ig_n^1 - 2i(n-1)g_n^2 + 2(n-1))^2 = (2n + 2ig_n^2 - 2)^2 \quad (3.13).$$

Using equalities (3.12) and (3.13), then it becomes trivial to check (by elementary computation and the fact that $i^2 = -1$ and by using (3.10) and (3.11)) that

$$\phi_n - \phi_n = -4g_{n+1}^2 = 4g_{n+1}^2 - 4isg_{n+1}^2 - 8isg_{n+1}^2 - 8n + 4 + 8ng_{n+1}^4 + 16ng_{n+1}^2.$$

(1) $\Rightarrow$ (2) Otherwise (we reason by reduction to absurd), let $n$ be an integer $\geq 1$ such that

$$g_{n+1} \neq 2n + 2 \quad (3.14)$$

(observe that such a $n$ exists, by remarking that the Goldbach conjecture is false and by using Theorem 2.1). Clearly

$$n > 5 \quad (3.15)$$

(use (3.14) and property (2.2.1) of Remark 2.2). Using (3.14) and (3.15), then it becomes immediate to deduce all the hypotheses of Proposition 1.3 are satisfied for such a $n$; therefore, all the conclusion of Proposition 1.3 are satisfied for such a $n$; in particular, property (3.1.1) of Proposition 1.3 is satisfied for such a $n$. Consequently

$$\phi_{n-1} - \phi_n = -4g_{n+1}^2 = 4g_{n+1}^2 - 4isg_{n+1}^2 - 8isg_{n+1}^2 - 8n + 4 + 8ng_{n+1}^4 + 16ng_{n+1}^2,$$

and the previous equality gives rise to a serious contradiction.

Theorem 3.2 is an original reformulation of the Goldbach conjecture via complex arithmetic calculus and is stronger than all the investigations that have been done on the Goldbach conjecture in the past. Indeed, Theorem 3.2 clearly says that: if for every integer $n \geq 2$, $\phi_{n-1} - \phi_n = -4g_{n+1}^2 = 4g_{n+1}^2 - 4isg_{n+1}^2 - 8isg_{n+1}^2 - 8n + 4 + 8ng_{n+1}^4 + 16ng_{n+1}^2$, then the Goldbach conjecture immediately follows. Visibly, Theorem 3.2 is not related to all the investigations that have been done on the Goldbach conjecture in the past and can be transformed to attack the Goldbach conjecture in an original way.

References


